Mathematics 105B Winter 2001, V. Ginzburg

Midterm

Print Your Name:

All questions should be answered on this exam using the backs of the sheets if necessary. The exam has 6 pages.

Show all your work and justify all claims when a proof is required. Good Luck!!

1	
2	
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5	
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Total	/100

1. For each of the questions below, indicate if the statement is *true* or *false*. You do **not** need to justify the answer. Each correct answer is worth 2 points.

a	
b	
c	
d	
e	

f	
g	
h	
i	
j	

(a) Let $f(x) = 1/x^2$ for $x > 0$ and $f(0) = 0$. Then f is integrable on $[0, 1]$.

(b) Assume that f is a differentiable function and f' is continuous. Then $f(x) = \int_0^x f'(t)\,dt + f(0)$.

(c) Let f be an integrable function on $[0, 1]$. Set $F(x) = \int_0^x f(t)\,dt$. Then $F' = f$.

(d) Let f be an integrable function on $[0, 1]$ such that $\int_0^1 f(t)^2\,dt = 0$. Then $f(x) = 0$ at every point $x \in [0, 1]$.

(e) Assume that f is differentiable function $[a, b] \to \mathbb{R}^n$. Then there exists a constant M such that $\| f(b) - f(a) \| \leq M(b - a)$.

(f) The series $\sum \frac{1}{n(\log n)^2}$ converges.

(g) Assume that $f\colon [a, b] \to \mathbb{R}^n$ is a differentiable function and $n \geq 2$. Then there exists $x \in (a, b)$ such that $f(b) - f(a) = f'(x)(b - a)$.

(h) Every continuous function on $[0, 1]$ is differentiable at some point of $[0, 1]$.

(i) Assume that f is integrable and g is continuous. Then $g \circ f$ is integrable.

(j) Let f be monotone and bounded on $[a, b]$ and let α be continuous on $[a, b]$. Then $f \in \mathcal{R}(\alpha)$.

2. (10 points) Suppose that f is twice differentiable on some neighborhood of c and that $f''(c)$ exists. Prove that

$$\frac{f''(c)}{2} = \lim_{h\to\infty} \frac{f(c+h) - f(c) - f'(c)h}{h^2}.$$

Hint: Use l'Hôpital's rule.

3. (10 points) Suppose that f' exists and is continuous on $[a, b]$. Prove that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(t)}{x - t} - f'(x) \right| < \epsilon$$

whenever $0 < |x - t| < \delta$ and x and t are in $[a, b]$.

4. (15 points) Let n be a positive even integer. Prove that

$$\sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{x^{2n+1}}{(2n+1)!}$$

for any $x \in [0, \pi]$.

5. (15 points). Define a function f on $[0,1]$ as follows:

$$f(x) = \begin{cases} x, & x \text{ is rational}, \\ 0, & x \text{ is irrational}. \end{cases}$$

Find the upper and lower integrals for f and determine whether f is integrable or not.

6. (a) (10 points) Prove that

$$\lim_{h \to 0} \int_{-a}^{a} \frac{h}{h^2 + x^2}\, dx = \pi$$

for any $a > 0$.

(b) (30 points) Assume that f is continuous on $[-1, 1]$. Prove that

$$\lim_{h \to 0} \int_{-1}^{1} \frac{h}{h^2 + x^2} f(x)\, dx = \pi f(0).$$