Mathematics 203 Fall 1999/2000, V. Ginzburg

Final Exam

Print Your Name:

When working on this exam you may use the textbook ("Topology" by J. Munkres).

Show all your work and justify all claims when a proof is required. Good Luck!!

1	
2	
3	
4	
5	
6	
7	
Total	/100

1. For each of the questions below, indicate if the statement is *true* or *false*. You do **not** need to justify the answer. Each correct answer is worth 2 points.

a	
b	
c	
d	
e	

f	
g	
h	
i	
j	

(a) Let $F_1 \supset F_2 \supset F_3 \supset \ldots$ be a nested sequence of closed subsets of a complete metric space such that $\operatorname{diam} F_n \to 0$ as $n \to \infty$. Then $\bigcap F_n \neq \emptyset$.

(b) The following two spaces are homeomorphic:

$$\mathbb{R}^2 \setminus \{(n, 0) \mid n \in \mathbb{Z}\} \text{ and } \mathbb{R}^2 \setminus \{(n, 0) \mid n \in \mathbb{Z},\ n > 0\}$$

(c) A Hausdorff topological space is compact if and only if it is limit point compact.

(d) Assume that X and Y are normal. Then $X \times Y$ is normal.

(e) Every metric space has a countable dense subset.

(f) A bounded closed subset of a complete metric space is compact.

(g) For $f \in C([0,1], \mathbb{R})$, set $\| f \| = \sup |f(x)|$. Let $D \subset C([0,1], \mathbb{R})$ be the set of continuously differentiable functions such that $\| f \| \leq 1$ and $\| f' \| \leq 1$. Then the closure of D with respect to the uniform metric is compact in $C([0,1], \mathbb{R})$.

(h) The space $\mathbb{R}^\omega$ is connected in the box topology.

(i) The space $\mathbb{R}^\omega$ is path connected in the product topology.

(j) The space $\mathbb{R}^J$ with product topology is not metrizable, when J is uncountable.

2. (15 points) Prove that the two-dimensional sphere S^2 is not homeomorphic to any subset of $\mathbb{R}^2$.

3. (15 points) Let Y be a compact Hausdorff space. A function $f\colon X \to Y$ is continuous if and only if the graph $\Gamma_f = \{(x, f(x)) \mid x \in X\}$ is closed in $X \times Y$.

4. (10 points) Let X be a compact Hausdorff space and $f_n \colon X \to \mathbb{R}$ an equicontinuous sequence of continuous functions. Prove that $f_n \to f$ pointwise if and only if $f_n \to f$ uniformly.

5. (15 points) Let X be a complete metric space and let $\mathcal{F}$ be a subset of $C(X, \mathbb{R})$ such that for every $x \in X$, the set $\mathcal{F}_x = \{f(x) \mid f \in \mathcal{F}\}$ is bounded. Show that there is a non-empty open subset $U \subset X$ and a constant M such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$ and all $x \in U$.

6. (15 points) Let $K \subset [0,1]$ be the Cantor set. Does there exist a continuous function $f\colon [0,1] \to [0,1]$ such that $f(K) = [0,1]$? Construct such a function or show that it does not exist.

7. (10 points) Does the set of irrational numbers contain a subset which is uncountable and closed in $\mathbb{R}$? Construct such a subset or show that it does not exist.