§8 Arnold's conjecture and all that

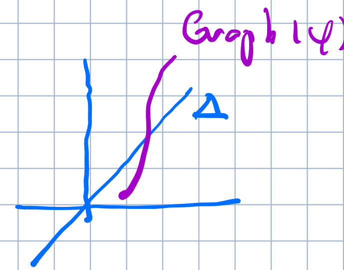
Lecture 16
11/28-2021

Some definitions - non-degeneracy

$\varphi : M \circlearrowleft$

Def • $x \in \mathrm{Fix}(\varphi)$ is non-deg
if $\det(D\varphi_x - I) \neq 0$
$T_x M \circlearrowleft$

No eigenvectors with eigenvalue 1.



$\Leftrightarrow$ $\mathrm{Graph}(\varphi) \pitchfork_{\text{at } x} \Delta \subset M \times M$

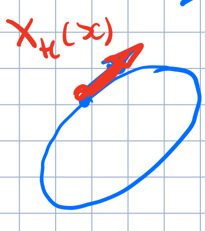
• φ is non-deg if all $x \in \mathrm{Fix}(\varphi)$ are non-deg

Rmk: In any reasonable class (all smooth, symplectic, vol. pres, Hamiltonian) non-deg is a generic condition

Rmk: Warning $H: M \to \mathbb{R}$ symplectic

$\varphi^t(x) \subset \{H = c\}$ • T-periodic
• non-constant

$\Rightarrow$ x is deg fixed pt for φ^T
$X_H(x)$ is an eigenvector with eigenvalue 1.



Arnold's conjecture

Conj (AC, I) $\varphi_H : (M, \omega) \circlearrowleft$ Ham

closed

our focus

$$\Rightarrow \quad \# \operatorname{Fix}(\varphi) \geq \begin{cases} \sum \dim H_i(M), & \varphi_H \text{ non-deg} \\ CL(M)+1, & \text{in general} \end{cases}$$

As in morse theory

Next page

same as for critical pts of smooth functions

Rmk
- One of the most important questions in sympl. geometry $\Rightarrow$ Floer theory
- Motivation: Poincaré's work in celestial mechanics (non-trivial paths)
- Arnold stated it somewhat differently
- - Non-deg case is pretty much proved by now: Floer - .... - Fukaya et al
  - Deg case is still open for many manifolds
    $S^2 \times S^2 \times S^2 \leftarrow$ with areas $(1, \sqrt{2}, \sqrt{3})$ ...

A word on Lusternik-Schnirelmann theory

Cup-length of M : $cL(M)$ over $\mathbb{F}$

$$cL(M) = \max\{k \mid \exists\, \alpha_1, \dots, \alpha_k \in H^{*>0}(M) : \alpha_1 \cup \dots \cup \alpha_k \neq 0\}$$

Note: $cL(M) \leq \dim M$

<u>Ex</u>

- $cL(S^n) = 1$
- $cL(\mathbb{C}P^n) = n$
- $cL(\mathbb{R}P^n) = n$ over $\mathbb{F}_2 = \mathbb{Z}/_2$
- $cL(\mathbb{T}^n) = n$
- M closed symplectic $\Rightarrow cL(M) \geq \frac{1}{2}\dim M$

<u>Thm</u> (LS)

$f : \underset{\text{closed}}{M} \longrightarrow \mathbb{R}$ smooth

$\Rightarrow |\mathrm{Crit}(f)| \geq cL(M) + 1$

Some other variants: Lagrangian intersections

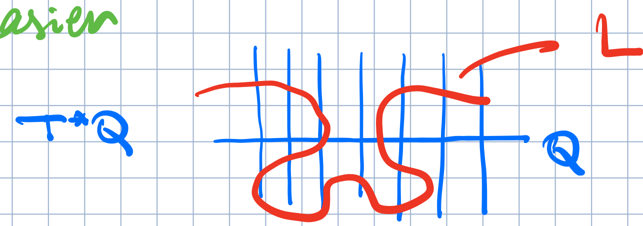
(Thm of Laudenbach - Sikorav) – Also conj by A.

$\varphi : T^*Q \circlearrowleft$ Hamiltonian diffeo

$L = \varphi(Q)$ zero section

$\Rightarrow \# L \cap Q \geq \begin{cases} \sum \dim H_i(L) : & \overset{\text{non-deg}}{L \pitchfork Q} \\ cl(L)+1, & \text{general} \end{cases}$

much easier



Rmk – Ex $f : Q \longrightarrow \mathbb{R}$

f Morse $\Longleftrightarrow$ $\underbrace{\text{graph}(df)}_{\text{Lagn } L} \pitchfork \underset{\text{zero section}}{Q} \subset T^*Q$

Crit pts $\longleftrightarrow$ graph$(df) \cap Q$

$L = \text{graph}(df) \subset T^*Q$

$L = \varphi(Q) \quad : \quad \varphi(q,p) = (q, p + df)$

$H(q,p) = f(q)$

- Prove for $Q = S^1$ — explain

Area "swept" by L

Digression

$M, \ \omega = d\lambda$ exact symplectic manifold

φ_t = symplectic isotopy

$\varphi_t^* \omega = \omega \quad , \ \varphi_0 = id$

X_t = generating v.f. $\quad \alpha_t = i_{X_t}\omega$ closed

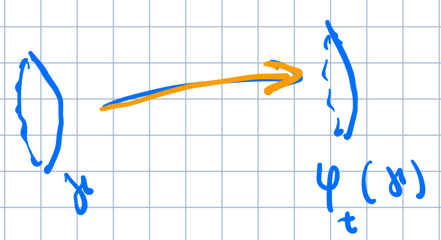
Q How do we tell if φ_t is Ham, i.e. α_t is exact?

Rmk $\alpha, \ d\alpha = 0$

α is exact $\iff \displaystyle\int_\gamma \alpha = 0 \ \ \forall$ loops γ

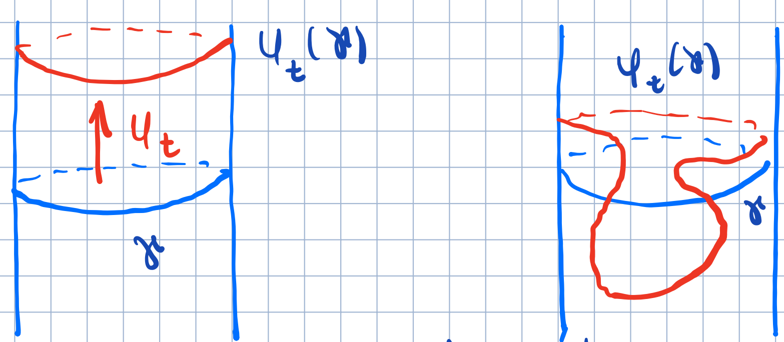
Lemma φ_t is Ham

$\iff \forall$ loop $\gamma \ \ \forall t \qquad \displaystyle\int_{\varphi_t(\gamma)} \lambda = \int_\gamma \lambda$



Ex

$M = T^*S^1$



Not Ham $\qquad\qquad$ Area above = Area below

Ham

Pf

$\frac{d}{dt}:$ $$\int_{\varphi_t(\gamma)} \lambda = \int_\gamma \varphi_t^* \lambda$$

$$\frac{d}{dt}\int_{\varphi_t(\gamma)} \lambda = \int_\gamma \frac{d}{dt}\varphi_t^* \lambda = \int_\gamma \varphi_t^* L_{X_t} \lambda$$

$$= \int_{\varphi_t(\gamma)} i_{X_t} d\lambda + \int_{\varphi_t(\gamma)} \cancelto{0}{d\lambda(X_t)}$$

$$= \int_{\varphi_t(\gamma)} \alpha_t$$ $\leftarrow$ also ranges through all loops

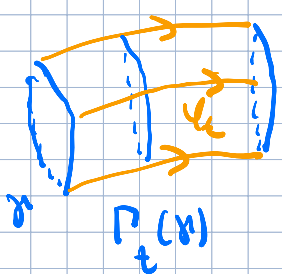
φ_t Ham $\Leftrightarrow$ α_t exact $\Leftrightarrow$ $\int_{\varphi_t(\gamma)} \alpha_t = 0$ $\forall \gamma$ $\forall t$

$\Leftrightarrow$ $\int_{\varphi_t(\gamma)} \lambda = \text{const}(\gamma)$

$\triangleleft$

Rmk A variant

(M,ω) $\leftarrow$ not exact



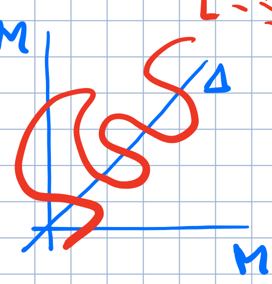
φ_t is Ham $\Leftrightarrow$ $\underbrace{\int_{\Gamma_t(\gamma)} \omega}_{\text{Flux}} = 0$ $\forall t$ $\forall \gamma$

Lecture 17 11/23-2021

AC, II $\quad \Phi : \underset{(\omega, -\omega)}{M \times M} \circlearrowleft$ Ham. diffeo

$$L = \Phi(\Delta) \leftarrow \text{Lagr}$$

$$\# L \cap \Delta \geq \ldots\ldots$$



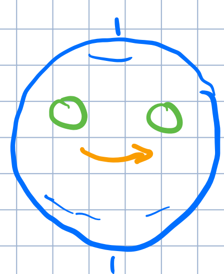
AC, II $\Rightarrow$ AC, I:

$$\Phi(x, y) = (x, \varphi(y))$$

Bottom line: in many instances one can expect L_0 & $\varphi(L_0)$ have many intersections

But not always:

Ex



$L_0 \subset S^2$ small circle (Lagrangian)

φ = rotation

$\varphi(L_0) \cap L_0 = \emptyset$

Back to AC, I

Ex $\quad H : M \to \mathbb{R}$ autonomous

$\mathrm{Crit}(H) \subset \mathrm{Fix}(\varphi)$

Note: $x \in \mathrm{Crit}(H)$ can be non-deg as a crit pt but deg as a fixed pt

E.g. $H(p, q) = \pi(p^2 + q^2)$ 

But $\varphi_H = \mathrm{id}$

Further evidence

Thm (Weinstein, 70s)

$\varphi : M \circlearrowleft$ C¹-close to id, Ham

$\Rightarrow$ Fix$(\varphi) \geq \ldots$

two versions

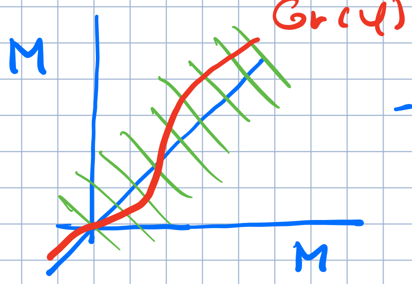
just $\varphi = \varphi^1_H$ or all φ^t_H, $0 \leq t \leq 1$

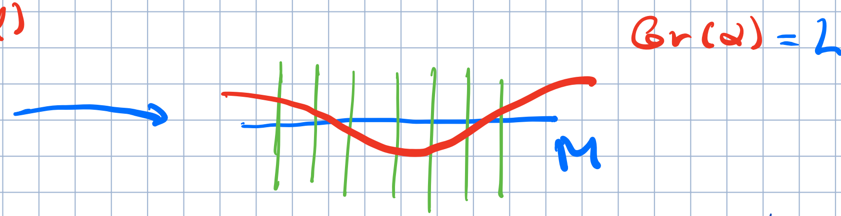
Focus on this

Pf

- $Gr(\varphi) \subset M \times M$ is C¹-close to Δ

Weinstein tub nbd thm





$M \times M$

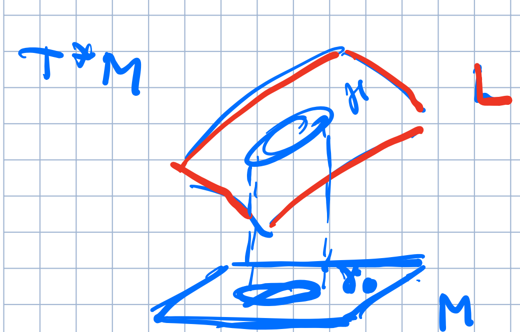
image$(Gr(\varphi)) \underset{C^1}{\approx}$ zero section

$\Rightarrow$ a section

T^*M

Lagrangian $\Leftrightarrow$ α is closed

- φ Hamiltonian $\Rightarrow$ $\underbrace{\alpha \text{ is exact}}$



$\Leftrightarrow \int_{\gamma_0} \alpha = 0$

$\gamma_0 \leftarrow$ any loop in M

$\Leftrightarrow \int_{\gamma} \lambda = p\,dq$

$\gamma \leftarrow$ any loop in L

- But $\int_\gamma \lambda = \mathrm{Flux}(\varphi)(\gamma_0) = 0$

Nuance: ∘ easy when $\varphi_H^t \overset{C^1}{\simeq} id$

• Need work when only $\varphi \overset{C^1}{\simeq} id$
($\Leftarrow$ Flux conjecture)

$\Rightarrow \quad \alpha = df$

$L \cap M \longleftrightarrow \mathrm{Crit}(f)$

Apply Morse Theory or LS Theory $\triangleleft$

Toward the Pf of AC:

The least Action Principle

Lecture 18

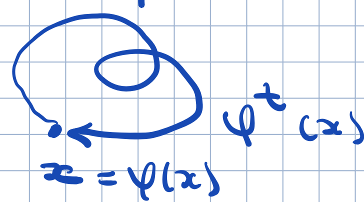
11/30 - 2021

* M exact $\omega = d\lambda$

E.g. $\mathbb{R}^{2n}$ or T^*M

$H: M \times S^1 \longrightarrow \mathbb{R}$, $\varphi = \varphi^1_H$

$\mathrm{Fix}(\varphi) \longleftrightarrow$ 1-periodic orbits of φ^t_H



$\Lambda = C^\infty(S^1, M)$

$A_H : \Lambda \longrightarrow \mathbb{R}$ action functional

$$A_H(\gamma) = -\int_\gamma \lambda + \int_0^1 H_t(\gamma_t)\,dt$$

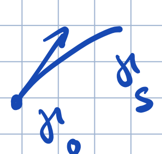
Thm (Least Action Principle)

$$\boxed{\underbrace{\mathrm{Crit}(A_H)} = \text{1-periodic orbits of } \varphi^t_H}$$

$\gamma_0 \in \mathrm{Crit}(A_H)$: $\forall\ \gamma_s \leftarrow$ var. of γ_0

$$\frac{d}{ds} A_H(\gamma_s)\Big|_{s=0} = 0$$

Discuss in more detail



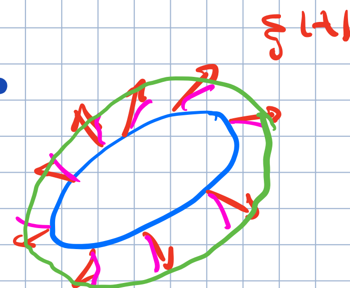
Analogy with finite dim

(83)

Rmk $T_\gamma \Lambda = \{\text{vector fields } \xi \text{ along } \gamma\}$

$= \{\xi : S^1 \xrightarrow{C^\infty} TM \mid \xi(t) \in T_{\gamma(t)} M\}$

$$L_\xi A_H(\gamma) = \frac{d}{ds} A_H(\gamma_s) \Big|_{s=0}$$



Ex: independent of γ_s when

$$\frac{\partial \gamma_s(t)}{\partial s}\Big|_{s=0} = \xi(t)$$

Note $\forall \, \xi \ \exists \ \gamma_s$ s.t.

Take $\gamma_s(t) = \exp_{\gamma(t)}(s\,\xi(t))$

Pf of LHP

Need to show:

$$\gamma \in \mathrm{Crit}(A_H) \iff \dot\gamma(t) = X_{H_t}(\gamma(t))$$

$$\Updownarrow$$

$$L_\xi A_H(\gamma) = 0 \quad \forall \xi$$

$$\Updownarrow$$

$$\frac{d}{ds} A_H(\gamma_s)\Big|_{s=0} = 0 \quad \forall \gamma_s$$

$$A_H(\gamma_s) = -\int_{\gamma_s} \lambda + \int_0^1 H_t(\gamma_s(t))\,dt$$

$$\frac{d}{ds} A_H(\gamma_s)\Big|_{s=0} = \underbrace{-\frac{d}{ds}\int_{\gamma_s} \lambda \Big|_{s=0}}_{2} + \underbrace{\frac{d}{ds}\int_0^1 H_t(\gamma_s(t))\,dt}_{1}\Big|_{s=0}$$

1) $$\frac{d}{ds}\int_0^1 H(\gamma_s)\,dt\Big|_{s=0} = \int_0^1 \frac{d}{ds} H_t(\gamma_s(t))\,dt\Big|_{s=0}$$

$$= \int_0^1 L_{\xi(t)} H_t \,dt$$

$$= \int_0^1 dH_t(\xi(t))\,dt$$

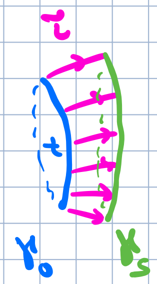
$i_{X_{H_t}}\omega = -dH_t$

suppress

$$= \int_0^1 \omega(\xi(t), X)\,dt$$

2) $$\frac{d}{ds}\int_{\gamma_s}\lambda\Big|_{s=0} = \lim_{s\to 0}\frac{1}{s}\left(\int_{\gamma_s}-\int_{\gamma_0}\right)\lambda$$

$$= \lim_{s\to 0}\frac{1}{s}\int_u \omega$$



$$u : \ \begin{array}{ccc} S^1\times[0,s] & \longrightarrow & M \\ t \quad \tau & \longmapsto & \gamma_\tau(t) \end{array}$$

$$= \lim_{s\to 0}\frac{1}{s}\int_0^1\int_0^s \omega\left(\frac{\partial u}{\partial\tau},\frac{\partial u}{\partial t}\right)ds\,dt$$

$$= \int_0^1 \lim_{s\to 0}\frac{1}{s}\int_0^s \omega\Big(\underbrace{\frac{\partial u}{\partial\tau}}_{\xi(t)},\underbrace{\frac{\partial u}{\partial t}}_{\dot\gamma(t)}\Big)\,d\tau\,dt$$

$$= \int_0^1 \omega(\xi(t),\dot\gamma(t))\,dt$$

$\Rightarrow \quad \gamma\in \mathrm{Crit}(A_H)$

$\Leftrightarrow \quad -\int_0^1\omega(\xi(t),\dot\gamma(t))dt+\int_0^1\omega(\xi(t),X)\,dt=0$

$\forall\,\xi$

$$\Longleftrightarrow \quad \int_0^1 \omega(\xi \quad , -\dot{\gamma} + X \,)\, dt = 0 \quad \forall\, \xi$$

Non-degeneracy of ξ $\overset{(Ex)}{\Longrightarrow}$

$$\Longleftrightarrow \quad -\dot{\gamma}(t) + X_{H_t}(\gamma(t)) = 0$$

$$\Longleftrightarrow \quad \underbrace{\dot{\gamma}(t) = X_{H_t}(\gamma(t))}_{\gamma \text{ is a 1-periodic orbit of } H}$$

$\triangleleft$

* M closed, $\omega|_{\pi_2} = 0$: $\int_{S^2 \to M} \omega = 0$: E.g. T^{2n}, $\Sigma_{g \geq 1}$

But not $\mathbb{C}P^n$

$\Lambda_0 =$ contractible loops

$$A_H(\gamma) = \int_{\substack{\gamma \\ D^2 \to M}} \omega + \int_0^1 H_t(\gamma(t))\,dt$$

well-defined : $\int_{S^2} \omega = 0$

$$A_H : \Lambda_0 \to \mathbb{R}$$

contr. 1-periodic orbits of φ_H^t $\subset \mathrm{Fix}(\varphi)$

Thm (LAP)

$$\boxed{\mathrm{Crit}(A_H) = \text{Contr. 1-periodic orbits of } \varphi_H^t}$$

Pf Similar. Essentially can assume ω is exact near γ.

Ham Floer Theory: Outline

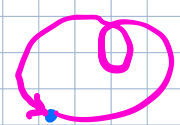
Lecture 19
12/02-2021

Assume: $\pi_2(M) = 0$ or at least $\omega|_{\pi_2} = 0$

$\Rightarrow$ $\mathcal{A}_H : \Lambda \longrightarrow \mathbb{R}$ is well defined

contractible loops

Recall: $Fix(\varphi_H) \longleftrightarrow Crit(\mathcal{A}_H)$



$\leftarrow \varphi_H^t(x)$

$x = \varphi_H(x)$

$\Rightarrow$

To prove AC, I : $|Fix(\varphi)| \geq \dots$

suffices to show that $|Crit(\mathcal{A}_H)| \geq \dots$

- **Assume:** φ_H is non-deg $\Leftrightarrow$ $\mathcal{A}_H$ is **Morse**

Idea: Do Morse theory for $\mathcal{A}_H$ on Λ for an L^2-metric

Does not literally work

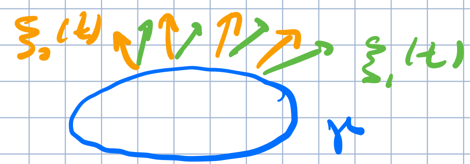
But let's try

Need a Riemannian metric on Λ

- $\langle , \rangle$ a R. metric on M

  $T_\gamma \Lambda =$ v.f. ξ along γ

  $$\langle \xi_0, \xi_1 \rangle_{L^2} = \int_0^1 \langle \xi_0(t), \xi_1(t) \rangle \, dt$$



Need to chose $\langle , \rangle$ on M carefully

- → Recall J on (V, ω) is compatible with ω if $\langle X, Y \rangle = \omega(X, JY)$ is an inner product

  → Also call $\langle , \rangle$ compatible

  J compatible $\Longleftrightarrow$ $\langle , \rangle$ compatible

→ Exist and form a contractible set (E.g. McDuff-Solamon)

$\Rightarrow$ **Lemma** $\exists$ $J : TM \to TM$, $J^2 = -I$ which is compatible with ω at every pt:

$$\langle \cdot , \cdot \rangle_{\mathbb{C}} = \underbrace{\omega(\cdot , J\cdot)}_{\langle , \rangle} + i\omega(\cdot , \cdot)$$

is a Hermitian at every pt

$\langle , \rangle$ ← Take this

Discuss the pf

Rmk $\omega(\underset{\sim}{X}, JY) = \langle \underset{\sim}{X}, Y \rangle$

$\Rightarrow \quad \omega(\underset{\sim}{X}, Y) = \langle \underset{\sim}{X}, -JY \rangle$

$\Rightarrow \quad X_H = -J\nabla H \iff \nabla H = JX_H$

- The gradient of $\mathcal{A}_H$

The Pf of LAP:

$$\underbrace{d\mathcal{A}_H(\xi)}_{\mathcal{L}_\xi \mathcal{A}_H(\gamma)} = \int_0^1 \omega\left(\xi, -\dot{\gamma} + X_H\right) dt$$

$$= \int_0^1 \langle \xi, -J(-\dot{\gamma} + X_H) \rangle \, dt$$

Recall: ∇f : $\langle \nabla f, \xi \rangle = df(\xi)$

$$= \langle +J\dot{\gamma} \underbrace{-JX_H}_{+\nabla H}, \xi \rangle_{L^2}$$

Ex

$$\Rightarrow \boxed{\nabla \mathcal{A}_H(\gamma) = J\dot{\gamma} + \nabla H}$$

The anti-grad flow equation

- $u : \underset{s}{\mathbb{R}} \longrightarrow \Lambda =$ contr. loops in M
  $\{S^1 \longrightarrow M\}$

  $$\underbrace{\frac{du}{ds}}_{\substack{\cap \\ T_{u(s)}\Lambda}} = -\nabla A_H(u(s))$$

- $u : \underset{s}{\mathbb{R}} \times \underset{t}{S^1} \longrightarrow M$

$$\boxed{\begin{gathered} \frac{\partial u}{\partial s} = -J\frac{\partial u}{\partial t} - \nabla H(u) \\ \text{or} \\ \underbrace{\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t}}_{\text{Cauchy-Riemann}} = -\nabla H(u) \end{gathered}}$$

Floer equation

First order elliptic equation

Ex. 1 H autonomous
$u : \mathbb{R} \to M$ ind of t
anti-grad trajectory solves FE

Ex. 1 $H \equiv 0$, J true complex str

$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = \bar{\partial} u$: u sol of FE
$\Leftrightarrow u : \mathbb{R}\times S^1 \xrightarrow{\text{hol}} M$

Difficulties

- the flow does not exist $\Leftarrow$ Ex 2

  $H = 0 \Rightarrow u$ hol

  $\Rightarrow u(s)$ is real analytic

  $\Rightarrow$ no initial value sol for most initial values

- not bounded from below
- infinite index and coindex

Ex. $M = \mathbb{R}^2 = \mathbb{C}$; $H = 0$

$$\Lambda := \{ S^1 \xrightarrow{z} \mathbb{C} \}$$

$$A_H(z) = -\int_z p\,dq = A(z)$$

Fourier expansion

$$z = \sum z_k e^{2\pi i k t}$$

Ex. $$A(z) = \underbrace{-2\pi \sum_{k=-\infty}^{+\infty} k |z_k|^2}_{\infty \,\#\text{ of pos \& neg squares}}$$

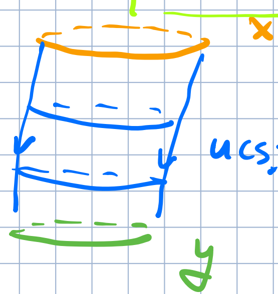
Solution

Even though the flow is not defined the mod spaces $\mathcal{M}(x,y)$ are and have reasonable properties

$$\mathcal{M}(x,y) = \left\{ u : \mathbb{R} \to M \;\middle|\; u \begin{cases} \to x & s \to -\infty \\ \to y & s \to +\infty \end{cases} \right\}$$

Crit

$$= \left\{ u : \mathbb{R} \times S^1 \to M \text{ s.t. } \boxed{u(s,t) \begin{cases} \to x(t) & s \to -\infty \\ \to y(t) & s \to +\infty \end{cases}} \right\}$$



$u(s,t)$

$$\bar{\partial} u = -\nabla H(u)$$

Asymptotic boundary value problem for a 1st order elliptic equation

$\Rightarrow$ solutions

$\Rightarrow$ Can do Morse Theory

Lots of technical difficulties ...

— FIN —