MATH 234B, Winter 2000

Riemannian Geometry I

Final Exam

1. (10 points.) For X and Y in $so(n)$ set $\langle X, Y\rangle_I = -\mathrm{tr}(XY)$. Extend $\langle , \rangle_I$ from $T_I\,\mathrm{SO}(n) = so(n)$ to a metric $\langle , \rangle$ on $\mathrm{SO}(n)$ by left translations. Clearly, the resulting metric $\langle , \rangle$ is left-invariant. Prove that $\langle , \rangle$ is in fact bi-invariant.

2. (10 points.) Let $\varphi^t\colon M \to M$ be the flow of a vector field X on a manifold M. (We assume that φ^t is defined for all $t \in \mathbb{R}$.) Prove that for every closed differential form α on M, the forms $(\varphi^t)^*\alpha$ and α belong to the same cohomology class.

Hint: $(\varphi^t)^*\alpha - \alpha = \int_0^t \frac{d}{d\tau}\left((\varphi^\tau)^*\alpha\right)\, d\tau$.

3. (10 points.) Denote by D the two-dimensional disc. Let $f\colon D \to \mathbb{R}^2$ be a smooth mapping such that the closed curve $f|_{\partial D}$ has a non-zero winding number with respect to the origin: $W(f|_{\partial D}, 0) \neq 0$. Prove that $f^{-1}(0)$ is non-empty, i.e., the equation $f(z) = 0$ has at least one solution $z \in D$.

Hint: f is a homotopy between $f|_{\partial D}$ and a constant mapping.

Remark: It is very instructive (and not so difficult) to derive the Fundamental Theorem of Algebra from Problem 3.

4. (10 points.) For every $n \in \mathbb{Z}$ give an example of a map $f\colon S^2 \to S^2$ which has degree n.

Hint: $S^2 \subset \mathbb{R} \times \mathbb{C}$ is given by the equation $x^2 + |z|^2 = 1$, where $x \in \mathbb{R}$ and $z \in \mathbb{C}$.

5. (5 points.) Let M be a Riemannian manifold. Prove the Bianchi identity:

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

for any vectors X, Y, and Z.

6. (5 points.) Give an example of a metric on $\mathbb{T}^2$ which has conjugate points. Hint: look for an appropriate embedding $\mathbb{T}^2 \hookrightarrow \mathbb{R}^3$. As a solution of this problem, it suffices to draw a picture with an adequate explanation.

7. Let M be a surface with a Riemannian metric. By definition the geodesic polar coordinates (r, θ) near $p \in M$ are obtained by exporting the genuine polar coordinates on T_pM to M by means of the exponential map.

- **a.** (5 points.) Show that near p the metric on M has the form $(dr)^2 + g(r, \theta)(d\theta)^2$.
- **b.** (10 points.) Prove that for $r > 0$,
$$K = -\frac{\frac{\partial^2 \sqrt{g}}{\partial r^2}}{\sqrt{g}}$$
is the curvature of M.

8. Let M be a Riemannian manifold such that $R = 0$ for the Levi-Civita connection on M. The goal of this exercise is to show that M is locally diffeomorphic to $\mathbb{R}^n$ with its standard Euclidean metric.

- **a.** (15 points.) Let γ_0 and γ_1 be smooth curves in M connecting points p and q. Assume that these curves are homotopic with fixed end-points. Prove that the parallel transport from T_pM to T_qM along γ_0 is equal to the parallel transport along γ_1.

  Hint: Use the formula
$$\frac{D}{\partial t}\frac{D}{\partial u}V - \frac{D}{\partial u}\frac{D}{\partial t}V = R(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial t})V$$
for the parametrized surface f giving a homotopy between γ_0 and γ_1.
- **b.** (5 points.) Let now U be a small neighborhood of a point $p \in M$, which is diffeomorphic to an open ball in $\mathbb{R}^n$. Fix $X(p) \in T_pM$. For $q \in U$, let $X(q) \in T_qM$ be the vector obtained by the parallel transport of $X(p)$ to q. By Part (a), $X(q)$ is well defined. Prove that $\nabla_Y X = 0$ for every vector Y.
- **c.** (5 points.) Apply Part (b) to vectors $X_1(p), \ldots, X_n(p)$ which form an orthonormal basis in T_pM. Prove that $[X_i, X_j] = 0$ for all i and j.
- **d.** (10 points.) Show that there exist coordinates $(x_1, \ldots, x_n)$ near p such that $X_i = \partial/\partial x_i$ for all i and that in these coordinates the metric is the standard Euclidean metric.