MATH 234B, Winter 2000

Riemannian Geometry I

Midterm

1. (10 points.) Let M be a connected Riemannian manifold. For two points p and q in M, set

$$d(p,q) = \inf_{\gamma} l(\gamma),$$

where γ is a smooth curve with end–points p and q and $l(\gamma)$ denotes the length of γ.

Give a detailed proof of the facts that d is a metric on M and that the topology induced by d coincides with the original topology of M.

2. (20 points) Consider the plane $\mathbb{R}^2$ with coordinates $x = (x_1, x_2)$ and the standard Euclidean metric $\langle\ ,\ \rangle$. Let

$$\alpha(x) = Bx_1 \frac{\partial}{\partial x_2} - Bx_2 \frac{\partial}{\partial x_1},$$

where B is a constant, and let

$$L(\gamma) = \int_0^1 \frac{1}{2} \langle \dot\gamma(t) - \alpha(\gamma(t)), \dot\gamma(t) - \alpha(\gamma(t)) \rangle \, dt,$$

where $\gamma\colon [0,1] \to \mathbb{R}^2$. Find, if it exists, a curve γ from $(-1,0)$ to $(1,0)$ which is a critical point of L.

3. (20 points) Prove that $so(3)$ is the Lie algebra of $\mathrm{SO}(3)$.

Hint: You need to show two things: (1) $so(3) = T_I \mathrm{SO}(3)$, where I is the identity matrix; (2) the Lie bracket $[A,B] = AB - BA$ on $so(3)$ is equal to the bracket on $T_I \mathrm{SO}(3)$ obtained by identifying the tangent space with the space of right invariant vector fields.

4. Let G be a $\langle\ ,\ \rangle$ be a bi-invariant metric on a Lie group G and let ∇ be the Levi-Civita connection. Let X, Y, Z be unit left invariant vector fields.

- **a.** (10 points) Show that $\nabla_X Y = [X,Y]/2$. (Hint: first show that $\nabla_Z Z = 0$ for any left invariant vector field and then use the fact that the connection is symmetric.)
- **b.** (5 points) Prove that $R(X,Y)Z = [[X,Y],Z]/4$.

5. Let M be a manifold with a Riemannian metric $g = \langle\ ,\ \rangle$ and let ∇ be a connection compatible with g.

a. (5 points.) Let A be a one-form with values in skew-symmetric operators on tangent spaces, i.e., for every point $p \in M$, we have a linear mapping (smooth in p) $T_pM \to so(T_pM)$. Denote this mapping by $Z \mapsto A_Z$ so that $A_Z \colon T_pM \to T_pM$ is skew-symmetric for every Z.

Prove that $\nabla' = \nabla + A$ (i.e., $\nabla'_Z X = \nabla_Z X + A_Z X$) is a connection compatible with g.

b. (10 points.) Conversely, assume that ∇' is compatible with g. Prove that there exists a unique skew-symmetric A such that $\nabla' = \nabla + A$.

c. (15 points.) Assume now that $M = \mathbb{R}^2$ with the standard Euclidean metric g and let ∇ be the Levi-Civita connection for g. Write in the standard coordinates

$$A = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix},$$

where α is a one-form on $M = \mathbb{R}^2$. By (a), $\nabla' = \nabla + A$ is compatible with g.

Prove that the curvature of ∇' is

$$R = \begin{bmatrix} 0 & d\alpha \\ -d\alpha & 0 \end{bmatrix}.$$

d. (5 points.) Find the torsion tensor of ∇'.

This exercise shows, in particular, that there are many connections compatible with a given Riemannian metric. In fact, as can be seen from (a) and (b), such connections, when one of them is fixed, form an infinite-dimensional vector space.