MATH 234A, Fall 2000

Differential Topology

Midterm

1. Let Π be the square $\max\{|x|, |y|\} = 1$ on $\mathbb{R}^2$ with coordinates x and y.

- **a.** (5 points) Show in detail that Π is not a smooth submanifold of $\mathbb{R}^2$.
- **b.** (5 points) Prove that Π is a topological submanifold which is homeomorphic to S^1.

2. Let M and N be n-dimensional manifolds and let $F\colon M \to N$ be an immersion.

- **a.** (5 points) Show that F is an open map, i.e., the image of an open set is open.
- **b.** (5 points) Assume additionally that M is compact and N is connected. Prove that then F is onto.

3. (15 points) Prove that $\mathbb{R}P^2$ is not orientable, i.e., the tangent bundle to $\mathbb{R}P^2$ is not orientable. (Hint: First prove that the Möbius strip is not orientable. Then find an open subset in $\mathbb{R}P^2$ diffeomorphic to the Möbius strip.)

4. (10 points) Let E be a (real) line bundle over a manifold M. Prove that E is orientable if and only if E is trivial.

5. (15 points) Let E be a vector bundle over a compact manifold M. Prove that there exists a vector bundle monomorphism from E to a trivial line bundle $M \times \mathbb{R}^k$. (Hint: adapt the proof of the fact that a compact manifold can be embedded into some $\mathbb{R}^l$.)

6.

- **a.** (10 points) Let v be a vector field on M. Denote by ϕ^t the (local) flow of v. Let $F\colon M \to N$ be a diffeomorphism. Show that $F\phi^t F^{-1}$ is the (local) flow of $F_* v$. (Hint: prove that $F(\phi^t(F^{-1}(q)))$ is the integral curve of $F_* v$ through $q \in N$.)
- **b.** (5 points) Let $F\colon M \to M$ be a diffeomorphism and let, as above, v be a vector field on M with (local) flow ϕ^t. Prove that $F_* v = v$ if and only if $F\phi^t = \phi^t F$ for all t.

c. (10 points) Let v and w be two vector fields on M. Denote their local flows by ϕ^t and ψ^s, respectively. Prove that ϕ^t and ψ^s commute, i.e., $\phi^t\psi^s = \psi^s\phi^t$ for all t and s, if and only if $[v, w] = 0$.

7. Consider the following vector fields on $\mathbb{R}^3$:

$$v = \frac{\partial}{\partial x} \quad \text{and} \quad w = x\frac{\partial}{\partial z} + \frac{\partial}{\partial y} \ .$$

(a) (5 points) Find $[v, w]$.

(b) (10 points) Assume that f is a smooth function on $\mathbb{R}^3$ such that

$$L_v f = L_w f = 0$$

at every point. Prove that $f = const$. Hint: first show that $L_u f = 0$ for any vector u. Then use (and prove) that

$$f(\gamma(1)) - f(\gamma(0)) = \int_0^1 L_{\dot\gamma(t)} f \, dt$$

for any smooth curve $\gamma\colon [0, 1] \to \mathbb{R}^3$.