Mathematics 203 Fall 1999/2000, V. Ginzburg
Midterm

| Print Your Name: |
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All 6 questions should be answered on this exam using the backs of the sheets if necessary. The exam has 6 pages.

Show all your work and justify all claims when a proof is required. Good Luck!!

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| Total | /100 |

1. For each of the questions below, indicate if the statement is *true* or *false*. You do **not** need to justify the answer. Each correct answer is worth 2 points.

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| a | |
| b | |
| c | |
| d | |
| e | |

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| f | |
| g | |
| h | |
| i | |
| j | |

(a) The boundary of the set $\{(x, \sin(1/x)) \mid x > 0\} \subset \mathbb{R}^2$ is $\{0\} \times [0, 1]$.

(b) A function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous if and only if it is continuous with respect to each variable.

(c) The space $\mathbb{R}^\omega$ with product topology is metrizable.

(d) A product of Hausdorff spaces is Hausdorff.

(e) Let $A$ be a connected subset of a topological space $X$ and let $A \subset B \subset \bar{A}$. Then $B$ is connected.

(f) The following two spaces are homeomorphic:

$X = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0\} \cup \{(0, 0)\}$ and $Y = \{(x, y) \in \mathbb{R}^2 \mid x > 0\} \cup \{(0, 0)\}$.

(g) Let $f\colon X \to Y$ be continuous and onto and let $X$ be path connected. Then $Y$ is path connected.

(h) A bounded closed subset of a metric space is compact.

(i) A subset of a locally compact metric space is locally compact.

(j) A sequentially compact metric space is compact.

2. Let $A$ be a set and $\{X_\alpha\}$, $\alpha \in J$, an indexed family of spaces and let $f_\alpha \colon A \to X_\alpha$, be $\alpha \in J$, an indexed family of functions.

   (a) (5 points) Prove that there is a unique coarsest topology $\mathcal{T}$ on $A$ with respect to which each of the functions $f_\alpha$ is continuous.

   (b) (5 points) prove that $g \colon Y \to A$ is continuous with respect to $\mathcal{T}$ if and only if each of the functions $f_\alpha \circ g$ is continuous.

3. (15 points) Let $X$ be a compact Hausdorff space and let $Y = X \setminus \{x\}$, where $x$ is a point of $X$. Prove that the one point compactification of $Y$ is homeomorphic to $X$.

4. (20 points) Let $X$ be the complement of the set $\{n \mid n > 0,\ n \in \mathbb{Z}\}$ in $\mathbb{R}^2$ and let $Y$ be the complement of the set $\{1/n \mid n > 0,\ n \in \mathbb{Z}\}$ in $\mathbb{R}^2$. Are $X$ and $Y$ homeomorphic? Prove your claim.

Hint: Is $X$ locally compact? Is $Y$ locally compact?

5. (20 points) Let $X$ be a compact Hausdorff space and $A_1 \supset A_2 \supset \ldots$ a nested sequence of closed connected subsets of $X$. Prove that $Y = \bigcap A_n$ is connected.

Hint: Assume that $Y = B \sqcup C$, where $B$ and $C$ are closed in $Y$. First note that $B$ and $C$ have disjoint open neighborhoods, say, $U$ and $V$. (Why?) Show that $\bigcap (A_n \setminus (U \cup V))$ is non-empty.

6. (15 points) Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a contraction. (In other words, there exists an $\alpha < 1$ such that $\mathrm{d}(f(x), f(y)) \leq \alpha \mathrm{d}(x, y)$ for all $x$ and $y$ in $\mathbb{R}^n$.) Show that $f$ has a unique fixed point, i.e. there is a unique $p \in \mathbb{R}^n$ such that $f(p) = p$.

Hint: Pick an arbitrary point $x \in \mathbb{R}^n$. Then $p$ is the limit of the sequence $f^k(x)$.