Mathematics 117 Fall 1999/2000, V. Ginzburg

Midterm I

| Print Your Name: |
|---|

All 6 questions should be answered on this exam using the backs of the sheets if necessary. The exam has 6 pages.

Show all your work and justify all claims when a proof is required. Good Luck!!

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| Total | /100 |

1. For each of the questions below, indicate if the statement is *true* or *false*. You do **not** need to justify the answer. Each correct answer is worth 2 points.

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| a | |
| b | |
| c | |
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| e | |

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| f | |
| g | |
| h | |
| i | |
| j | |

   (a) Let $W$ be a subspace of a vector space $V$. Then $W + V = V$.

   (b) Let $W$ and $U$ be subspaces of a finite–dimensional vector space $V$ such that $\dim W + \dim U = \dim V$. Then $W + U = V$ if and only if $W \oplus U = V$.

   (c) Every linearly independent list of vectors in a finite–dimensional vector space can be extended to a basis.

   (d) The space of all polynomials is finite–dimensional.

   (e) Let $v_1, \ldots, v_n$ be a basis in $V$ and let $U$ be a subspace of $V$. Then the list $v_1, \ldots, v_n$ can be reduced to a basis in $U$.

   (f) A vector space is isomorphic to $\mathbb{R}^n$ if and only if it has a basis of $n$ elements.

   (g) A linear map $T \in \mathcal{L}(V, W)$ is invertible if and only if there exists a linear map $S \in \mathcal{L}(W, V)$ such that $ST = I$.

   (h) Assume that $V$ is finite–dimensional and $V = W + U$. Then $\dim V = \dim W + \dim U$.

   (i) Let $V$ be finite–dimensional. A linear map $T \in \mathcal{L}(V, V)$ is invertible if and only if it is surjective.

   (j) Let $V$ and $U$ be finite–dimensional. Then $\dim \mathcal{L}(V, W) = \dim V \cdot \dim W$.

2. Consider the space $\mathcal{P}_n(\mathbb{R})$ of polynomials of degree less than or equal to $n$.

   (a) (5 points) Let $U$ be the set of all polynomials $f \in \mathcal{P}_n(\mathbb{R})$ such that $f'(0) = f''(0) = 0$. (Here $f'$ denotes the derivative of $f$ and $f''$ is the second derivative of $f$.) Prove that $U$ is a subspace of $\mathcal{P}_n(\mathbb{R})$.

   (b) (10 points) Find a subspace $W$ of $\mathcal{P}_n(\mathbb{R})$ such that $U \oplus W = \mathcal{P}_n(\mathbb{R})$.

   (c) (5 points) What is $\dim U$? (Prove your claim.)

3. Let $V = \mathcal{P}_3(\mathbb{R})$. Consider the following vectors in $V$:

$$\begin{aligned} v_1 &= 1, \\ v_2 &= z + 1, \\ v_3 &= z^2 + z + 1, \\ v_4 &= z^3 + z^2 + z + 1. \end{aligned}$$

(a) (5 points) Prove that $v_1, \dots, v_4$ is a basis in $V$.

(b) (10 points) Find the matrix of $T \in \mathcal{L}(V)$ such that $Tf = f'$ in this basis.

4. Let $T \in \mathcal{L}(V, W)$. Let $v_1, \dots, v_n$ be a basis in $V$ such that $v_1, \dots, v_k$, where $k \leq n$, is a basis in *null* $T$.

   (a) (15 points) Prove that the collection $Tv_{k+1}, \dots, Tv_n$ is linearly independent in $W$.

   (b) (5 points) Prove that $\dim W \geq n - k$.

5. Let $V$ be finite–dimensional and $T \in \mathcal{L}(V, V)$.

(a) (10 points) Prove that $T$ is surjective if and only if there exists $S \in \mathcal{L}(V, V)$ such that $TS = I$.

(b) (5 points) Prove that $T$ is invertible if and only if there exists $S \in \mathcal{L}(V, V)$ such that $TS = I$.

6. Let $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$. Assume that $ST$ is injective

(a) (5 points) Is $T$ necessarily injective? Prove or give a counterexample.

(b) (5 points) Is $S$ necessarily injective? Prove or give a counterexample.