Mathematics 105B Winter 2001, V. Ginzburg

Final

Print Your Name:

All questions should be answered on this exam using the backs of the sheets if necessary. The exam has 9 pages.

Show all your work and justify all claims when a proof is required. Good Luck!!

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Total	/100

1. For each of the questions below, indicate if the statement is *true* or *false*. You do **not** need to justify the answer. Each correct answer is worth 2 points.

a	
b	
c	
d	
e	

f	
g	
h	
i	
j	

(a) Assume that f is an infinitely differentiable function on $\mathbb{R}$. Then the Taylor expansion of f converges to f at every point.

(b) Assume that f_n are integrable functions on $[a,b]$ and $f = \sum_{n=1}^{\infty} f_n$ uniformly on $[a,b]$. Then f is also integrable and

$$\int_a^b f\,dx = \sum_{n=1}^{\infty} \int_a^b f_n\,dx.$$

(c) Assume that f_n are differentiable functions on $[a,b]$ and $f = \sum_{n=1}^{\infty} f_n$ uniformly on $[a,b]$. Then f is also differentiable and $f' = \sum_{n=1}^{\infty} f_n'$.

(d) Let $\sum_{n=0}^{\infty} c_n x^n$ have radius of convergence $R > 0$. Then this series converges uniformly on $(-R, R)$.

(e) Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $|x| < R$ where $R > 0$. Then f is differentiable on $(-R, R)$.

(f) The series $\sum_{n=0}^{\infty} (-1)^n x^n/(n2^n)$ has radius of convergence $R = 2$.

(g) The series $\sum_{n=0}^{\infty}(-1)^n x^n/(n2^n)$ converges if and only if $|x| < 2$.

(h) Let f and g be integrable functions. Then the composition $f \circ g$ is necessarily integrable.

(i) Let f be an integrable function on $[-\pi, \pi]$ whose Fourier series is $\sum c_n e^{inx}$. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 \, dx = \sum |c_n|^2.$$

(j) Assume that f is continuous and g is monotone increasing on $[a, b]$. Then $f \in \mathcal{R}(g)$.

2. (5 points) Assume that f is differentiable on $\mathbb{R}$ and that f' is bounded. Prove that f is uniformly continuous on $\mathbb{R}$.

3. (5 points) Let $0 < a < b$. Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that $f(b) - f(a) = cf'(c)\ln(b/a)$.

4. (15 points) Suppose that f is a continuous function on $[0,1]$. Prove that there exists a sequence g_n of piecewise linear continuous functions such that $g_n \to f$ uniformly on $[0,1]$.

Hint: Let g_n be a function which is linear on every interval $[k/n,(k+1)/n]$ and such that $g_n(k/n) = f(k/n)$ for $k = 0, \dots, n-1$. Use the fact that f is uniformly continuous to show that $g_n \to f$ uniformly.

5. (5 points) Let $g(x) = \tan^{-1}(x)$. Find

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n x \, dg(x).$$

6. Let

$$f_k(x) = \frac{xe^{-x/k}}{k}.$$

(a) (5 points) Prove that $f_k \to 0$ uniformly on any interval $[0, a]$.

(b) (10 points) Prove that f_k does *not* converge uniformly to zero on $[0, \infty)$.

(c) (5 points) Let $a > 0$. Find $\lim_{k\to\infty} \int_0^a f_k(x)\,dx$.

(d) (10 points) Let $k > 0$. Find $\lim_{a\to\infty} \int_0^a f_k(x)\,dx$.

(e) (5 points) Determine whether

$$\lim_{a\to\infty} \lim_{k\to\infty} \int_0^a f_k(x)\,dx = \lim_{k\to\infty} \lim_{a\to\infty} \int_0^a f_k(x)\,dx.$$

7. (a) (5 points) For $n = 0, \pm 1, \pm 2, \ldots$ evaluate

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx}\, dx.$$

(b) (10 points) Prove that

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{n^2}$$

for all $x \in [-\pi, \pi]$.