MATH 234A, Fall 2000

Differential Topology

Final

1. (10 points) Let N be a smooth manifold and $M = TN$. Prove that N is orientable as manifold, regardless of whether N is orientable or not. (In fact, M has a canonical orientation.)

2.

a. (5 points) Prove that the antipodal map $S^n \to S^n$ is orientation preserving if and only if n is odd.

b. (5 points) Prove that the real projective space $\mathbb{R}P^n$ is orientable if and only if n is odd.

3. (Poincaré's Lemma) Let v be a vector field on a manifold M. Assume that the flow φ_t of v is defined for all $t \in \mathbb{R}$.

a. (5 points) Prove that $i_v \varphi_t^* \alpha = \varphi_t^* i_v \alpha$ for any differential form α.

b. (5 points) Prove that

$$\frac{d}{dt}\varphi_t^* \alpha = L_v \varphi_t^* \alpha = \varphi_t^* L_v \alpha$$

for all $t \in \mathbb{R}$ and any differential form α.

Let now $M = \mathbb{R}^n$ and v be given by $v(x) = -x$, $x \in \mathbb{R}^n$. (Here we identify $T_x\mathbb{R}^n$ and $\mathbb{R}^n$.) Let $k \geq 1$. Consider the linear operator $\Phi_k : \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n)$ defined by

$$(\Phi_k \alpha)(w_1, \ldots, w_{k-1}) = -\int_0^\infty (\varphi_t^* i_v \alpha)(w_1, \ldots, w_{k-1})\, dt$$

for a k-form α on $\mathbb{R}^n$.

c. (5 points) Prove that the improper integral in the definition of $\Phi_k \alpha$ converges.

d. (10 points) Complete the proof of Poincaré's Lemma by showing that $d\Phi_k \alpha + \Phi_{k+1} d\alpha = \alpha$ for any k-form α on $\mathbb{R}^n$. (Thus, if α is closed, $\alpha = d\Phi_k \alpha$ and, hence, α is exact.)

4. Let M be connected oriented n-dimensional manifold without boundary. Let α be a compactly supported n-form on M. The goal of this series of problems is to show that $\int_M \alpha = 0$ implies that $\alpha = d\beta$ for some compactly supported $(n-1)$-form β. As a consequence, $H^n(M) = \mathbb{R}$, when M is compact.

a. (5 points) Prove the converse of the above assertion for a compact M.

b. (5 points) Prove the assertion for $M = \mathbb{R}$.

c. (10 points) Prove the assertion for $M = \mathbb{R}^n$.

Hint: Use induction in n as follows. Let $\alpha = f\, dx_1 \wedge \ldots \wedge dx_n$. Consider the form $\bar{\alpha} = \bar{f}\, dx_1 \wedge \ldots \wedge dx_{n-1}$ on $\mathbb{R}^{n-1}$, where

$$\bar{f}(x_1, \ldots, x_{n-1}) = \int_{-\infty}^{\infty} f(x_1, \ldots, x_{n-1}, t)\, dt.$$

By the induction hypothesis, $\bar{\alpha} = d\lambda$ on $\mathbb{R}^{n-1}$. Let g be any compactly supported function in one variable with $\int_{-\infty}^{\infty} g(t)\, dt = 1$. Consider the form

$$\alpha_1 = \alpha - d\big(g(x_n)\pi^*\lambda \wedge dx_n\big)$$

on $\mathbb{R}^n$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection along the x_n-axis. Arguing as in **b** prove that $\alpha_1 = d\beta_1$ for some compactly supported $(n-1)$-form β_1 on $\mathbb{R}^n$.

d. (15 points) Prove the assertion for an arbitrary M.

5. (The Fundamental Theorem of Algebra) Let $f : \mathbb{C} \to \mathbb{C}$ be the polynomial $f(z) = z^n + a_1 z^{n-1} + \ldots + a_n$.

a. (10 points) Prove that $f : \mathbb{C} \to \mathbb{C}$ is proper.

b. (5 points) Prove that $\deg f = n$. Hint: Show that $f_t(z) = tf(z) + (1-t)z^n$, where $t \in [0,1]$, is a proper homotopy.

c. (5 points) Conclude that for any $w \in \mathbb{C}$ the equation $f(z) = w$ has at least one solution.