Symplectic Geometry, Math 248
2021 F

Lecture 1

09/23 - 2021

→ Go through basic info

* No exams, no hw
* Problems stated in lectures

Up to them how much they take home

* OH: TBA

→ why is SG important? Two aspects:

* Math Language (like algebra or diff. manifolds) connecting disparate areas:
  - classical mechanics, quantum physics, diff geometry, string and mirror theory etc ... Keep coming up
* Genuinely deep results ...

In this class - always a choice - a bit of both ...

Math 248, Symplectic Geometry, Fall 2021

- **Lectures:** TTh 9:50 AM - 11:25 AM, McHenry Clrm 4130
- **Instructor:** Viktor Ginzburg; office: McHenry 4124
  email: ginzburg(at)ucsc.edu
- **Office Hours:** TBA or by appointment
- **Text:** There will be no "official" textbook in this course. Suggested reading:
  - *Introduction to Symplectic Topology* by Dusa McDuff and Dietmar Salamon,
  - *Lectures on Symplectic Geometry* by Ana Canas da Silva,
  - *Morse Theory and Floer Homology* by Michelle Audin and Mihai Damian
- **Tentative Syllabus:** The course will cover fundamentals from symplectic geometry and touch upon Morse theory with an eye on applications of modern symplectic topological techniques to Hamiltonian dynamics. We will begin with an (ideally, brief) discussion of basic concepts of symplectic geometry: symplectic manifolds, Hamiltonian diffeomorphisms and flows, Lagrangian submanifolds, the least action principle, etc. We will also introduce several classes of dynamical systems of interest, such as geodesic flows and twisted geodesic (or magnetic) flows, and formulate the main problems in dynamics (e.g., Arnold's and Weinstein's conjectures, i.e., the existence of fixed points and periodic orbits) studied by symplectic techniques. Then we turn to a very brief review of Morse theory. In contrast with previous iterations of this course, this time I plan to focus more on Lagrangian submanifolds -- one of the most fundamental objects in symplectic geometry. Time permitting, we will touch upon symplectic topological methods (e.g., Lagrangian and Hamiltonian Floer homology) and/or conclude the course with student presentations.

  It should be said that this is not a comprehensive course in symplectic geometry and many important concepts (mainly those concerning symmetries) will be entirely omitted or just briefly mentioned.

  COVID-19 Information: Please take care to comply with all university guidelines about masking in indoor settings, performing daily symptom and badge checks, testing as required by the campus vaccine policy, self-isolating in the event of exposure, and respecting others' comfort with distancing. Please do not come to class if your badge is not green. If you are ill or suspect you may have been exposed to someone who is ill, or if you have symptoms that are in any way similar to those of COVID-19, please err on the side of caution and stay home until you are well or have tested negative after an exposure.

§1. Symplectic manifolds

– Def's and basic examples

Origins – Hamiltonian dynamics
to be discussed later

Def A real finite-dim symplectic v.s. (V, ω)

skew-symmetric form

$\omega : V \times V \longrightarrow \mathbb{R}$ $\quad \omega(X,Y) = -\omega(Y,X)$

* non-degenerate

- $\forall X \neq 0 \, \exists Y : \quad \omega(X,Y) \neq 0$
- $e_1, \dots, e_m$ basis $\quad \omega = \sum \omega_{ij} e_i^* \wedge e_j^*$
  $\det \omega_{ij} \neq 0$

$\Rightarrow \quad \dim V = \text{even} = 2n$: $\quad$ **Pf:** $\det \omega = \det \omega^T$
$= \det(-\omega)$
$= (-1)^{\dim V} \det(\omega)$

- $\omega^\# : V \xrightarrow{\cong} V^*$

Ex $\exists$ basis $v_1, w_1, \dots, v_n, w_n$ "Darboux"
s.t. $\quad \omega = \sum v_i^* \wedge w_i^*$

$$\text{Matrix}(\omega) = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & 0 \\ & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \\ 0 & & \ddots \end{pmatrix}$$

"Linear Darboux Theorem"

Def $(M^m, \omega) \leftarrow$ symplectic manifold

$\swarrow$ symplectic form

$\omega \in \Omega^2(M)$

- $d\omega = 0$
- ω non-deg: every T_pM is a s.v.s.
  - $\omega^\#: TM \xrightarrow{\simeq} T^*M$
    $X \longmapsto i_X\omega$
  - $\omega = \sum \omega_{ij}\, dx_i \wedge dx_j \leftarrow$ locally
    $\det(\omega_{ij}) \neq 0$

Note $\Rightarrow \dim M = \text{even} = 2n$

Non-deg $\Leftrightarrow \omega^n \neq 0$

Examples

0. (V, ω) symplectic v.s

$$\cong (\mathbb{R}^{2n}, \omega_{st}) \; ; \quad \omega = \Sigma \, dp_i \wedge dq_i = \text{"} dp \wedge dq \text{"}$$

$(p_1, \dots, p_n, q_1, \dots, q_n)$

- obviously $d\omega = 0$

$\omega^n \neq 0 \Leftarrow dp_1 \wedge \dots \wedge dp_n \wedge dq_1 \dots$

Standard s.s. on $\mathbb{R}^{2n}$

1. $\mathbb{T}^{2n} = \mathbb{R}^{2n} / \mathbb{Z}^{2n}$ same formula

$p_1, \dots, q_n$ mod 1

Or ω_{st} is transl inv $\Rightarrow$ descends to $\mathbb{T}^{2n}$

2. M^2 orientable surface $\supset$ orientability

ω = area form : $\omega \neq 0 \leftrightarrow$ non. deg

$d\omega = 0$ – $\dim M = 2$

Can be associated with a R. metric

3. Kähler manifolds

M , $\langle \, , \rangle_{\mathbb{C}} = (\, ,) + i\omega(\, ,)$

skew

$\Uparrow$

R.S. symmetric

complex

- ω non.deg $\Leftarrow \langle \, , \rangle_{\mathbb{C}}$ non-deg
- $d\omega = 0 \Leftrightarrow$ Kahler

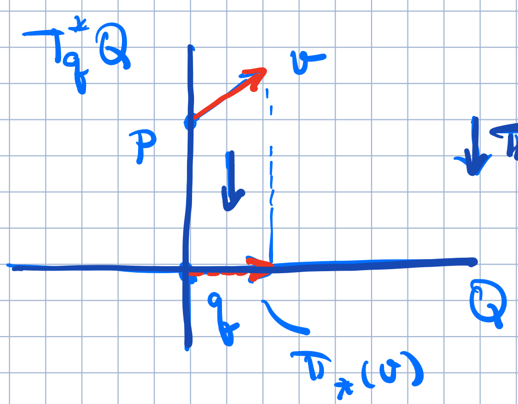
Hermitian

To be discussed in detail later

4. Cotangent bundles { first time confusing

$M = T^*Q$ contest a canonical s.f.

construction



$\pi : T^*Q \longrightarrow Q$

$\lambda_p(v) = p(\pi_* v)$

v in $T_p T^*Q$, $p \in T_q^*Q$, $\pi_* v \in T_q Q$

$\Rightarrow$ Get $\lambda \in \Omega^1(T^*Q)$

By def: $\omega = d\lambda$. $(d^2 = 0 \Rightarrow d\omega = 0)$

Non-degeneracy – write λ in local coordinate

Local expression

$q_1, \dots, q_n \leftarrow$ local coord on Q

$p_1, \dots, p_n \leftarrow$ "dual coord" : $\underbrace{T^*Q}_{\text{subset}} \longrightarrow \mathbb{R}$

$\alpha = p_1(\alpha)\, dq_1 + \dots + p_n(\alpha)\, dq_n$

$\rightarrow$ coord on T^*Q (should be $q_i \circ \pi \dots$)

$\Rightarrow \boxed{\lambda = \sum p_i\, dq_i}$ $(*)$

Then $\omega = \sum dp_i \wedge dq_i \leftarrow$ as for $\mathbb{R}^{2n}$

$\Rightarrow$ non-degeneracy $\boxed{\text{Rmk. } \mathbb{R}^{2n} = T^*\mathbb{R}^n}$

Pf of (*)

$$v = \sum a_i \frac{\partial}{\partial q_i} + \cancel{b_i \frac{\partial}{\partial p_i}}$$

killed by π_*

$\pi : (p,q) \longmapsto q$

$$\alpha = \sum p_i \, dq_i \longleftarrow \text{dd of } p_i\text{'s}$$

$$\pi_*(v) = \sum a_i \frac{\partial}{\partial q_i}$$

$$\underbrace{\lambda(v)}_{\alpha(\pi_* v)} = \sum p_i a_i = \sum p_i \, dq_i(v)$$

$\triangleleft$

5. Twisted cotangent bundle

$$\big(T^*Q, \ \omega = \underbrace{d\lambda}_{\text{standard}} + \pi^*\sigma\big)$$

$\sigma \in \Omega^2(Q)$

$d\sigma = 0$

$$d\omega = d(d\lambda + \pi^*\sigma) = 0$$

Non-deg: $\omega = \sum dp_i \wedge dq_i + \overbrace{\sum_{i,j} \sigma_{ij} \, dq_i \wedge dq_j}^{\sigma}$

$$\begin{array}{c|cc} & p & q \\ \hline p & 0 & -I \\ q & -I & \sigma \end{array} \longleftarrow \text{non-deg no matter what } \sigma \text{ is}$$

- More examples later

Non-examples

- To admit a s.f. M has to be orientable:

ω sympl $\Rightarrow$ $\underbrace{\omega^n \neq 0 \leftarrow \text{"volume form"}}_{\text{orientation}}$

- $\boxed{\omega \text{ symplectic, } M \text{ closed} \Rightarrow [\omega] \neq 0 \text{ in } H^2(M;\mathbb{R})}$ (**)

compact $\partial M = \emptyset$

Cor. $S^{2n \geq 4}$ does not admit a sympl. form

Pf. Assume not: $[\omega] = 0$: $\omega = d\lambda$

$$\int_M \omega^n = \int_M (d\lambda)^n = \int_M d\left(\lambda \wedge (d\lambda)^{n-1}\right)$$

$$= \int_{\partial M \neq 0} \lambda \wedge d\lambda^{n-1} = 0$$

But $\omega^n \neq 0 \Rightarrow \int_M \omega^n \neq 0$ ← sign depends on the orientation

$\triangleleft$

- More to follow...

Symplectomorphisms

Lecture 8

09/28-2021

- $\varphi : (M_0, \omega_0) \longrightarrow (M_1, \omega_1)$
  is symplectic
  if $\varphi^* \omega_1 = \omega_0$ $\Bigg\} \overset{Ex}{\Rightarrow}$ φ is an immersion
  $\operatorname{rk} D\varphi = \dim M_0$

$\Rightarrow$: $\dim M_0 = \dim M_1 \Rightarrow \varphi$ is a local diffeo

- $\varphi : (M_0, \omega_0) \longrightarrow (M_1, \omega_1)$
  is a symplectomorphism
  if φ is a diffeo & $\varphi^* \omega_1 = \omega_0$

- Group $\operatorname{Sympl}(M, \omega) < \operatorname{Diff}_{\omega^n}(M)$

symplectomorphism $\Rightarrow$ vol preserving

$$\varphi^* \omega_1^n = \omega_0^n$$

Discussion: Gromov's thm
symplectic vs volume preserving

§2 Darboux and Moser's thm's

Darboux: all sympl forms (of the same dim) are locally symplectomorphic

More rigorously

Thm (Darboux) (M^{2n}, ω) symplectic, $M \ni p$

$\exists$ nbd. $U \ni p$ and a diffeo

$\varphi : (U, x) \longrightarrow (B^{2n}, 0)$ s.t. $B^{2n} \subset \mathbb{R}^{2n}$

$\omega = \varphi^* \omega_{st}$

Thm' (Darboux) (M_0^{2n}, ω_0) & (M_1^{2n}, ω_1) sympl, $M_0 \ni x_0$, $M_1 \ni x_1$

$\Rightarrow$ $U_0 \ni x_0$ & $U_1 \ni x_1$ and a diffeo

$\varphi : (U_1, x_1) \longrightarrow (U_0, x_0)$ s.t.

$\omega_1 = \varphi^* \omega_0$

Rmk Contrast with diff geometry: sympl str's are more like diff str's

Pf Moser's homotopy method
– extremely important

- Result is local can assume
  $M^{2n} = \mathbb{R}^{2n}, \quad x = 0$
- Linear Darboux theorem $\omega_0 = \omega$ on $T_0\mathbb{R}^{n} = \mathbb{R}^{2n}$ can be made standard by a lin transf.
- Consider $\omega_t = (1-t)\omega_0 + t\omega$
  - ω_t at 0 is $\omega_0 \Rightarrow$ sympl on a nbd of 0
  - $\omega_0 \overset{\omega_t}{\rightsquigarrow} \omega_1 = \omega$
- Looking for φ_t : nbd of $0 \longrightarrow$ nbd of 0
  $\omega_0 = \varphi_t^* \omega_t$

Then φ_1 does the job

- φ_t is generated by the time dependent v.f. v_t : $\frac{d}{dt}\varphi_t(x) = v_t(\varphi_t(x))$

Discuss? Digression: isotopies, etc → p. 11

Looking for $v_t \rightsquigarrow \varphi_t$
↑
uniqueness and existence of solutions of ODE

- $\frac{d}{dt}$ of $\varphi_t^* \omega_t = 0$

$$\varphi_t^* L_{v_t}\omega_t + \varphi_t^* \frac{d}{dt}\omega_t = 0$$

Apply $(\varphi_t^*)^{-1}$: $\quad L_{v_t}\omega_t + \frac{d}{dt}\omega_t = 0 \quad (*)$

$$L_{v_t}\omega_t = \cancel{i_{v_t} d\omega_t}^{\,=0} + d\, i_{v_t}\omega_t$$

$d\omega = 0$

$(1-t)\omega_0 + t\omega_1$

$$(*) \iff d\, i_{v_t}\omega_t = -\frac{d}{dt}\omega_t = \underbrace{\omega_1 - \omega_0}_{d\lambda}$$ Poincaré

$$\Longleftarrow \quad i_{v_t}\omega_t = \lambda$$

$$\iff \quad v_t = (\omega_t^\#)^{-1}\lambda$$ ← Non-degeneracy

<u>Nuance</u>: need to know that φ_t is defined for $t \in [0,1]$

$\iff$ solutions of $\dot{x} = v_t(x)$ with initial conditions near 0 exist for $[0,1]$

Does <u>not</u> follow automatically from existence & uniqueness

<u>From ODE's sufficient to have</u> $v_t(0) = 0 \quad \forall t$

$\Longleftarrow \quad \lambda|_0 = 0$

Modify λ: $\lambda \rightsquigarrow \lambda - \lambda_0$ ← linear extension

$\triangleleft$

<u>**Rmk.**</u> Stark contrast with Riemannian geometry:
- ω does not have local invariants
- R.m. (symmetric tensors) do: curvature

Extra discussion: time dependent v.f.

- v ind of time $\rightsquigarrow$ φ^t flow
  - $\varphi^{t_1+t_2} = \varphi^{t_1}\varphi^{t_2}$
  - $\varphi^0 = id$

$t \mapsto \varphi^t(x)$ = sol of $\dot{x} = v(x)$ with $\varphi^0(x) = x$

- v_t depends on t $\rightsquigarrow$ φ_t <u>isotopy</u>

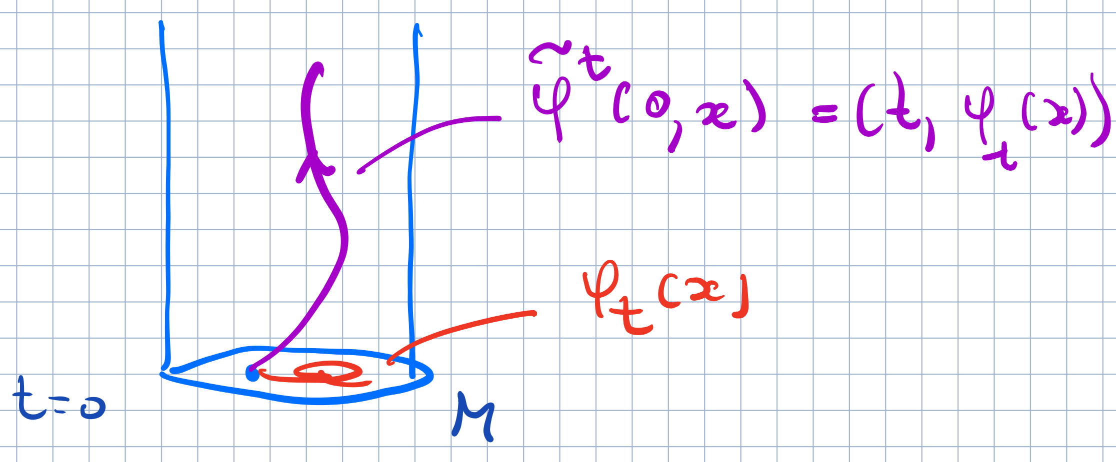
  $\varphi_0 = id$

<u>Construction</u>: pass to $\tilde{M} = \mathbb{R} \times M$

ind of time $\longrightarrow$ $\tilde{v} = \frac{\partial}{\partial t} + v_t$

flow $\tilde{\varphi}^t$ then $\tilde{\varphi}^t(0,x) = (t, \varphi_t(x))$

or $t \mapsto \varphi_t(x)$ is a sol of $\dot{x} = v_t(x)$ with initial condition x at $t=0$



To be more precise, incorporating time:

$$\varphi_{t_1}^{t_2}\varphi_{t_0}^{t_1} = \varphi_{t_0}^{t_2}$$

$$\frac{d}{d\tau}\varphi_t^\tau(x)\Big|_{\tau=t} = v_t(x)$$

We have here $t_0 = 0$ $\varphi^t = \varphi_0^t$

$$\varphi_t^\tau\varphi^t = \varphi^\tau$$

$$\Rightarrow \quad (\varphi^\tau)^* = (\varphi^t)^*(\varphi_t^\tau)^*$$

$$\Rightarrow \quad \frac{d}{d\tau}(\varphi^\tau)^*\alpha\Big|_{\tau=t} = (\varphi^t)^* L_{v_t}\alpha$$

Global Rigidity: Moser's thm's

Lecture 3
09/30-21

- Volume form = non-vanishing top deg form
  - E.g. ω symplectic $\Rightarrow \omega^n$ vol. form
- η_1 & η_0 vol. form $\quad \eta_1 = \underset{\neq 0}{f}\,\eta_0$

  $f > 0$: η_1 & η_0 have the same sign

  $f < 0$ —— . —— . - opposite signs
- Existence of a vol. form $\Rightarrow$ orientability
- $\int_M : H^m(M) \xrightarrow{\cong} \mathbb{R}$ ← Discuss

Moser: total volume is the only inv of a volume form

Discuss: Local Moser = Darboux for vol forms

Easy by a level set pf

Go through

Thm (Moser) M closed (orientable)

η_1 & η_0 vol. forms and

$$\int_M \eta_1 = \int_M \eta_0 \quad (\Rightarrow \text{same sign})$$

$\Rightarrow \exists\ \varphi : M \circlearrowleft$ diffeo: $\eta_1 = \varphi^* \eta_0$

Pf Set

$$\eta_t = (1-t)\eta_0 + t\eta_1 \leftarrow \text{all volume forms}$$

$$= \underbrace{(1-t+tf)}_{>0}\eta_0 \qquad \Uparrow \text{ same sign}$$

Discuss

Note $\int_M \eta_t = \text{const} \Leftrightarrow \boxed{[\eta_t] = \text{const}}$

As before: looking for φ_t $\leftarrow$ generated by v_t

$$\varphi_t^* \eta_t = \eta_0$$

See p 11a

$$\frac{d}{dt}: \quad \varphi_t^* L_{v_t} \eta_t + \varphi_t^* \frac{d}{dt} \eta_t = 0$$

$$\underbrace{L_{v_t} \eta_t}_{d i_{v_t} \eta_t} = - \frac{d}{dt} \eta_t = \underbrace{\eta_0 - \eta_1}_{\int_M (\eta_0 - \eta_1) = 0} = d\lambda$$

Discuss

$$\Uparrow \qquad\qquad \Uparrow$$

$$i_{v_t} \eta_t = \lambda \in \Omega^{n-1}(M) \qquad (*)$$

Ex: linear alg

V^n v.s. $\quad \eta \in \Lambda^n V^* \neq 0$ vol. form

$$V \xrightarrow{\cong} \Lambda^{n-1} V^* \qquad \text{isomorphism}$$
$$v \longmapsto i_v \eta$$

$\Rightarrow \quad \exists! \ v_t \quad$ solving $(*)$

M closed $\Rightarrow$ the flow exists for $t \in [0, 1]$,

φ_1 does the job

$\triangleleft$

Rmk. Rigidity in general: deforming a str results in an equivalent str.

E.g. η_t family of vol. forms, M closed

$[\eta_t] = \text{const} \Rightarrow \exists \ \varphi_t : \ \varphi_t^* \eta_t = \eta_0$

Difficulty: Hodge theory $\longrightarrow$ Discuss

Similar rigidity for symplectic forms

But $\exists$ some complications:

(1) ω_1 & ω_0 symplectic on M^{2n}

$\not\Rightarrow$ $\omega_t = (1-t)\omega_0 + t\,\omega_1$ symplectic: sum of non-deg matrices need not be non-deg

Nothing like that can possibly be true for R.M.

Thm (Moser)

- M^{2n}, $\omega_t \leftarrow$ family of sympl forms; M closed
- $[\omega_t] =$ const !

$\Rightarrow$ $\exists$ $\varphi : M \circlearrowleft$ diffeo s.t. $\varphi^*\omega_1 = \omega_0$

On the pf: look for $\varphi_t^*\omega_t = \omega_0$

$\Leftarrow$
- $d\iota_{v_t}\omega_t = -\dot{\omega}_t := -\frac{d}{dt}\omega_t$
- $[\omega_t] =$ const $\Rightarrow$ all $\dot{\omega}_t$ exact

  discuss: cycles & $\frac{d}{dt}[\cdot] = [\frac{d}{dt}\cdot]$
- As before $-\dot{\omega}_t = d\lambda_t$

  $\iota_{v_t}\omega_t = \lambda_t \Rightarrow \dots$ as before

(2) Need λ_t to be smooth (or cont) in t

Not obvious at all.

Hodge theory or de Rham = Čech

Cor (Local rigidity of sympl. forms)

M closed

ω_0 symplectic

Assume that • ω is sufficiently close to ω_0

• $[\omega] = [\omega_0]$

$\Rightarrow \exists\ \varphi : M \circlearrowleft$ s.t. $\omega = \varphi^* \omega_0$

Pf $\omega_t = (1-t)\omega_0 + t\omega_1$

• symplectic $\Leftarrow$ $\left\{ \begin{array}{l} \forall \text{ any two } X, Y \\ \quad \|X\| = \|Y\| = 1 \\ \omega(X,Y) \approx \omega_0(X,Y) \end{array} \right.$

• $[\omega_t] = \text{const}$

Apply Global Moser thm for symplectic forms

$\triangleleft$

§ 3. Hamiltonian Dynamics: Definitions, Basic facts, Examples

Lecture 4
10/05-2021

(M^{2n}, ω) symplectic

- $H: \mathbb{R} \times M \longrightarrow \mathbb{R}$ — Hamiltonian
  t $\qquad H(t, \cdot) = H_t$
- Autonomous if ind of t: $H: M \longrightarrow \mathbb{R}$
- often 1-periodic in t, $S^1 = \mathbb{R}/\mathbb{Z}$
  $H: S^1 \times M \longrightarrow \mathbb{R}$ $\qquad H_{t+1} = H$

Def. • Hamiltonian v.f. generated by H:

$$i_{X_H}\omega = -dH \quad \exists!\, X_H - \text{non-deg}$$

• $X_H \rightsquigarrow$ "$\underbrace{\text{time-dependent flow}}_{\text{isotopy}}$"

time dependent

$\varphi_H^t \leftarrow$ Hamiltonian flow generated by H

Need not be defined for all t, and is not sometimes (collisions) but we will assume it is. (E.g. M is compact, etc)

Rmk. Doing dynamics, usually interested in • φ_H^t, $t \in \mathbb{R}$, H autonomous
or φ_H^{kT}, $k \in \mathbb{N}$, H time-dependent

Examples

Ex 1. $\mathbb{R}^{2n}$, $\omega_{st} = dp\wedge dq = \sum dp_i \wedge dq_i$

$$\begin{cases} \dot p = -\dfrac{\partial H}{\partial q} \\ \dot q = \dfrac{\partial H}{\partial p} \end{cases} \iff X_H = -\frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}$$

Checking $i_{X_H}\omega = -\dfrac{\partial H}{\partial q}dq - \dfrac{\partial H}{\partial p}dp = -dH$

Subexample $\mathbb{R}^{2n} = T^*\mathbb{R}^n = \underset{q}{\mathbb{R}^n} \times \underset{p}{\mathbb{R}^n}$

$H = \frac{1}{2m}\|p\|^2 + V(q)$ = kinetic + potential

$$\begin{cases} \dot p = -\dfrac{\partial V}{\partial q} \iff \boxed{m\ddot q = -\nabla V} \leftarrow \text{cons force} \\ \dot q = \dfrac{1}{m}p \iff p = m\dot q \leftarrow \text{momentum} \end{cases}$$

Newton's eq

2. **Cotangent bundle** $M = T^*Q$, ω_{st}

Fix R.M. on Q $\langle , \rangle$ $\quad TQ \longleftrightarrow T^*Q$

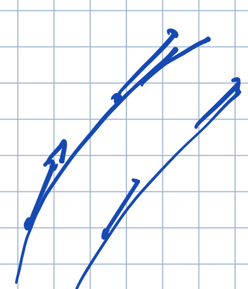
$v \longleftrightarrow \langle v, \cdot \rangle$

$H = \frac{1}{2}\langle , \rangle : T^*Q \longrightarrow \mathbb{R}$

X_H = geodesic spray

φ^t_H = geodesic flow

Describe.



Motion of a free particle on Q.

3. Twisted cotangent bundle

$$M = T^*Q \,, \quad \omega = \omega_{st} + \pi^*\sigma$$
$$\downarrow \pi$$
$$Q \,, \ \sigma \,, \ d\sigma = 0$$

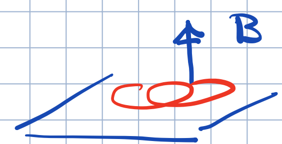
magnetic field

$$H = \tfrac{1}{2}\langle \, , \, \rangle$$

The flow governs the motion of a charge on Q in magnetic field σ.

Subexample a) $Q = \mathbb{R}^2$ $\longrightarrow (q_1, q_2)$, $\quad \sigma = B\, dq_1 \wedge dq_2$

conf space



unit charge, unit mass

$\vec{B} \perp (q_1, q_2)$ plane, charge in $\mathbb{R}^2 (x,y)$ $\quad q_1, q_2 = (x, y)$

H.E. $\iff$ $\ddot{q} = B(q)\, J \dot{q}$; $\quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b) $Q = \mathbb{R}^3$, $\vec{B}$ = v.f. on $\mathbb{R}^3$ ← magn. field

$$\sigma = i_{\vec{B}}\, dq_1 \wedge dq_2 \wedge dq_3$$

$$\left. d\sigma = 0 \iff \operatorname{div} B = 0 \right\}$$ one of the maxwell eq

H.E. $\iff$ $\ddot{q} = \dot{q} \times \vec{B}(q)$ ← Lorentz force

(unit charge, unit mass)

Energy and ω-conservation

Let φ_H^t be the Ham flow of H_t.

Prop

essential

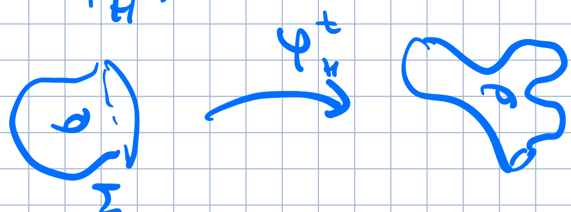
(a) Energy conservation

Assume that H is autonomous. Then

$$(\varphi_H^t)^* H = H \quad : \quad H(\varphi_H^t(p)) = \text{const} \ \forall p$$

(b) "Integral invariant": φ_H^t is symplectic

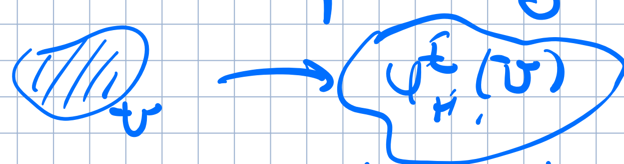
$$(\varphi_H^t)^* \omega = \omega$$



$$\varphi_H^t(\Sigma) = \Sigma'$$

$$\int_{\Sigma'} \omega = \int_{\Sigma} \omega$$

Cor. φ_H^t is vol. preserving: $(\varphi_H^t)^* \omega^n = \omega^n$



$$\int_{\mho} \omega^n = \int_{\varphi_H^t(\mho)} \omega^n$$

$\Rightarrow$ Huge restrictions on dynamics

Pf. (a) $\frac{d}{dt} H(\varphi_H^t(p)) = (L_{X_H} H)(\underbrace{\varphi_H^t(p)}_{x})$

$$= (i_{X_H} \cancel{d} dH + d\, i_{X_H} dH)(\ldots)$$

p. 11a

$$= d(-\underbrace{i_{X_H} i_{X_H} \omega}_{\text{HE:}}) = d\,\omega(X_H, X_H) = 0$$

$i_{X_H}\omega = -dH$

(b) $\frac{d}{dt}(\varphi_H^t)^* \omega = (\varphi_H^t)^* L_{X_H} \omega$ see p 11a

$= (\varphi_H^t)^* \left(d i_{X_H} \omega + \cancel{i_{X_H} d\omega} \right)$

$= (\varphi_H^t)^* \underbrace{d(-dH)}_{HE} = 0$

$\triangleleft$

Ex. Prove the Prop using a direct calculation in Darboux coordinates.

Cor $\dim M = 2 \leftarrow$ surface
H autonomous

sol of ODE

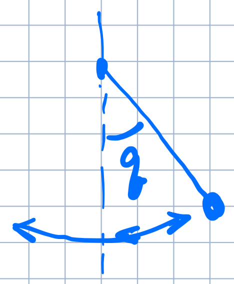
$\Rightarrow$ "integral curves of φ_H^t (unparametrized) "are" levels $\underbrace{H = \text{const}}_{\text{"alg equation"}}$.

Rmk Newtonian mechanics:

- $\ddot{q} = F(q)$ Energy consv $\Leftarrow \begin{cases} F \text{ is consv:} \\ F = -\nabla V \leftarrow \text{indep of } t \end{cases}$
- $\ddot{q} = F(q, \dot{q})$ as in Lorentz
  Energy conservation $\Leftarrow F \perp \dot{q}$

Continuing Examples

4. Investigating the pendulum



$$M = T^*S^1 = \underset{p}{\mathbb{R}} \times \underset{q \leftarrow \text{mod } 2\pi}{S^1}$$

$$\text{or} \quad \mathbb{R}^2 = \underset{p}{\mathbb{R}} \times \underset{q}{\mathbb{R}}$$

$$H = \frac{1}{2}|p|^2 - \cos q + 1$$

$$= \underbrace{\frac{1}{2}(p^2 + q^2)}_{} + \cdots \longleftarrow \text{Harmonic oscillator}$$

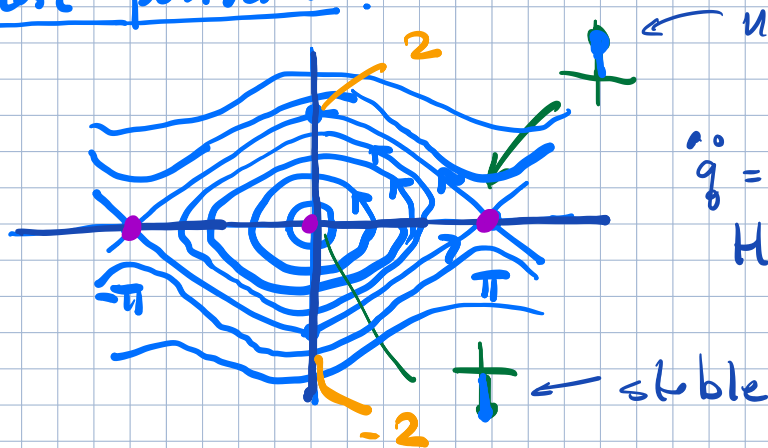
$m = 1,\ k = 1$

$\ddot{q} = -\sin q$

Hooke's law:

$\ddot{q} = -q$

Phase portrait:



Gives the behavior of integral curves up to parametrization

Further details - Ex (Not easy)

(a) $q \in (0, \pi)$ $\quad H(0, q) = \cos q - 1 = h$

integral curve through $\underbrace{(q, 0)}_{x}$ is $\{H = h\}$

$T(h)$ its period

- Show that $T(h)$ monotone increasing function from $\underset{h=0}{2\pi}$ to $\underset{h=2}{\infty}$ as $h \to 2$

(Compare with the harmonic oscillator
$H = \frac{1}{2}(p^2+q^2)$ & $T = \text{const}$)

- Find the Taylor exp of $T(h)$ at $h=0$

(b) Consider $D\varphi_H^T : T_x\mathbb{R}^2 \circlearrowleft$

Show that $D\varphi_H^T = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ $\lambda \neq 0$

(c) Find explicitly $\varphi_H^t(0,2)$ in elementary functions

Ex Hint to a): Area-period relation

$H: \mathbb{R}^2 \longrightarrow \mathbb{R}$ proper

- $\{H \leq h\}$ connected
- h - regular

$A(h) = \text{area of } \{H \leq h\} = \int_{H\leq h} \omega$

$T(h) = \text{period of } \{H = h\}$

Show that $\frac{dA}{dh} = T(h)$

5. Positive-Def quadratic Ham

Lecture 5
10/07-2021

$\mathbb{R}^{2n} = \mathbb{C}^n \qquad z_j = p_j + i q_j \ , \ z = (z_1, \dots, z_n)$

$$H = \frac{1}{2} \sum \underset{\substack{\vee \\ 0}}{\lambda_j} (p_j^2 + q_j^2) = \frac{1}{2} \sum \lambda_j |z_j|^2$$

$\sim$ or just $\neq 0$

Governs n uncoupled oscillators with frequencies λ_j

$E = \{H = h\}$ is an ellipsoid

- Find X_H and show that $\varphi_H^t(z) = (e^{\lambda_1 i t} z_1, \dots, e^{\lambda_n i t} z_n)$
- "Coordinate axis" $(0, \dots, 0, z_j, 0, \dots 0) \cap E$ are periodic orbits of φ_H^t with $T_j = \frac{2\pi}{\lambda_j}$
- Are there other periodic orbits? (The answer depends on $(\lambda_1, \dots, \lambda_n)$.)

6. Linear HE

$M = \mathbb{R}^{2n}$ $\qquad \omega = \omega_{st} = dp \wedge dq$

$= \mathbb{C}^n,$ $\qquad$ Matrix of ω:

$$J = \begin{pmatrix} \begin{array}{cc|c} 0 & -1 & \\ 1 & 0 & 0 \\ \hline & & \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} \\ 0 & & \ddots \end{array} \end{pmatrix} \leftarrow \text{multiplication by } i \text{ in } \mathbb{C}^n = \mathbb{R}^{2n}$$

$H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ quadratic form

$H(x) = \frac{1}{2} \langle Ax, x \rangle, \quad A^T = A$

$A = \nabla H$

HE: $\quad \dot{x} = J \nabla H(x) = JAx = X_H(x)$

$\varphi_H^t(x) = \exp(tJA)x$ $\qquad$ Discuss a bit Lie alg & Lie gps

$\exp : sp(2n) \longrightarrow Sp(2n)$

$\qquad X_H \longmapsto \varphi_H$

$sp(2n) = \text{lin Ham v.f.}$ $\qquad XJ + JX^T = 0$

$= \text{quadratic forms on } \mathbb{R}^{2n}$

$X \longrightarrow H = -\frac{1}{2} \langle JXx, x \rangle$

Normal forms

Discuss linear alg
Sp(2n) sp(2n)
vs SL

$PJP^T = J$

$\underbrace{}_{P \in Sp(2n)}$

- solving ODE's: $\dot{x} = Px$
  Bring A to a Jordan form
  to calculate $\exp(Pt)x$

- symplectic normal forms are more complicated

  Q symmetric

$$\underset{\substack{\cap \\ GL(n)}}{S} Q S^T \rightsquigarrow \mathrm{diag}(1, \dots, 1, 0 \dots 0)$$

$$\underset{\substack{\cap \\ O(n)}}{S} Q S^T \rightsquigarrow \mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

$$\underset{\substack{\cap \\ Sp(2n)}}{S} Q S^T \rightsquigarrow$$ much more complicated list

Not just $\sum \lambda_j |z_j|^2$

Rmk. Unless $A > 0$, then it can be diagonalized

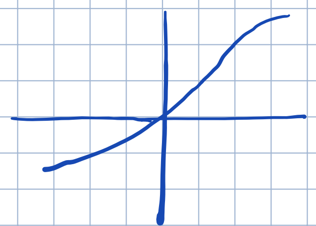
Time - change

Lecture 6

10/12 - 2021

$H : \mathbb{R} \times M \to \mathbb{R}$ Hamiltonian

$\underbrace{\lambda : \mathbb{R} \to \mathbb{R}}_{\text{time-change}}$



Does not have to be monotone but usually is

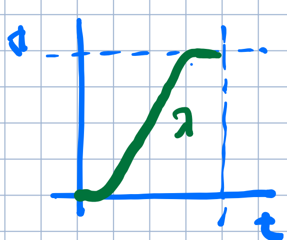
set $K_t(x) = \lambda'(t) H_{\lambda(t)}(x)$

Ex. Show that $\varphi^t_K = \varphi^{\lambda(t)}_H$

Ex. 1) $\lambda(t) = T \cdot t$: $K = T H_{Tt}$

$\Rightarrow \varphi^1_K = \varphi^T_H$: Looking at φ^T_H can always assume that $T = 1$

2)



$\varphi^1_K = \varphi^1_H$ but $K \equiv 0$ when ~~when~~ $t \approx 0$ & $t \approx 1$

$\Rightarrow$ Looking at φ^K_H can always assume H is 1-periodic in time $H_{t+1} = H_t$

§4 Relevant Groups: Ham v.s. Symp

Def. $\varphi_H = \varphi_H^1$ is called a Hamiltonian diffeo

Rmks
- Can assume that H is 1-periodic in t
- Can replace 1 by anything
- Hamiltonian $\Rightarrow$ Symplectic: $\varphi^* \omega = \omega$
- When M is not compact, need to assume smth about H at ∞. We'll usually assume that H is compactly supported $\Rightarrow$ supp φ is compact $\not\Leftarrow$

Prop The collection of Ham diffeos $\mathrm{Ham} = \mathrm{Ham}(M, \omega)$ is a gp.

Rmk Not obvious: H is not autonomous

$$(\varphi_H^t)^{-1} \neq \varphi_H^{-t}$$

and it's not clear why

$$\varphi_H^t \varphi_K^t \text{ is Hamiltonian}$$

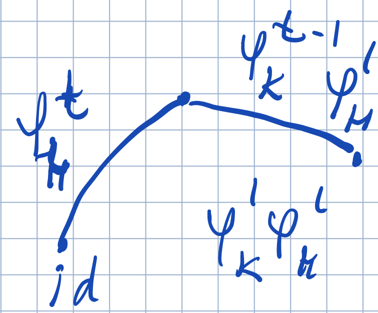
Focus on the product:

Pf 1. Consider $\left.\begin{array}{ll} H_t, & t\in[0,1] \\ K_{t-1}, & t\in[1,2]\end{array}\right\} F_t$

smooth in t when say

$\left.\begin{array}{lll} H_t \equiv 0 & \text{for} & t\approx 1 \\ K_t \equiv 0 & \text{for} & t\approx 0\end{array}\right\}$ can be achieved by time chg

Then F_t generates



φ_H^t for $t\in[0,1]$

$\varphi_K^{t-1}\varphi_H^1$ for $t\in[1,2]$

So over $t\in[0,2]$ it generates $\varphi_K^1\varphi_H^1$

$\Rightarrow$ $Ham(M,\omega)$ is closed under the product

Ex: generate $(\varphi_H^1)^{-1}$ $\triangleleft$

Pf 2 - Ex

- $\varphi_K^t\varphi_H^t$ is generated by $K_t + H_t\circ(\varphi_K^t)^{-1}$

$\Downarrow$

- $(\varphi_H^t)^{-1}$ — . — . — $-H_t\circ(\varphi_H^t)^{-1}$ $\triangleleft$

Now we have

usually strict (Examples) T^{2n}!

$$\boxed{Ham(M,\omega) \subset \underbrace{Symp_0(M,\omega)}_{\text{connected component of the id}}} \subset Sympl(M,\omega)$$

$$Sympl(M,\omega) \subset Diff_{\omega^n}(M)$$

should think of these as ∞-dim Lie groups

On the level of Lie algebras: vector fields

$$Ham \subset Symp_0$$

Discuss in more detail

Lie algebras

$$\boxed{\begin{array}{ccc} \text{Ham v.f.} & \subset & \text{Sympl. v.f.} \\ i_X\omega = \text{exact} & & i_X\omega \text{ closed} \iff L_X\omega = 0 \end{array}}$$

$$\begin{array}{ccc} \updownarrow \; X \mapsto i_X\omega & & \updownarrow \; X \mapsto i_X\omega \\ \text{exact 1-forms} & \subset & \text{closed 1-forms} \end{array}$$

$$\Rightarrow \boxed{\frac{\text{Sympl. v.f.}}{\text{Ham v.f}} \Longrightarrow H^1(M;\mathbb{R})}$$

$$\boxed{\underline{\text{Cor}} \quad H^1 = 0 \Rightarrow \text{Symp. v.f} = \text{Ham. v.f.}}$$

Ex. Shifts of $\mathbb{T}^2$

$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (x, y) "coordinates"

$\varphi : (x,y) \mapsto (x+a, y)$ $\omega = dx \wedge dy$

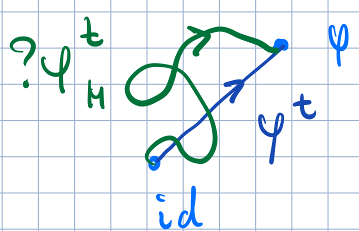
Generated by $X = a\frac{\partial}{\partial x}$, $\varphi = \varphi^1$

Symplectic but not hamiltonian:

$i_X \omega = a\, dy$ closed but not exact

$\not\Rightarrow$ $\varphi \notin \mathrm{Ham}$

what if $\exists$ some other φ_H^t from id to φ?



E.g. $a = 1$, $\varphi = \mathrm{id}$ by $X = \frac{\partial}{\partial x} \neq 0$

But can take $H = 0$

In fact, in this case $\varphi \notin \mathrm{Ham}$

and $\mathrm{Symp}_0 / \mathrm{Ham} = H^1(\mathbb{T}^2; \mathbb{R}) / H^1(\mathbb{T}^2; \mathbb{Z})$

$= \mathbb{R}^2/\mathbb{Z}^2$

$= \mathbb{T}^2$

Non-obvious: flux, cf McDuff-Salamon

Note: Ham v.f. $= \underbrace{C^\infty(M)}/\mathbb{R}$ ← in the center

Lie algebra with Poisson bracket

$$\{H, K\} := \omega(X_H, X_K) = -dH(X_K)$$

Ex.
- Check the Jacobi id
- Prove that

$$H \longmapsto X_H$$

is a Lie alg homo: $\{H,K\} \mapsto [X_H, X_K]$

$$C^\infty(M) \longrightarrow \text{Ham. v.f.}$$

- For $\mathbb{R}^{2n}$

$$\left\{\begin{matrix}\text{quadratic} \\ \text{forms}\end{matrix}\right\} \xrightarrow{\cong} sp(2n)$$

§5 Submanifolds of symplectic manifolds

Lecture 7

10/14-2021

Linear algebra

(V^{2n}, ω) symplectic v.s. : $\mathbb{R}^{2n} = \mathbb{C}^n$, $i = J$

$L \subset V$ linear subspace, $d = \dim L$

Def symplectic orthogonal

$$\boxed{L^\omega = \{X \in V \mid \omega(X, Y) = 0 \ \forall Y \in L\}}$$

Obvious properties

- $\dim L^\omega = 2n - d$
- $(L^\omega)^\omega = L$

Def
- L is isotropic if $L \subset L^\omega \iff \omega|_L = 0 \implies d \leq n$
- L is coisotropic if $L^\omega \subset L \implies d \geq n$
- L is Lagrangian if $L = L^\omega$ (coiso & iso) $\implies d = n$

most important

- L is symplectic if $\omega|_L$ is non-deg $\iff L^\omega \cap L = 0$

Ex. • $\dim L = 1 \Rightarrow$ isotropic
• $\operatorname{codim} L = 1 \Rightarrow$ coisotropic
• $L \subset \mathbb{C}^n$ complex $\Rightarrow$ symplectic
$JL = L$ $\not\Leftarrow$

• L Lagr $\Rightarrow$ L is real: $JL \cap L = 0$

Prop Given $L \Rightarrow$

$\exists$ Darboux basis $e_1, f_1, \dots, e_n, f_n$ s.t.

• L isotropic: $L = \operatorname{span}(e_1, e_2, \dots, e_d)$
• L coiso : $L = \operatorname{span}(e_1, \dots, e_n, f_1, \dots, f_k)$
• L Lagr : $L = \operatorname{span}(e_1, \dots, e_n)$
• L sympl : $L = \operatorname{span}(e_1, f_1, \dots, e_k, f_k)$

Cor All Lagr. subspaces a conj. by $Sp(2n)$
(Likewise for other types with d fixed)

Rmk. $V = L \oplus L'$ $\leftarrow$ Lagr

$$\left.\begin{array}{l} \Rightarrow\ L' \cong L^* \\ \quad x \mapsto i_x\omega|_L \end{array}\right\} \Rightarrow V = T^*L = L \times L^*$$

Ex. L coisotropic $\supset L^\omega$

L/L^ω symplectic : ω_{red}

$\omega_{red}(X, Y) = \omega(\tilde{X}, \tilde{Y})$ lifts

More generally: $L/L \cap L^\omega$ is symplectic

Lagr. Grassmannian

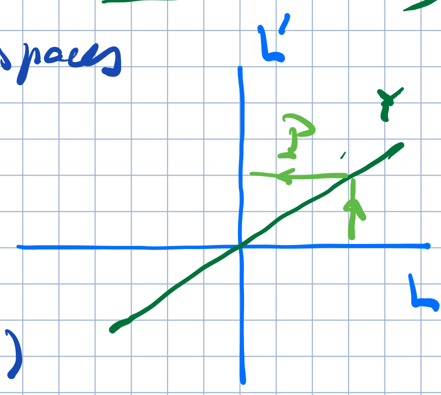
$\Lambda = \{ L \subset \mathbb{R}^{2n} \mid \text{Lagr} \}$

A manifold

$\mathbb{R}^{2n} = \mathbb{C}^n$
$\cup$
$\mathbb{R}^n$

$\boxed{\text{Discuss first general Grassmannians}}$

Charts: $L \oplus L' = \mathbb{R}^{2n}$ – fixed
from some collection
e.g. coordinate subspaces



$\mathcal{U} = \mathcal{U}_{L,L'} = \{ Y \pitchfork L' \mid Y \text{ Lagr} \}$

$Y = \text{Graph}(P : L \to L' = L^*)$

$\{ P : L \to L^* \} \longleftrightarrow \{ \text{bilinear forms } \beta \text{ on } L \}$

$P \longleftrightarrow (X,Y) \mapsto P(X)(Y) = \beta$

Y Lagr $\Leftrightarrow$ β is symmetric

$\mathcal{U}_{L,L'} \longleftrightarrow$ quadratic forms on L
(symmetric matrices)

$\Rightarrow \dim \Lambda = \frac{n(n+1)}{2}$

Lecture 8
10/19-2021

Digression:
complex & symplectic v.s.'s

complex str: V real v.s.
$J: V \to V$, $J^2 = -I$

Then $(a+ib)v = a + bJv$

Complex linear maps: $A: V \circlearrowleft$: $JA = AJ$

Ex: $A: V \circlearrowleft$ complex linear
$\Rightarrow A$ is $\mathbb{R}$-linear
Prove that $\det_{\mathbb{R}} A_{\mathbb{R}} = |\det_{\mathbb{C}} A|^2$

Cor: • $GL(n, \mathbb{C}) \subset \underbrace{GL^+(2n, \mathbb{R})}_{\det > 0}$
orientation pres

• $U(n) \subset O(2n)$

Complexification: $V = W \otimes \mathbb{C}$
W real v.s.
$\left.\begin{array}{l} V = W_{\mathbb{C}} = V \oplus V \circlearrowleft J \\ J(x,y) = (-y, x) \end{array}\right\}$ complex

But it also has an extra str
conjugation: $(x,y) \mapsto (x, -y)$

$e_1, \dots, e_n$ basis in $W \Rightarrow$ also a basis in V

Functorial

- $A: W \to W$ $\Rightarrow$ $A_{\mathbb{C}}: V \to A$
  $\mathbb{R}$-linear $\qquad\qquad$ $\mathbb{C}$-linear

  same matrix

- inner product on W $\Rightarrow$ Hermitian product on V

$\Rightarrow \quad O(n) \subset U(n)$ and $\boxed{O(n) = U(n) \cap GL(n, \mathbb{R})}$

Symplectic v.s. Hermitian

(V, ω) symplectic v.s.

J complex str.

ω & J are compatible if

Ex J always exists

1) $\omega(X, JX) \geq 0$
$\qquad\qquad > 0 \quad X \neq 0$

2) $J \in Sp(V, \omega)$

$\Leftrightarrow \quad \langle X, Y \rangle = \omega(X, JY)$ is an inner product
and $J \in O(V, \langle , \rangle)$

$\Leftrightarrow \quad \langle X, Y \rangle_{\mathbb{C}} = \langle X, Y \rangle + i\,\omega(X, Y)$
is a Hermitian inner product

• $L \subset V$ Lagrangian, J compatible with ω

$\Rightarrow JL \cap L = \{0\}$

$V = L \otimes \mathbb{C}$

$A \in O(L) \Rightarrow A_{\mathbb{C}} \in U(V)$

minor generalization

$\Rightarrow U(n)$ acts transitively on Λ

$$L \overset{A}{\rightsquigarrow} L'$$

$$\underbrace{e_1, \dots, e_n}_{\text{orthogonal}} \rightsquigarrow \underbrace{e_1', \dots, e_n'}_{\text{orthogonal}}$$

Then $A_{\mathbb{C}}: V = L \otimes \mathbb{C} \longrightarrow L' \otimes \mathbb{C} = V$
is unitary

And $\mathrm{Stab}(L) \subset U(n)$ is $O(n)$
$= U(n) \cap GL(n, \mathbb{R})$

$\Rightarrow$

Ex. • $\Lambda = U(n)/O(n)$ ← Explain

• $\pi_1(\Lambda) \xrightarrow{\cong} \mathbb{Z}$ ← "Maslov class"

$A \longmapsto \det^2_{\mathbb{C}}(A)$

• $H_1(\Lambda) \longrightarrow \mathbb{Z}$: $\mathrm{Maslov} \in H^1(\Lambda; \mathbb{Z})$

Ex. $\Lambda_1 = \mathbb{RP}^1 = S^1$

$\Lambda_2 = S^1 \times S^2/\sim$ ← antipodal in both factors

Back to symplectic manifolds

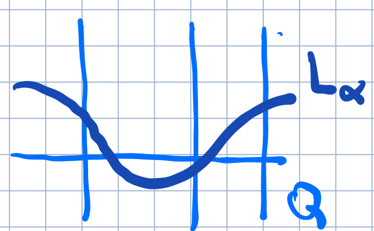
$(M^{2n},\omega) \supset L$

Def. L Lagr (iso, coiso, sympl) if
$T_x L \subset T_x M$ is Lagr (iso...) $\forall x \in L$

Ex. $\dim L = 1 \Rightarrow$ iso
$\operatorname{codim} L = 1$ (hypersurface) $\Rightarrow$ coiso

Focus on Lagr. submanifolds

Ex 1 • $M = T^*Q \longrightarrow Q$
$\alpha \in \Omega^1(Q)$ = section of T^*Q
$\Rightarrow \quad L_\alpha \subset T^*Q$



$\omega = d\lambda$

$d\alpha = 0 \iff L_\alpha$ Lagr

Pf $\alpha = \lambda|_{L_\alpha = Q}$

$d\lambda|_L = 0 \iff d\alpha = 0$ $\triangleleft$

Or if you wish

$$\psi : Q \longrightarrow T^*Q$$
$$x \longmapsto \alpha_x$$

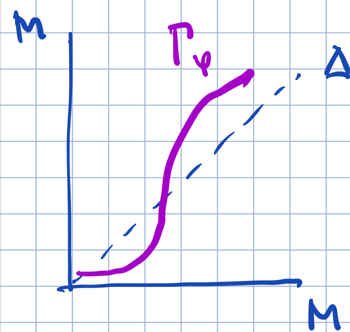
$\psi^*\lambda = \alpha$
$d\lambda|_{L_\alpha} = 0 \iff \psi^* d\lambda = 0$
$\iff d\alpha = 0$

Ex 2 $W = (M \times M, \underbrace{(-\omega, \omega)}_{\tilde{\omega}})$

$\varphi : M \longrightarrow M$

$L = \Gamma_\varphi \subset W$, the graph of φ

$L = \{(x, \varphi(x)) \mid x \in M\}$



Claim:

Γ_φ Lagr $\Longleftrightarrow$ φ is symplectic
$\varphi^* \omega = \omega$

Pf Identify $\varphi : M \longrightarrow \Gamma_\varphi$

$x \longmapsto (x, \varphi(x))$

$$\tilde{\omega}\big|_{\Gamma_\varphi} = 0 \Longleftrightarrow \varphi^* \tilde{\omega} = 0$$

$$\Longleftrightarrow -\omega + \varphi^* \omega = 0$$

$$\Longleftrightarrow \varphi^* \omega = \omega$$

$\triangleleft$

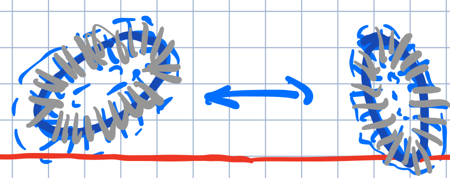
Lecture 9
10/21-2021

Thm (Weinstein's tubular nbd)

$L \xhookrightarrow{i} (M,\omega)$, $L \xhookrightarrow{i'} (M',\omega')$ — L closed — two Lagr. embeddings

$\Rightarrow \exists$ nbds $U \supset i(L)$ & $U' \supset i'(L)$

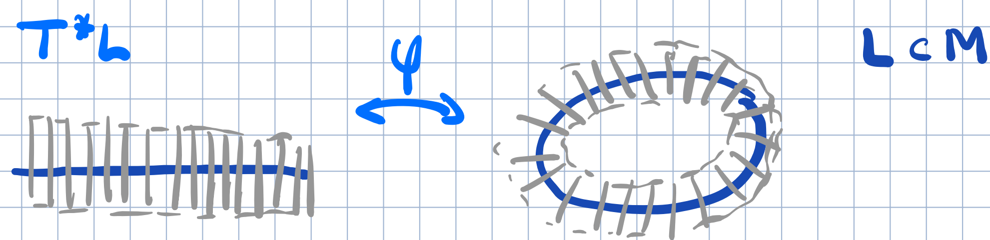
s.t. $(U, i(L))$ & $(U', i'(L))$ are symplecto...



Thm' (. . . . .)

closed $L \hookrightarrow M$ Lagr. embedding

$\Rightarrow$ nbd of L is symplectomorphic to a nbd $L \subset T^*L$



Discuss the tubular nbd thm?

On the pf: similar to Darboux

$L \subset M$ Lagr. closed

- Preliminary (lin. alg)

N_L normal budle: $N_L \oplus TL = T_L M$
can be chosen Lagr

$N_L \cong T^*L$

Now use ordinary tubular nbd thm

Discuss

to identify a nbd of $L \subset M$ with a nbd of L in T^*L

$U = \big($nbd of L in M $\cong$ nbd of L in $T^*L\big)$

$\Rightarrow$ $U \subset$ nbd of L in T^*L
ω_0 & ω_1 two symplectic forms
s.t. L is Lagr for both
and by construction

ω_0 & ω_1 $\underbrace{\text{agree on } T_L(T^*L)}_{(*)}$
standard

$\Downarrow$

symplectic

- Set $\omega_t = (1-t)\omega_0 + t\omega_1$

  Run the homotopy method

  X_t: $\quad i_{X_t}\omega_t = \lambda \qquad \underbrace{d\lambda = \omega_0 - \omega_1}_{\text{exists}}$

$H^2(U) \cong H^2(L)$

$[\omega_0 - \omega_1] \leftrightarrow [\underbrace{\omega_0|_L}_{= 0} - \underbrace{\omega_1|_L}_{= 0}]$

Nuance: !

Need $\lambda = 0$ at every pt of L

(To make sure φ_t is defined for $t \in [0,1]$)

Not only $\lambda|_L = 0$

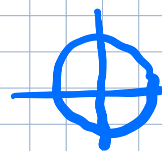
<u>Ex</u>: Such λ exists by $(*)$

Not obvious $\triangleleft$

Lagr. submanifolds of $\mathbb{R}^{2n}$

Lecture 10
10/26-2021

Important question in sympl topology
Some simple observations

- $S^1 \hookrightarrow \mathbb{R}^2$ Lagr  $\Rightarrow$

$T^n = S^1 \times \dots \times S^1 \hookrightarrow \mathbb{R}^{2n}$

A lot of different (non-equivalent)

$\omega_{st} = d\lambda$, $\lambda = \frac{1}{2}\sum (p_i\, dq_i - q_i\, dp_i)$

$\lambda|_{T^n}$ closed $[\lambda|_{T^n}] \in H^1(T^n; \mathbb{R})$ inv

- **Prop** $L \subset \mathbb{R}^{2n}$ closed Lagr $\Rightarrow$ $\chi(L) = 0$

Pf. $N_L = T^*L \Rightarrow L \cdot L = \chi(L)$

But $\underset{\text{in } \mathbb{R}^{2n}}{L \cdot L = 0}$ e.g. because $[L] = 0$

or by deformation invariance

Cor. $\Sigma_{g \neq 1}$ does not admit Lagr. embeddings into $\mathbb{R}^4$

Liouville class

Rmk $\exists$ much more subtle results

E.g. $L \subset \mathbb{R}^{2n}$ Lagr $\Rightarrow H^1(L; \mathbb{R}) \neq 0$

In fact $[\lambda|_L] \neq 0$ (Gromov)

$\Rightarrow S^3$ does not have Lagr. embedding into $\mathbb{R}^6$

Maslov class

$L \looparrowright \mathbb{R}^{2n}$ Lagn, immersed, closed

$\Rightarrow G: L \to \Lambda$ $\leftarrow$ Gauss map

$x \mapsto T_x L$

$\mu \in H^1(\Lambda; \mathbb{Z})$ Maslov

$\mu_L \in H^1(L; \mathbb{Z}) \leftarrow$ Maslov class of L.

$\shortparallel$

$G^* \mu$

Ex. L orientable $\Rightarrow \mu_L$ is even

Fact (Gromov) $L \subset \mathbb{R}^{2n}$ embedded

$\Rightarrow \mu_L \neq 0 \Rightarrow H^1 \neq 0$

$\Rightarrow$ again S^3 does not have a Lagr. emb

Rmk. μ_L can also be defined for $L \subset T^*Q$ (but it can be 0)

§6 Contact manifolds

Contact str = odd-dim sister of sympl str

M^{2n+1} ← odd dimensional

Def • $\alpha \in \Omega^1(M)$ is contact if

$\alpha \wedge (d\alpha)^n \neq 0$ ← vol. forms $\boxed{\Rightarrow M^{2n+1} \text{ orientable}}$

$\Leftrightarrow$ $d\alpha|_{\ker \alpha}$ is non-deg

• $\xi = \ker \alpha$ is a contact str

strictly speaking: a codim-1 distr ξ is contact if locally $\xi = \ker \alpha$ ← contact

can be made globally $\Leftrightarrow$ ξ is coorient-d

Note: α contact $\Rightarrow$ $f\alpha$ contact

$f \neq 0$ — ξ same contact str.

$$(f\alpha) \wedge [d(f\alpha)]^n$$
$$= (f\alpha) \wedge [df \wedge \alpha + f d\alpha]^n$$
$$\overset{\alpha^2 = 0}{=} f^{n+1} \alpha \wedge (d\alpha)^n$$

Cor M^{2n+1} admits a contact str ξ (not necessarily coorientable) and $n+1$ even $\Rightarrow$ M is orientable

Def-fact α contact form $\Rightarrow$

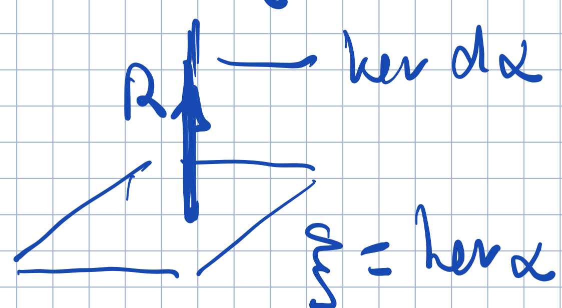
$\exists!$ v.f. R Reeb v.f :

$$\alpha(R) = 1$$

$$i_R \, d\alpha = 0$$

In fact α contact

$$\Longleftrightarrow \begin{cases} \ker(d\alpha) \ 1\text{-dim} \\ \ker d\alpha \pitchfork \underbrace{\ker \alpha}_{\xi} \end{cases}$$



Reeb v.f. $\rightsquigarrow$ Reeb flow

Examples

Ex 1. $\mathbb{R}^{2n+1}$ (p, q, z)

$\alpha = dz + p\,dq$ or $dz + \frac{1}{2}(p\,dq - q\,dp)$

are contact : st. contact form on $\mathbb{R}^{2n+1}$

$R = \frac{\partial}{\partial z}$ The st. contact form or str. on $\mathbb{R}^{2n+1}$

Visualize $n=1$

More generally $M = \mathbb{R}_z \times W \xrightarrow{\pi}$

$\alpha = dz + \pi^* \lambda$ ← contact $\quad \omega = d\lambda$

Ex 2

$\Sigma \subset \mathbb{R}^{2n}$, $\Sigma = \partial(\text{starshaped})$

e.g. convex

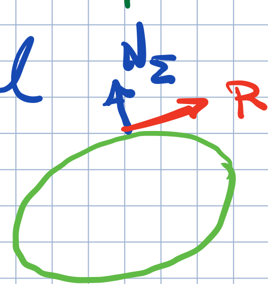
$\lambda = \frac{1}{2}(p\,dq - q\,dp)$

$\alpha = \lambda|_\Sigma$ is contact

Motivation coming from Ham dynamics

unit normal

- $R = J N_\Sigma$ ← normal
- $\Sigma = \{H = \text{const}\}$ ← reg



Then $R = f X_H$ on Σ

Lecture 11
10/28-2021

Move generally

$\Sigma^{2n-1} \subset (M^{2n}, \omega)$ symplectic

Def. Σ has contact type if
$\omega|_\Sigma$ has a contact primitive α:
$d\alpha = \omega|_\Sigma$, $\alpha \wedge (d\alpha)^{n-1} \neq 0$

Then: $\Sigma = \{H = \text{const}\}$ ← regular
$\Rightarrow R = f X_H$ on Σ

Rmk. Not every closed hypersurface in $\mathbb{R}^{2n}$ has contact type
Ex - Weinstein: two spheres

Ex 3 $\Sigma \subset T^*Q$ fiberwise starshaped
$\lambda = p\,dq$ Liouville form
$\alpha = \lambda|_\Sigma$ is contact

Motivation: geometric optics

Σ fiberwise convex: Finsler metric
Reeb flow = Finsler geodesic flow

Ex 4 - Fact every closed orientable 3-manifold admits a contact structure

necessary

Existence of contact str. Discuss in more detail?

Contact topology ← active area

Contact Darboux Thm and all that

Thm (contact Darboux)

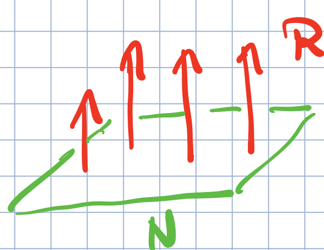
$\boxed{\text{Any two contact forms (fixed dim) are locally diffeomorphic}}$

Thm' (Contact Darboux)

Any contact form in $\dim = 2n+1$ is locally diffeo to the standard contact form on $\mathbb{R}^{2n+1}$: $\exists$ coord p, q, z such that $\alpha = dz + p\,dq$

Two ways to prove:

1) As a consequence of symplectic Darboux – Ex



crossection

$d\alpha|_N$ = symplectic $= d(\underbrace{p\,dq}_{\lambda})$

$d\alpha|_N = d\lambda|_N \Rightarrow \alpha - \lambda = df$ ← can assume 0 at 0

"z = time of Reeb flow from N + f"

2) Use Moser's homotopy method directly

Rmk No global version for contact forms

$M, \alpha_t \leftarrow$ a family of contact forms

cannot expect α_s to be diffeo to each other

$\alpha_s \rightsquigarrow R_s \rightsquigarrow$ dynamics changes with s

Ex. $\Sigma_t \subset \mathbb{R}^{2n}$ a family of ellipsoids

$\Sigma_t = \{H=1\} \leftarrow$ quadratic flow

$(\Sigma_t, \lambda|_\Sigma) \cong (S^{2n-1}_t, \alpha_t)$

$R_t = X_t \rightsquigarrow R_t$

we have seen that things depend on eigenvalues

$$\boxed{\begin{array}{l}\textbf{Thm (Gray's Thm)}\\ (M^{2n+1}, \xi_t)\ \text{closed, contact} \Rightarrow \varphi_t \ \text{s.t.}\ (\varphi_t)_* \xi_t = \xi_0\end{array}}$$

What is actually proved

$M^{2n+1}, \ \xi_t = \ker \alpha_t$

$\Rightarrow \exists \ \varphi_t \ \& \ f_t > 0 : \ \varphi_t^*(f_t \alpha_t) = \alpha_0$

Pf: Moser's homotopy method

Rmk Discuss symplectization

A glimpse of contact topology

ξ cooriented contact str

⇝ R_α Reeb : non-vanishing section of TM

⇝ The homotopy type of R is ind of α

A section of STM

⇝ A top inv. of ξ

homotopy type of ξ as cooriented distr.

E.g. $M = S^3$ $\quad ST S^3 = S^3 \times S^2$

homotopy types of sections

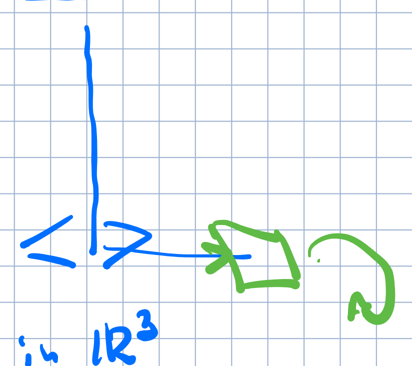
$\delta(\xi) \in [S^3, S^2] = \pi_3(S^2) = \mathbb{Z}$

Each of them can be realized by a contact str, and those contact str are not diffeo to each other

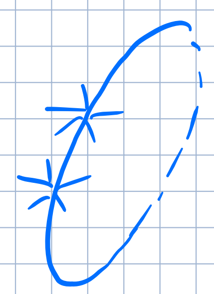
standard ⇝ $0 = \delta(\xi_{st})$

But $\exists$ (exactly one) contact str ξ_{ot} with $\delta(\xi_{ot}) = 0 = \delta(\xi_{st})$

Describe



in $\mathbb{R}^3$



in S^3

§7. Elements of Morse Theory

- Not directly related to symplectic geom but extr. important on its own.
- Connections with many things inc s.g., ODE's, PDE's, everything

General setting & motivation

X some space: a manifold, loop space, path space

$f: X \longrightarrow \mathbb{R}$ a function (smooth)

Looking for critical pts of f

Does it have them? How many, etc?

Ex a) x_0 & $x_1 \in Q \leftarrow$ Rieman. manifold

$X = \{$path connecting $x_0, x_1\}$

$= \{\gamma : [0,1] \longrightarrow Q \mid \gamma(i) = x_i\}$

Fix a R.m. metric on Q

$$\underset{p}{T^*Q} = \underset{v}{TQ}$$

potential energy

$$H = \frac{1}{2}\langle p, p\rangle + V(q)$$

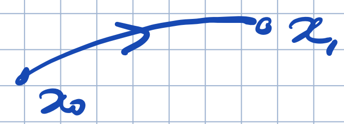
From mechanics → diff geometry → Original Morse theory

$$f(\gamma) = \int_0^1 H(\dot\gamma, \gamma)\, dt$$

__Least action principle (LAP)__

$\mathrm{Crit}(f)$ = integral curves connecting x_0 & x_1 in time-1

E.g. $V = 0$: geodesics from x_0 to x_1

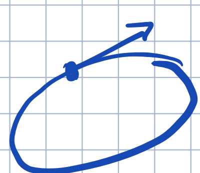


b) X = loop space
$= \{\gamma : S^1 \longrightarrow Q\}$

f = the same

__Relations to diff geometry:__
Hopf-Rinow, closed geodesics, Hadamard

LAP: $\mathrm{Crit}(f)$ = periodic traj of φ_H^t of period 1 in T^*Q



E.g. closed geodesics

__Etc__ : Everything of interest in physics is a crit pt of some functional

Calculus of variations

Finite dimensional setting

- $X = M$ a compact (or even closed) finite-dim. manifold
- $f : M \to \mathbb{R}$ $\quad$ ($f|\partial X = \text{const}$)

Q. How many crit pts does f have?

- Nontrivial even in simple cases.
- Assume M is closed:

$\max f \geq \min f$ — critical values

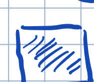
Anything else

- In general yes, unless $M \overset{\text{homeo}}{\simeq} S^n$

But might be few

| Ex. $\text{Crit} = \{\max, \min\}$ $\Rightarrow M \underset{\text{homeo}}{\simeq} S^n$ |

Ex. . Construct $f : \Sigma_{g \geq 1} \to \mathbb{R}$ with exactly 3 critical pts

- sketch the levels for $\mathbb{T}^2$:  $/\sim$

The third pt = Monkey saddle

- Situation changes when we impose a non-deg cond on f, satisfied generically $\leftarrow$ explain

Definitions

$p \in M$ critical pt of $f: M \to \mathbb{R}$

Def Hessian of f at p is the quadratic (or bilinear) form!

$$d^2 f: T_pM \times T_pM \longrightarrow \mathbb{R}$$
$$v\ ,\ w \longrightarrow (L_{\tilde{v}} L_{\tilde{w}} f)(p)$$

↑ Ext of v & w to v.f.

Ex. • Show that $d^2 f$ is well defined
• symmetric
• In local coordinates $x_1, \dots, x_n$

$$d^2 f = \sum \frac{\partial^2 f}{\partial x_i \partial x_j}(p)\, dx_i\, dx_j$$

Do some of it

Then assume that $d^2 f$ is non-deg

Morse index of p = index of $d^2 f$:

$$d^2 f = -(x_1^2 + \dots + x_k^2) + (x_{k+1}^2 + \dots + x_n^2)$$

Ex. $p = \max$: index $= n$
$p = \min$: index $= 0$

Def f is Morse if all its critical pts are non-deg

Note: Morse functions form an open and dense subset of $C^\infty(M)$

Thm (**Morse Lemma**) f is C^3

Near a non-deg critical pt p of a function f is diffeo to its Hessian $H = d^2 f$:

- $\exists\ \varphi : (V, p) \to (V, p)$ s.t.
  $f \circ \varphi = \varphi^* f = H + f(p)$

$\Updownarrow$

- In some coordinates $x_1, \dots, x_n$
  $f(x) = f(p) + \sum a_{ij} x_i x_j$

Rmk
- One of the normal form results
- Another example:
  $df_p \neq 0 \quad \exists\ (x_1, \dots, x_n)$ s.t.
  $f(x) = f(p) + x_1$ } Ex prove directly

  More generally, the local norm form for submersions (also immersion)

- Similar questions for other objects: vector fields, maps, etc
- E.g. v v.f.
  $$v(p) \neq 0 \Rightarrow \exists\ x_1, \dots, x_n$$
  $$v(x) = \frac{\partial}{\partial x_n}$$
  what if $v(p) = 0$?

Pf: Moser's homotopy method...

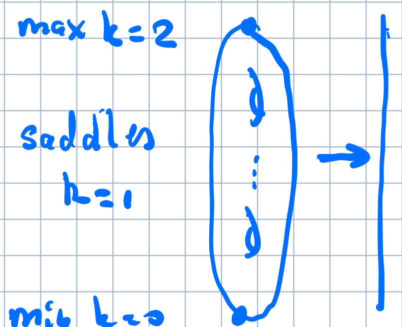
Morse Homology

Lecture 13
10/04-2021
Ref: Audin-Damian
Banyaga-Hurtubise

- $f: M \longrightarrow \mathbb{R}$ morse function
- $\mathrm{Crit}_k(f) \longleftarrow$ the collection of critical pts of index k

- Fix a ground ring $\mathbb{F}$: $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{Z}_2$, etc
- $CM_k(f)$ = free module over $\mathbb{F}$ generated by $\mathrm{Crit}_k(f)$

E.g. Height function on Σ_g



$CM_0 = \mathbb{F}$
$CM_1 = \mathbb{F}^{2g}$
$CM_2 = \mathbb{F}$

$\boxed{\partial^2 = 0}$

Goal: turn CM_k into a complex

$$0 \to CM_n \xrightarrow{\partial} CM_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} CM_1 \xrightarrow{\partial} CM_0 \to 0$$

so that

$$\boxed{\underbrace{H_*(CM_*(f),\partial)}_{=:\ HM_*(f)} \cong \underbrace{H_*(M;\mathbb{F})}_{\text{Homology of of } M}}$$

Morse Homology

Cor (**Morse inequalities**)

$\mathbb{F}$ a field

(a) $\underbrace{\# \mathrm{Crit}_k(f)}_{c_k} \geq \underbrace{\dim H_k(M)}_{b_k}$

(b) $c_k - c_{k-1} + c_{k-2} - \ldots \pm c_0 \geq b_k - b_{k-1} + \ldots \pm b_0$

Pf (a) $c_k = \dim CM_k \geq b_k = \dim HM_k$

(b) $\Leftarrow$ Ex $\Leftarrow$ Purely algebraic statement

C_* a complex over $\mathbb{F}$

show that C_* can be decomposed as a sum of elementary complexes explain

check (b) for an elementary complex $\triangle$

Yet a different formulation. Set

$Q(t) = \sum c_k t^k$, $\quad P(t) = \sum b_k t^k \leftarrow$ Poincaré Pol

Then

$$Q(t) = P(t) + (1+t)R(t) \qquad R(t): \text{coeff} \geq 0$$

Ex. Prove this - alg, same method

Ex A Morse function on Σ_g has at least $2g+2$ critical pts.

Rmk. Morse inequalities can be further refined

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Construction of the Morse differential ∂: Preliminaries

- While $CM_*(f)$ is completely determined by f, ∂ depends on an extra str: a R. metric on M

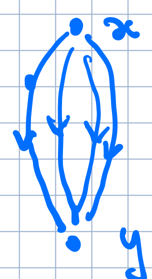
- Fix a R. m. on M (has to be from a certain open and dense set of R.M.'s)

Consider the antigradient flow of f:

$$\dot{x} = -\nabla f(x) \quad : \quad \varphi_t$$

Set
$$\mathcal{M}(x,y) = \left\{ z \;\middle|\; \varphi_t(z) \begin{array}{l} \nearrow x \quad t \to -\infty \\ \searrow y \quad t \to +\infty \end{array} \right\}$$

crit

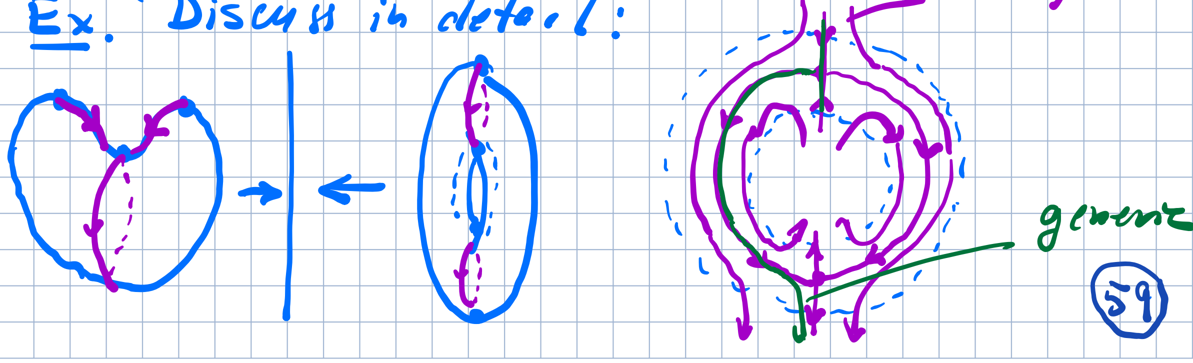


Denote the index of x by $\mu(x)$.

Note: "$\dim \mathcal{M}(x,y) \geq 1$" if $\neq \emptyset$

Thm For a generic metric, $\mathcal{M}(x,y)$ is a smooth manifold of dimension $\mu(x)-\mu(y)$

Ex. Discuss in detail:

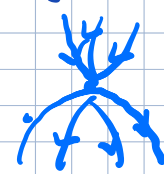


One way to prove the theorem:

$$W^u(x) = \{ z \mid \varphi^t(z) \longrightarrow x,\ t \to -\infty \}$$

$$W^s(x) = \{ \text{——} \ . \ \text{——} \ . \ \ldots\ t \to +\infty \}$$

stable, unstable manifolds



Morse Lemma: $\Rightarrow$ $W^u(x) \underset{\text{diffeo}}{\simeq} D^{\mu(x)}$

(Look at the examples) $\qquad W^s(x) \simeq D^{n-\mu(x)}$

$$\mathcal{M}(x,y) = W^u(x) \cap W^s(y)$$

If $\quad W^u(x) \pitchfork W^s(y)$,

$\mathcal{M}(x,y)$ is smooth and

$$\dim \mathcal{M}(x,y) = \dim W^u(x) + \dim W^s(y) - n$$
$$= \mu(x) + n - \mu(y) - n$$
$$= \mu(x) - \mu(y)$$

How to achive transversality

Look at $(W^u(x) \cap \{f=c\}) \cap (W^s(y) \cap \{f=c\})$

$f(y) < \underset{\text{reg}}{c} < f(x)$, perturb the metric slightly above c to alter

$\triangleleft$

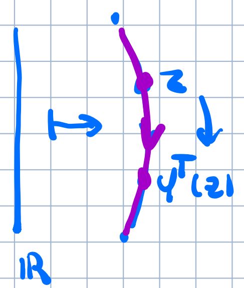
Cor By thm, for a generic metric

$$\mu(y) \geq \mu(x) \Rightarrow \mathcal{M}(x,y) = \emptyset$$

More modern & different perspective

Lecture 16
11/09-2021

$\mathcal{M}(x,y)$ = the space of parametrized trajectories $t \mapsto \varphi^t(z)$ from x to y



trajectory $\longleftrightarrow$ initial condition

$\varphi^t(z) \longleftrightarrow z = \varphi^0(z)$

Time shift: $t \mapsto \varphi^t(z)$ $\qquad z \mapsto \varphi^T(z)$

$$\begin{gathered} t \mapsto \varphi^t(z) \\ \wr \\ t \mapsto \varphi^{t+T}(z) \end{gathered}$$

$\Rightarrow$ free $\mathbb{R}$-action on $\mathcal{M}(x,y)$, $\quad x \neq y$

Space of unparametrized trajectories

$$\hat{\mathcal{M}}(x,y) = \mathcal{M}(x,y)/\mathbb{R}$$

Cor $\hat{\mathcal{M}}(x,y)$ is a smooth manifold of dim $\mu(x) - \mu(y) - 1$

E.g. $\mu(x) = \mu(y) + 1 \Rightarrow \hat{\mathcal{M}}$ is discr

$\mu(x) \leq \mu(y) \Rightarrow \hat{\mathcal{M}} = \emptyset$

Note
- $\mathcal{M}$ & $\hat{\mathcal{M}}$ are usually non-compact
- Geometrically, $\hat{\mathcal{M}}$ can be identified with $\mathcal{M} \cap \{f = c\}$

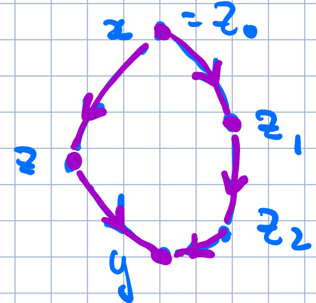
$f(y) < c < f(x)$

$\uparrow$
regular

(For a generic metric)

Thm $\hat{\mathcal{M}}(x,y)$ has a compactification formed by broken trajectories

Make it precise



Such trajectories $x = z_0 \rightsquigarrow z_1 \rightsquigarrow \dots \rightsquigarrow z_k = y$ form a compact manifold with corners.

Rmk $f(x) > f(z_1) > \dots > f(y)$

$\mu(x) > \mu(z_1) > \dots > \mu(y)$

Cor. $\mu(x) = \mu(y) + 1$

$\Rightarrow \hat{\mathcal{M}}$ = compact $\Rightarrow$ finite collection of pts

$\Leftrightarrow \exists$ finite many traj from x to y (for a generic metric)

Definition of ∂

Fix a generic metric so that all the things hold

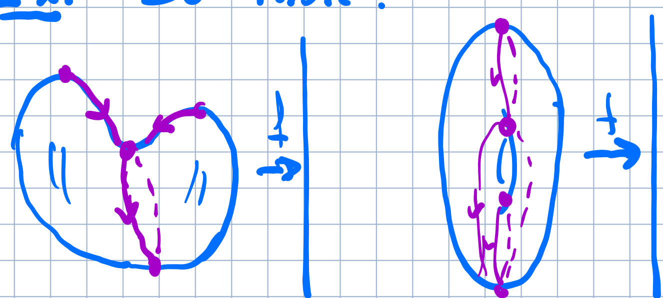
$$\mu(x) = \mu(y) + 1$$

- Over $\mathbb{Z}_2$, set

$$\mathbb{Z}_2 \ni m(x,y) = \# \hat{\mathcal{M}}(x,y) \mod 2$$

$$\boxed{\partial x = \sum_{\substack{y \\ \mu(x) = \mu(y)+1}} m(x,y)\, y} \qquad (*)$$

Ex. Do these:



- Over $\mathbb{Z}$ (and hence any ring)

Need to take into account orientations

Fix orientations of $T_x W^u(x)$ $\forall x$

$\Rightarrow$ coorientations of $T_x W^s(x)$

$\Rightarrow \begin{cases} \text{orientations of } W^u(x) \\ \text{coorientation of } W^s(x) \end{cases}$

$\Rightarrow$ orientations of

$$\mathcal{M}(x,y) = W^u(x) \cap W^s(x)$$

when $\mu(x) = \mu(y)+1$

$\mathcal{M}(x,y)$ = disj union of finite # of trajectories

Each trajectory γ is also oriented by the flow

$\Rightarrow$ Two orientations

$$\text{sign}(\gamma) = \begin{cases} +1 & \text{orientations agree} \\ -1 & \text{— " — " disagree} \end{cases}$$

And

$$m(x,y) = \sum_{x \overset{\gamma}{\rightsquigarrow} y} \text{sign}(\gamma)$$

$*$: $\boxed{\partial x = \sum m(x,y)\, y}$

Ex. Look at the torus example again.

Checking that $(CM(f), \partial)$ is a complex

$\boxed{\textbf{Thm } \partial^2 = 0}$

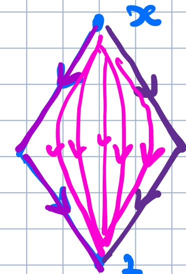
Pf. For the sake of simplicity over $\mathbb{Z}_2$

$$\partial^2 x = \partial \sum_y m(x,y)\, y$$

$\mu(x) = \mu(y)+1$
$\mu(y) = \mu(z)+1$

$$= \sum_y m(x,y) \sum_z m(y,z)\, z$$

$$= \sum_z \Big(\underbrace{\sum_y m(x,y)\, m(y,z)}_{\substack{\#\text{ of broken trajectories (one break)} \\ \text{from } x \text{ to } z \ (\text{mod } 2)}} \Big) \quad \text{mod } 2$$



But $\hat{M}(x,y)$ one-dim manifold
its compactification: S^1 or I closed interval

$\Rightarrow$ broken trajectories come in pairs

$\Rightarrow$ # is even

$$\Rightarrow \sum_y m(x,y)\, m(y,z) = 0 \mod 2$$

$\Rightarrow \partial^2 = 0$ $\triangleleft$

Set $HM_*(f) = H_*(CM(f), \partial)$; fixed coefficients

$$\boxed{\begin{array}{l} \underline{\text{Thm}} \text{ (Morse theory)} \\ \qquad HM_*(f) = H_*(M) \end{array}}$$

<u>Rmk</u> • As a consequence, r.h.s is independent of f

• We have already seen some consequences: Morse inequalities, etc

<u>Outline of the pf:</u>

"Classical" Morse theory

<u>Morse function f on M</u> $\rightsquigarrow$ <u>Cellular decomposition of M</u>

$Crit_k(f) \rightsquigarrow W^u(x) \leftarrow$ cells

$ Crit_*(f)$

Morse complex $CM_k(f)$ $\leftrightsquigarrow$ Cellular complex of M: $C(M)$

$\partial_M = \partial_{cw}$

$$\Rightarrow \quad H_*(CM(f), \partial_M) = \underbrace{H_*(C(M), \partial_{cw})}_{H_*(M)} \qquad \triangleleft$$

<u>Details:</u> Audin-Damian

Important:

we could have established the isomorphism

$$HM_*(f_0) \longleftrightarrow HM_*(f_1)$$

without going throug $H_*(M)$

Method

$$f_0 \overset{s}{\sim} f_1 : f_s \leftarrow \text{not necessarily Morse}$$

$$\Rightarrow \quad CM_*(f_0) \xrightarrow{\not\cong} CM_*(f_1)$$

$$\Rightarrow \quad HM_*(f_0) \xrightarrow{\cong} HM_*(f_1)$$

an isom

Applications: A quick look

Lecture 15
11/16-2021

Two types: Lower bounds on $\# \operatorname{Crit}(f)$ ← Morse

Calculation of $H_*(M)$
Or more generally diff top of M

(1) Lower bounds: Morse inequalities

$$\# \operatorname{Crit}_k(f) \geq b_k = \dim H_k(M)$$
$$\# \operatorname{Crit}(f) \geq \sum b_k$$

Morse

E.g. • f on $\mathbb{T}^n$ $\quad \# \operatorname{Crit}(f) \geq 2^n$

• f on $\mathbb{CP}^n$ or $\mathbb{RP}^n$

$\# \operatorname{Crit}(f) \geq n+1$

• f on Σ_g $\quad \# \operatorname{Crit}(f) \geq 2 + 2g$

Etc.

(2) Calculations of $H_*(M)$ using Morse homology.

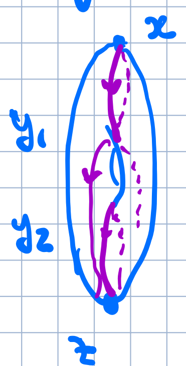
$$H_*(M) = H_*(\,HM_*(f),\, \partial_m)$$

very difficult to deal with in general

Works well when $\partial = 0$

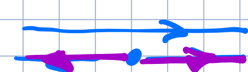
Examples (over $\mathbb{Z}$ or $\mathbb{F}$)

1) Σ_g on T^2



$$\partial x = y_1 + -y_1 + y_2 + -y_2 = 0$$

← work out unstable traj of y_1 & y_2 should come from x (Orientations)



$$\partial y_1 = 0 = \partial y_2$$

$\Rightarrow$

$$H_*(\Sigma_g) = \begin{cases} \mathbb{F} & k=2 \\ \mathbb{F}^{2g} & k=1 \\ \mathbb{F} & k=0 \end{cases}$$

2) $\mathbb{C}P^n$ (over $\mathbb{Z}$ or $\mathbb{F}$)

$$\mathbb{C}P^n = \{ (z_0 : \dots : z_n) \mid \sum |z_j|^2 = 1 \}$$

$$f(z) = \sum \lambda_j |z_j|^2 \quad \text{or} \quad \frac{\sum \lambda_j |z_j|^2}{\sum |z_j|^2}$$

distinct

$$\lambda_0 < \lambda_1 < \dots < \lambda_n \leftarrow \text{convenient to not assume}$$

Ex. a) $\mathrm{Crit}(f) =$ "coordinate axes"

Work out in detail

$$= \{ (0, \dots, 0, \underset{j}{1}, 0, \dots, 0) = x_j \}$$

b) In coordinates

$$\underset{j}{u} = (u_0, \dots, u_{j-1}, 1, u_{j+1}, \dots, u_n\}$$

near x_j the Hessian is

$$(\lambda_0 - \lambda_j)|u_0|^2 + (\lambda_1 - \lambda_j)|u_1|^2 + \dots \text{ skip } (\lambda_j - \lambda_j)$$

$\Rightarrow$ f is morse & $\mu(z_j) = 2j \leftarrow z_j$ is a complex #.

& $\partial = 0$

$$\Rightarrow \quad H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{F} & 0 \leq k = 2j \leq 2n \\ 0 & \text{otherwise} \end{cases}$$

Some details

a) Lagrange multipliers

$$\begin{tikzcd} \mathbb{C}^{n+1} \supset S^{2n+1} \arrow[d, "S^1"'] \arrow[dr, "\tilde f"] & \\ \mathbb{C}P^n \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$S^{2n+1} = \{g = 1\}$

Crit pts:

$\nabla g = c \cdot \nabla \tilde f$

$$\left.\begin{aligned} \nabla g &= 2z = 2(z_1, \dots, z_n) \\ \nabla \tilde f &= 2(\lambda_1 z_1, \dots, \lambda_n z_n) \end{aligned}\right\} \Rightarrow \begin{aligned} &\text{Crit pts} \\ &\quad \Updownarrow \\ &\text{Coord axes} \end{aligned}$$

b) Hessian say at $x_0 = (1, 0 \dots 0)$: $j = 0$

U_1 : $u_1 = \frac{z_1}{z_0}$, $u_2 = \frac{z_2}{z_0}$, ...

$$f(u) = \frac{\sum \lambda_k |z_k|^2}{\sum |z_k|^2}$$ Here $k = 0, \dots, n$

$$= \frac{\lambda_0 + \sum \lambda_k |u_k|^2}{1 + \sum |u_k|^2}$$ But here $k = 1, \dots, n$

$$= (\lambda_0 + \sum \lambda_k |u_k|^2)(1 + \sum |u_k|^2)^{-1}$$

$$= \lambda_0 + \sum \lambda_k |u_k|^2 - \lambda_0 \sum |u_k|^2$$

$$= \lambda_0 + \sum_{k=1}^{n} (\lambda_k - \lambda_0) |u_k|^2$$

3) $\mathbb{RP}^n$ over $\mathbb{Z}_2$

Similarly $\mathbb{RP}^n = \{ (y_0 : \dots : y_n) \mid \sum |y_j|^2 = 1 \}$

$$f(y) = \sum \lambda_j |y_j|^2$$

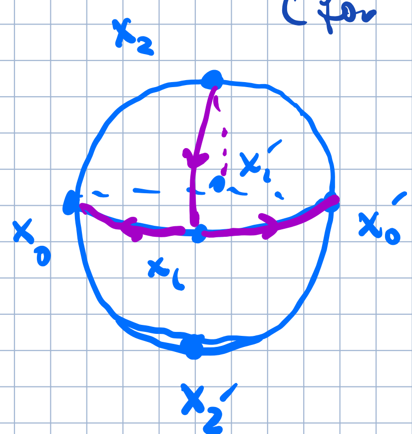
Ex. similarly

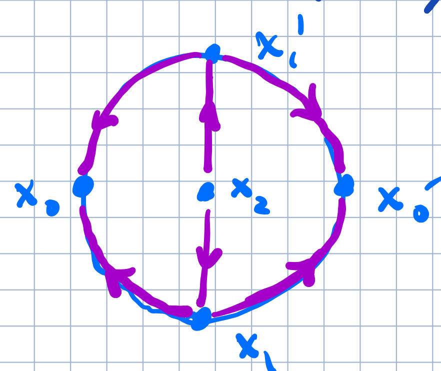
a) $x_j = (0, \dots, 0, \underset{j}{1}, 0, \dots, 0) \leftarrow$ Critical pts

b) Hessian: similar — same calculation

$$\Rightarrow \quad \mu(x_j) = j$$

c) $\partial = 0$ over $\mathbb{Z}_2$: exactly two trajectories from x_{j+1} to x_j (for the round metric)





on S^2

$\downarrow$

$\mathbb{RP}^2$

$$\Rightarrow \quad H_k(\mathbb{RP}^n; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

Rmk. Over $\mathbb{Z}$, harder — orientations

Other applications:

- **General** : Poincaré duality

$$CM_*(f) = CM_{m-*}(f)^* \qquad m = \dim$$

$$\Rightarrow \quad H_*(M) = H_{m-*}(M)^* = H_{m-*}(M)$$

- Diff Topology: · Handle body decomposition
  · 3-manifolds
  · classification of surfaces

The Poincaré conj & classification of manifolds

- More homology calculations
- Loop spaces & closed geodesics

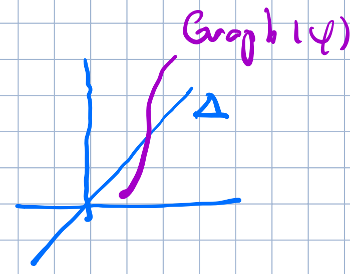
§8 Arnold's conjecture and all that

Lecture 16
11/18-2021

Some definitions - non-degeneracy

$\varphi : M \circlearrowleft$

Def • $x \in \mathrm{Fix}(\varphi)$ is non-deg
if $\det(D\varphi_x - I) \neq 0$
$T_xM \circlearrowleft$



No eigenvectors with eigenvalue 1.

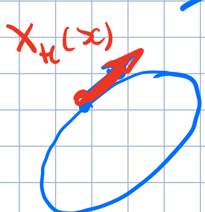
$\iff$ $\mathrm{Graph}(\varphi) \pitchfork_{\text{at } x} \Delta \subset M \times M$

• φ is non-deg if all $x \in \mathrm{Fix}(\varphi)$ are non-deg

Rmk: In any reasonable class (all smooth, symplectic, vol. pres, Hamiltonian) non-deg is a generic condition

Rmk: Warning $H: M \to \mathbb{R}$ symplectic

$\varphi^t(x) \subset \{H = c\}$ • T-periodic
• non-constant



$\Rightarrow$ x is deg fixed pt for φ^T
$X_H(x)$ is an eigenvector with eigenvalue 1.

Arnold's conjecture

Conj (AC, I) $\varphi_H : (M, \omega) \circlearrowleft$ Ham

closed

our focus

$$\Rightarrow \quad \# \operatorname{Fix}(\varphi) \geq \begin{cases} \sum \dim H_i(M), & \varphi_H \text{ non-deg} \\ CL(M)+1, & \text{in general} \end{cases}$$

As in morse theory

Next page

same as for critical pts of smooth functions

Rmk
- One of the most important questions in sympl. geometry $\Rightarrow$ Floer theory
- Motivation: Poincaré's work in celestial mechanics (non-trivial path)
- Arnold stated it somewhat differently
- - Non-deg case is pretty much proved by now: Floer - .... - Fukaya et al
  - Deg case is still open for many manifolds
    $S^2 \times S^2 \times S^2 \leftarrow$ with areas $(1, \sqrt{2}, \sqrt{3})$ ...

A word on Lusternik-Schnirelmann theory

Cup-length of M : $cL(M)$ over $\mathbb{F}$

$$cL(M) = \max\{k \mid \exists\, \alpha_1, \dots, \alpha_k \in H^{*>0}(M) : \alpha_1 \cup \dots \cup \alpha_k \neq 0\}$$

Note: $cL(M) \leq \dim M$

Ex
- $cL(S^n) = 1$
- $cL(\mathbb{C}P^n) = n$
- $cL(\mathbb{R}P^n) = n$ over $\mathbb{F}_2 = \mathbb{Z}/_2$
- $cL(\mathbb{T}^n) = n$
- M closed symplectic $\Rightarrow cL(M) \geq \frac{1}{2}\dim M$

Thm (LS)

$f : \underset{\text{closed}}{M} \longrightarrow \mathbb{R}$ smooth

$\Rightarrow \quad |\,\mathrm{Crit}(f)\,| \geq cL(M) + 1$

Some other variants: Lagrangian intersections

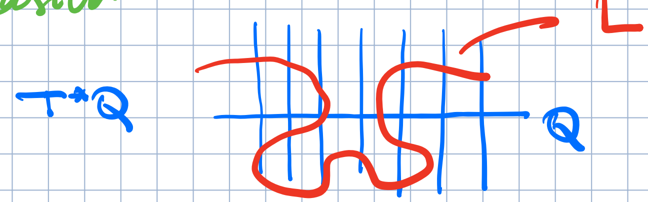
(Thm of Laudenbach - Sikorav) — Also conj by A.

$\varphi: T^*Q \circlearrowleft$ Hamiltonian diffeo

$L = \varphi(Q)$ zero section

$\Rightarrow \quad \# L \cap Q \geq \begin{cases} \sum \dim H_i(L) : & \overbrace{L \pitchfork Q}^{\text{non-deg}} \\ cl(L)+1, & \text{general} \end{cases}$

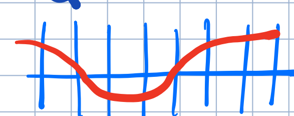
much easier



Rmk — Ex $\quad f: Q \longrightarrow \mathbb{R}$

f Morse $\iff \underbrace{\text{graph}(df)}_{\text{Lagn } L} \pitchfork \underset{\text{zero section}}{Q} \subset T^*Q$

Crit pts $\longleftrightarrow$ graph$(df) \cap Q$



$$L = \text{graph}(df) \subset T^*Q$$

$L = \varphi(Q) \; : \quad \varphi(q,p) = (q, p + df)$

$\qquad\qquad\qquad\quad H(q,p) = f(q)$

• Prove for $Q = S^1$ — explain

Area "swept" by L

Digression

M, $\omega = d\lambda$ exact symplectic manifold

φ_t = symplectic isotopy

$\varphi_t^* \omega = \omega$, $\varphi_0 = id$

X_t = generating v.f. $\alpha_t = i_{X_t}\omega$ closed

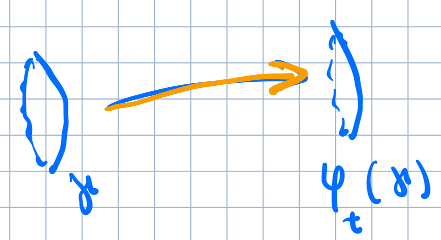
Q How do we tell if φ_t is Ham, i.e. α_t is exact?

Rmk α, $d\alpha = 0$

α is exact $\iff \int_\gamma \alpha = 0 \quad \forall$ loops γ

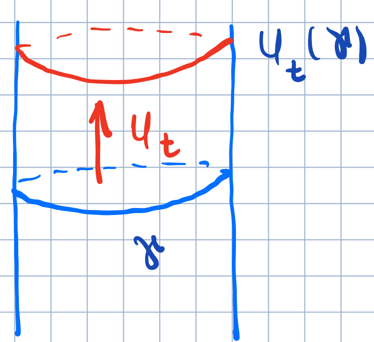
Lemma φ_t is Ham

$\iff \forall$ loop γ $\forall t$ $\quad \int_{\varphi_t(\gamma)} \lambda = \int_\gamma \lambda$



Ex

$M = T^*S^1$



Not Ham

Area above = Area below

Ham

Pf

$\frac{d}{dt}$: $\displaystyle\int_{\varphi_t(\gamma)} \lambda = \int_\gamma \varphi_t^* \lambda$

$$\frac{d}{dt}\int_{\varphi_t(\gamma)} \lambda = \int_\gamma \frac{d}{dt}\varphi_t^*\lambda = \int_\gamma \varphi_t^* L_{X_t}\lambda$$

$$= \int_{\varphi_t(\gamma)} i_{X_t} d\lambda + \cancelto{0}{\int_{\varphi_t(\gamma)} d\,\lambda(X_t)}$$

$$= \int_{\varphi_t(\gamma)} \alpha_t$$

$\varphi_t(\gamma)$ ← also ranges through all loops

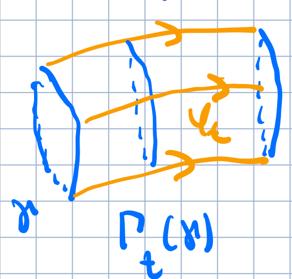
$$\varphi_t \text{ Ham} \Leftrightarrow \alpha_t \text{ exact} \Leftrightarrow \int_{\varphi_t(\gamma)} \alpha_t = 0 \quad \forall \gamma \ \forall t$$

$$\Leftrightarrow \int_{\varphi_t(\gamma)} \lambda = \text{const}(\gamma)$$

$\triangleleft$

Rmk A variant

(M,ω) ← not exact



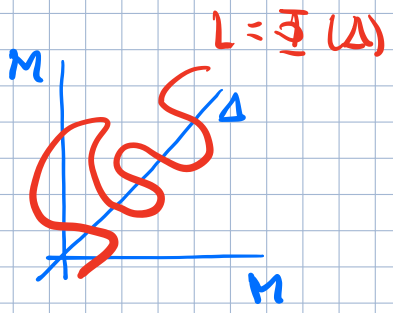
$$\varphi_t \text{ is Ham} \Leftrightarrow \underbrace{\int_{\Gamma_t(\gamma)} \omega}_{\text{Flux}} = 0 \quad \forall t \ \forall \gamma$$

Lecture 17
11/23-2021

$\underline{AC, II}$ $\Phi: \underset{(\omega, -\omega)}{M \times M} \circlearrowleft$ Ham. diffeo

$L = \Phi(\Delta) \leftarrow$ Lagr

$\# L \cap \Delta \geq \ldots$

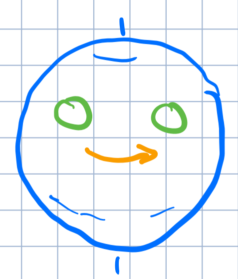


$AC, II \Rightarrow AC, I$:
$\Phi(x,y) = (x, \varphi(y))$

$\underline{\text{Bottom line:}}$ in many instances one can expect L_0 & $\varphi(L_0)$ have many intersections

$\underline{\text{But not always:}}$

$\underline{\text{Ex}}$



$L_0 \subset S^2$ small circle (Lagrangian)
φ = rotation
$\varphi(L_0) \cap L_0 = \emptyset$

$\underline{\text{Back to } AC, I}$

$\underline{\text{Ex}}$ $H: M \to \mathbb{R}$ autonomous
$\text{Crit}(H) \subset \text{Fix}(\varphi)$

$\underline{\text{Note:}}$ $x \in \text{Crit}(H)$ can be non-deg as a crit pt but deg as a fixed pt
E.g. $H(p,q) = \pi(p^2+q^2)$
But $\varphi_H = id$

Further evidence

Thm (Weinstein, 70s)

$\varphi : M \circlearrowleft$ C^1-close to id, Ham

$\Rightarrow$ $\text{Fix}(\varphi) \geq \ldots$

two versions

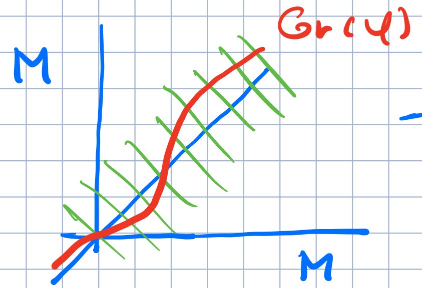
just $\varphi = \varphi^1_H$ or all φ^t_H, $0 \leq t \leq 1$

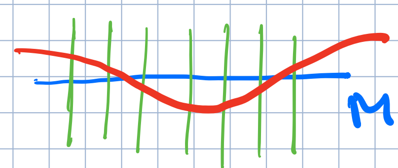
Focus on this

Pf

- $Gr(\varphi) \subset M \times M$ is C^1-close to Δ

Weinstein tub nbd thm

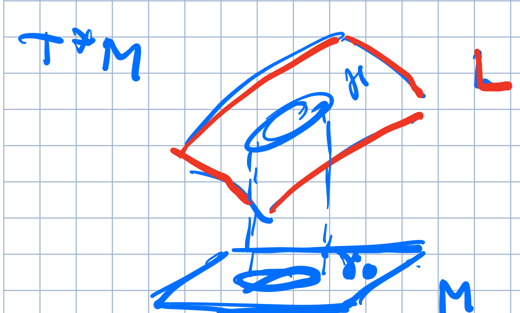




image$(Gr(\varphi)) \underset{C^1}{\approx}$ zero section $\Rightarrow$ a section

Lagrangian $\Leftrightarrow$ α is closed

- φ Hamiltonian $\Rightarrow$ $\underbrace{\alpha \text{ is exact}}$



$\Leftrightarrow \int_{\gamma_0} \alpha = 0$ $\gamma_0 \leftarrow$ any loop in M

$\Leftrightarrow \int_{\gamma} \lambda = p\,dq$ $\gamma \leftarrow$ any loop in L

- But $\int_\gamma \lambda = \mathrm{Flux}(\varphi)(\gamma_0) = 0$

Nuance: ◦ easy when $\varphi_H^t \overset{C^1}{\simeq} id$

- Need work when only $\varphi \overset{C^1}{\simeq} id$
(⇐ Flux conjecture)

$\Rightarrow \quad \alpha = df$

$L \cap M \longleftrightarrow \mathrm{Crit}(f)$

Apply Morse Theory or LS Theory

$\triangleleft$

Toward the Pf of AC:

Lecture 18

11/30 - 2021

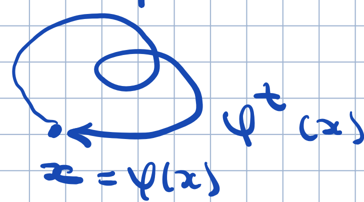
The least Action Principle

* M exact $\omega = d\lambda$

E.g. $\mathbb{R}^{2n}$ or T^*M

$H: M \times S^1 \longrightarrow \mathbb{R}$, $\varphi = \varphi^1_H$

$\mathrm{Fix}(\varphi) \longleftrightarrow$ 1-periodic orbits of φ^t_H



$\Lambda = C^\infty(S^1, M)$

$A_H: \Lambda \longrightarrow \mathbb{R}$ action functional

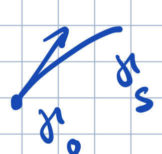
$$A_H(\gamma) = -\int_\gamma \lambda + \int_0^1 H_t(\gamma_t)\,dt$$

$$\boxed{\begin{array}{l}\underline{\text{Thm}}\ (\underline{\text{Least Action Principle}}) \\ \underbrace{\mathrm{Crit}(A_H)} = \text{1-periodic orbits of } \varphi^t_H\end{array}}$$

$\gamma_0 \in \mathrm{Crit}(A_H)$: $\forall\ \gamma_s \leftarrow$ var. of γ_0

$$\frac{d}{ds} A_H(\gamma_s)\Big|_{s=0} = 0$$

Discuss in more detail

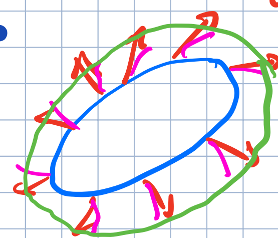


Analogy with finite dim

Rmk $T_\gamma \Lambda = \{\text{vector fields } \xi \text{ along } \gamma\}$

$$= \{ \xi : S^1 \xrightarrow{C^\infty} TM \mid \xi(t) \in T_{\gamma(t)} M \}$$

$$L_\xi A_H(\gamma) = \frac{d}{ds} A_H(\gamma_s) \Big|_{s=0}$$



Ex: independent of γ_s when

$$\frac{\partial \gamma_s(t)}{\partial s}\Big|_{s=0} = \xi(t)$$

Note $\forall \xi \ \exists \gamma_s$ s.t.

Take $\gamma_s(t) = \exp_{\gamma(t)}(s\,\xi(t))$

Pf of LHP

Need to show:

$$\gamma \in \mathrm{Crit}(A_H) \iff \dot\gamma(t) = X_{H_t}(\gamma(t))$$

$$\Updownarrow$$

$$L_\xi A_H(\gamma) = 0 \quad \forall \xi$$

$$\Updownarrow$$

$$\frac{d}{ds} A_H(\gamma_s)\Big|_{s=0} = 0 \quad \forall \gamma_s$$

$$A_H(\gamma_s) = -\int_{\gamma_s} \lambda + \int_0^1 H_t(\gamma_s(t))\,dt$$

$$\frac{d}{ds} A_H(\gamma_s)\Big|_{s=0} = \underbrace{-\frac{d}{ds}\int_{\gamma_s} \lambda\Big|_{s=0}}_{2} + \underbrace{\frac{d}{ds}\int_0^1 H_t(\gamma_s(t))\,dt}_{1}\Big|_{s=0}$$

1) $$\frac{d}{ds}\int_0^1 H(\gamma_s)\,dt\Big|_{s=0} = \int_0^1 \frac{d}{ds} H_t(\gamma_s(t))\,dt\Big|_{s=0}$$

$$= \int_0^1 L_{\xi(t)} H_t \,dt$$

$$= \int_0^1 dH_t(\xi(t))\,dt$$

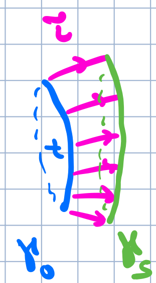
$$i_{X_{H_t}}\omega = -dH_t$$

supprem

$$= \int_0^1 \omega(\xi(t), X)\,dt$$

2) $$\frac{d}{ds}\int_{\gamma_s}\lambda\,\Big|_{s=0} = \lim_{s\to 0}\frac{1}{s}\left(\int_{\gamma_s}-\int_{\gamma_0}\right)\lambda$$

$$= \lim_{s\to 0}\frac{1}{s}\int_{u}\omega$$



$$u: \begin{array}{ccc} S^1\times[0,s] & \longrightarrow & M \\ t\quad\ \tau & \longmapsto & \gamma_\tau(t) \end{array}$$

$$= \lim_{s\to 0}\frac{1}{s}\int_0^1\int_0^s \omega\left(\frac{\partial u}{\partial \tau},\frac{\partial u}{\partial t}\right)\,ds\,dt$$

$$= \int_0^1 \lim_{s\to 0}\frac{1}{s}\int_0^s \omega\Big(\underbrace{\frac{\partial u}{\partial \tau}}_{\xi(t)},\underbrace{\frac{\partial u}{\partial t}}_{\dot\gamma(t)}\Big)\,d\tau\,dt$$

$$= \int_0^1 \omega(\xi(t),\dot\gamma(t))\,dt$$

$\Rightarrow \quad \gamma\in \mathrm{Crit}(A_H)$

$$\Leftrightarrow \quad -\int_0^1\omega(\xi(t),\dot\gamma(t))\,dt+\int_0^1\omega(\xi(t),X)\,dt=0$$

$$\forall\, \xi$$

$$\Longleftrightarrow \quad \int_0^1 \omega(\xi \quad , -\dot{\gamma} + X \;) \, dt = 0 \quad \underline{\underline{\forall \, \xi}}$$

Non-degeneracy of ξ $\overset{(Ex)}{\Longrightarrow}$

$$\Longleftrightarrow \quad -\dot{\gamma}(t) + X_{H_t}(\gamma(t)) = 0$$

$$\Longleftrightarrow \quad \underbrace{\dot{\gamma}(t) = X_{H_t}(\gamma(t))}_{\gamma \text{ is a 1-periodic orbit of } H}$$

$\triangleleft$

* M closed, $\omega|_{\pi_2} = 0$: $\int_{S^2 \to M} \omega = 0$: E.g. T^{2n}, $\Sigma_{g\geq 1}$

But not $\mathbb{C}P^n$

$\Lambda_0 =$ contractible loops

$$A_H(\gamma) = \int_{D^2 \to M} \omega + \int_0^1 H_t(\gamma(t))\,dt$$

well-defined : $\int \omega = 0$

$$A_H : \Lambda_0 \to \mathbb{R}$$

contr. 1-periodic orbits of φ_H^t $\subset$ $\mathrm{Fix}(\varphi)$

Thm (LAP)

$\mathrm{Crit}(A_H) =$ Contr. 1-periodic orbits of φ_H^t

Pf Similar. Essentially can assume ω is exact near γ.

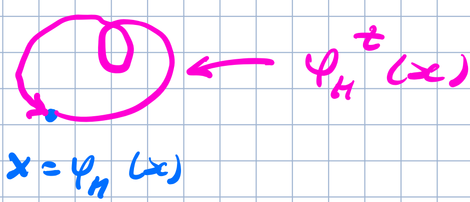
Ham Floer Theory: Outline

Lecture 19
12/02-2021

Assume: $\pi_2(M) = 0$ or at least $\omega|_{\pi_2} = 0$

$\Rightarrow$ $A_H : \Lambda \longrightarrow \mathbb{R}$ is well defined
contractible loops

Recall: $Fix(\varphi_H) \longleftrightarrow Crit(A_H)$



$\Rightarrow$

To prove AC, I: $|Fix(\varphi)| \geq \ldots$
suffices to show that $|Crit(A_H)| \geq \ldots$

- **Assume:** φ_H is non-deg $\Longleftrightarrow$ A_H is **Morse**

Idea: Do Morse theory for A_H on Λ for an L^2-metric

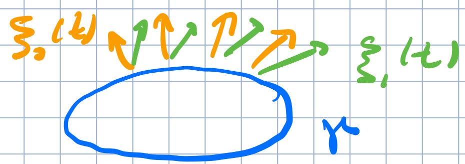
Does not literally work
But let's try

Need a Riemannian metric on Λ

- $\langle , \rangle$ a R. metric on M

  $T_\gamma \Lambda =$ v.f. ξ along γ

  $$\langle \xi_0, \xi_1 \rangle_{L^2} = \int_0^1 \langle \xi_0(t), \xi_1(t) \rangle \, dt$$



Need to chose $\langle , \rangle$ on M carefully

- → Recall J on (V, ω) is compatible with ω if $\langle X, Y \rangle = \omega(X, JY)$ is an inner product
  - → Also call $\langle , \rangle$ compatible
    J compatible $\longleftrightarrow$ $\langle , \rangle$ compatible

→ Exist and form a contractible set (E.g. McDuff-Salamon)

⇒ **Lemma** $\exists$ $J : TM \to TM$, $J^2 = -I$ which is compatible with ω at every pt:

$$\langle \cdot, \cdot \rangle_{\mathbb{C}} = \underbrace{\omega(\cdot, J\cdot)}_{\langle , \rangle} + i\omega(\cdot, \cdot)$$

$\langle , \rangle$ ← Take this

is a Hermitian at every pt

Discuss the pf

Rmk $\omega(\underset{\smile}{X}, JY) = \langle \underset{\smile}{X}, Y \rangle$

$\Rightarrow \quad \omega(\underset{\smile}{X}, Y) = \langle \underset{\smile}{X}, -JY \rangle$

$\Rightarrow \quad X_H = -J\nabla H \iff \nabla H = JX_H$

• The gradient of A_H

The Pf of LAP:

$$\underbrace{dA_H(\xi)}_{L_\xi A_H(\gamma)} = \int_0^1 \omega(\xi, -\dot{\gamma} + X_H)\, dt$$

$$= \int_0^1 \langle \xi, -J(-\dot{\gamma} + X_H)\rangle \, dt$$

Recall: ∇f : $\langle \nabla f, \xi \rangle = df(\xi)$

$$= \langle +J\dot{\gamma} \underbrace{-JX_H}_{+\nabla H}, \xi \rangle_{L^2}$$

Ex

$$\Rightarrow \boxed{\nabla A_H(\gamma) = J\dot{\gamma} + \nabla H}$$

The anti-grad flow equation

- $u : \underset{s}{\mathbb{R}} \longrightarrow \Lambda =$ contr. loops in M $\{S^1 \longrightarrow M\}$

$$\underbrace{\frac{du}{ds}}_{\substack{\cap \\ T_{u(s)}\Lambda}} = -\nabla A_H(u(s))$$

- $u : \underset{s}{\mathbb{R}} \times \underset{t}{S^1} \longrightarrow M$

$$\boxed{\begin{gathered} \frac{\partial u}{\partial s} = -J\frac{\partial u}{\partial t} - \nabla H(u) \\ \text{or} \\ \underbrace{\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t}}_{\text{Cauchy-Riemann}} = -\nabla H(u) \end{gathered}}$$

Floer equation

First order elliptic equation

Ex. 1 H autonomous
$u : \mathbb{R} \to M$ ind of t
anti-grad trajectory solves FE

Ex. 1 $H \equiv 0$, J true complex str

$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = \bar\partial u$: u sol of FE $\Leftrightarrow u : \mathbb{R}\times S^1 \xrightarrow{hol} M$

Difficulties

- the flow does not exist $\Leftarrow$ Ex 2

  $H = 0 \Rightarrow u$ hol

  $\Rightarrow u(s)$ is real analytic

  $\Rightarrow$ no initial value sol for most initial values

- not bounded from below
- infinite index and coindex

Ex. $M = \mathbb{R}^2 = \mathbb{C}$; $H = 0$

$$\Lambda := \{ S^1 \xrightarrow{z} \mathbb{C} \}$$

$$A_H(z) = -\int_z p\,dq = A(z)$$

Fourier expansion

$$z = \sum z_k e^{2\pi i k t}$$

Ex. $A(z) = \underbrace{-2\pi \sum_{k=-\infty}^{+\infty} k |z_k|^2}$

∞ # of pos & neg squares

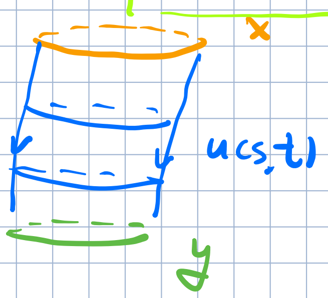
Solution

Even though the flow is not defined the mod spaces $\mathcal{M}(x,y)$ are and have reasonable properties

$$\mathcal{M}(x,y) = \left\{ u : \mathbb{R} \to \Lambda \;\middle|\; u \begin{cases} \to x & s \to -\infty \\ \to y & s \to +\infty \end{cases} \right\}$$

Crit

$$= \left\{ u : \mathbb{R} \times S^1 \to M \text{ s.t. } \boxed{u(s,t) \begin{cases} \to x(t) & s \to -\infty \\ \to y(t) & s \to +\infty \end{cases}} \right\}$$



$\bar{\partial} u = -\nabla H(u)$

Asymptotic boundary value problem for a 1st order elliptic equation

$\Rightarrow$ solutions

$\Rightarrow$ Can do Morse Theory

Lots of technical difficulties ...

— FIN —