Mathematics 117 Fall 1999/2000, V. Ginzburg

Final Exam

Print Your Name:

All 8 questions should be answered on this exam using the backs of the sheets if necessary. The exam has 9 pages.

Show all your work and justify all claims when a proof is required. Good Luck!!

1	
2	
3	
4	
5	
6	
7	
8	
Total	/100

1. For each of the questions below, indicate if the statement is *true* or *false*. You do **not** need to justify the answer. Each correct answer is worth 2 points.

a		f		k	
b		g		l	
c		h		m	
d		i		n	
e		j		o	

(a) Let W and U be subspaces of a finite-dimensional space V such that $W \cap U = \{0\}$ and $\dim V = \dim W + \dim U$. Then $V = W \oplus U$.

(b) Let U_1, U_2, and W be subspaces of V such that $U_1 \oplus W = V = U_2 \oplus W$. Then $U_1 = U_2$.

(c) The set

$$\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 \\ 1 \\ 8 \end{pmatrix} \right\}$$

is a basis in $\mathbb{R}^3$.

(d) Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.

(e) Let $T\colon V \to W$, where V and W are finite-dimensional vector spaces. Then

$$\dim \mathit{range}\, T + \dim \mathit{null}\, T = \dim V.$$

(f) A linear map $T\colon V \to W$ is invertible if and only if there exists a linear map $S\colon W \to V$ such that $TS = I$.

(g) Let $T\colon V \to V$, where V is finite-dimensional. Then *range* $T \oplus$ *null* $T = V$.

(h) Let V be finite–dimensional vector space. A linear operator $T\colon V \to V$ is invertible if and only if *null* $T = \{0\}$.

(i) $\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(A)\mathrm{Tr}\,(B)$ for any two square matrices A and B.

(j) $\det(A + B) = \det(A) + \det(B)$ for any two square matrices A and B.

(k) There exists a linear map $T\colon \mathbb{C}^4 \to \mathbb{C}^4$ whose characteristic polynomial is $(z-1)^3(z-3)$ and whose minimal polynomial is $(z-1)^3(z-4)$.

(l) Every linear operator $T\colon \mathbb{R}^5 \to \mathbb{R}^5$ has an eigenvalue.

(m) Let $N\colon V \to V$ be a nilpotent operator on a finite-dimensional vector space V. Then $N^n = 0$, where $n = \dim V$.

(n) Let $\lambda_1, \ldots, \lambda_r$ be distinct eigenvalues of $T\colon V \to V$, where V is a complex finite-dimensional vector space. Then

$$V = \mathit{null}\,(T - \lambda_1 I) \oplus \ldots \oplus \mathit{null}\,(T - \lambda_r I).$$

(o) Assume that the operator $T\colon \mathbb{C}^4 \to \mathbb{C}^4$ has three distinct eigenvalues, say λ_1, λ_2, and λ_3. Then

$$\mathbb{C}^4 = \mathit{null}\,(T - \lambda_1 I)^4 \oplus \mathit{null}\,(T - \lambda_2 I)^4 \oplus \mathit{null}\,(T - \lambda_3 I)^4.$$

2. (10 points) Let W be a subspace of a finite-dimensional space V. Prove that there exists a subspace U of V such that $V = W \oplus U$.

3. Let $T\colon V \to W$ and $S\colon W \to U$. Assume that ST is surjective.

   (a) (5 points) Is T necessarily surjective? Prove or give a counterexample.

   (b) (5 points) Is S necessarily surjective? Prove or give a counterexample.

4. Let 3, 2, and -1 be eigenvalues of $T\colon \mathbb{C}^3 \to \mathbb{C}^3$.

(a) (5 points) Find eigenvalues of $S = T^2 + T - I$.

(b) (5 points) Is S necessarily diagonalizable? (In other words: Does there exist a basis of eigenvectors of S?) Prove or give a counterexample.

5. (5 points) Give an example of $T\colon \mathbb{C}^5 \to \mathbb{C}^5$ whose characteristic polynomial is $z^2(z-1)^2(z-2)$ and whose minimal polynomial is $z^2(z-1)(z-2)$.

6. (10 points) Let V be a finite-dimensional real vector space and $T: V \to V$ an operator which has no eigenvalues. Prove that every subspace of V invariant under T has even dimension.

7. (15 points) Let V be a complex vector space of dimension $n \geq 2$ and $T\colon V \to V$ a linear operator such that $\mathit{null}\, T^{n-1} \neq \mathit{null}\, T^{n-2}$. Prove that T has at most two distinct eigenvalues.

Hint: Assume otherwise. Then $\dim \mathit{null}\, T^{n} \leq n-2$.

8. (10 points) Let V be a complex vector space and $T\colon V \to V$ an operator such that $ST = TS$ for all $S\colon V \to V$. Prove that $T = cI$ for some $c \in \mathbb{C}$.

Hint: Let v be an eigenvector of T with eigenvalue λ and let $ST = TS$. Then Sv is also an eigenvector of T with eigenvalue λ.