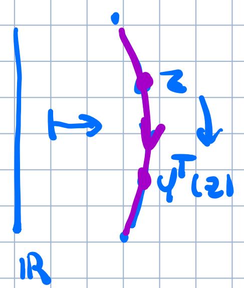
More modern & different perspective

$\mathcal{M}(x,y)$ = the space of parametrized trajectories $t \mapsto \varphi^t(z)$ from x to y



trajectory $\longleftrightarrow$ initial condition

$$\varphi^t(z) \longleftrightarrow z = \varphi^0(z)$$

Time shift: $t \mapsto \varphi^t(z) \qquad z \mapsto \varphi^T(z)$

$$\rotatebox{90}{$\leadsto$}$$

$$t \mapsto \varphi^{t+T}(z)$$

$\Rightarrow$ free $\mathbb{R}$-action on $\mathcal{M}(x,y)$, $x \neq y$

Space of unparametrized trajectories

$$\hat{\mathcal{M}}(x,y) = \mathcal{M}(x,y)/\mathbb{R}$$

Cor $\hat{\mathcal{M}}(x,y)$ is a smooth manifold of dim $\mu(x) - \mu(y) - 1$

E.g. $\mu(x) = \mu(y) + 1 \Rightarrow \hat{\mathcal{M}}$ is discr

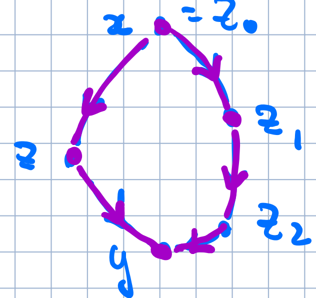
$\mu(x) \leq \mu(y) \Rightarrow \hat{\mathcal{M}} = \emptyset$

Note
- $\mathcal{M}$ & $\hat{\mathcal{M}}$ are usually non-compact
- Geometrically, $\hat{\mathcal{M}}$ can be identified with $\mathcal{M} \cap \{f = c\}$

$$f(y) < c < f(x)$$
$$\uparrow$$
regular

Thm $\hat{\mathcal{M}}(x,y)$ (For a generic metric) has a compactification formed by broken trajectories

Make it precise



Such trajectories $x = z_0 \rightsquigarrow z_1 \rightsquigarrow \dots \rightsquigarrow z_k = y$ form a compact manifold with corners.

Rmk $f(x) > f(z_1) > \dots > f(y)$

$\mu(x) > \mu(z_1) > \dots > \mu(y)$

Cor. $\mu(x) = \mu(y) + 1$

$\Rightarrow \hat{\mathcal{M}}$ = compact $\Rightarrow$ finite collection of pts

$\Leftrightarrow \exists$ finite many traj from x to y (for a generic metric)

Definition of ∂

Fix a generic metric so that all the things hold

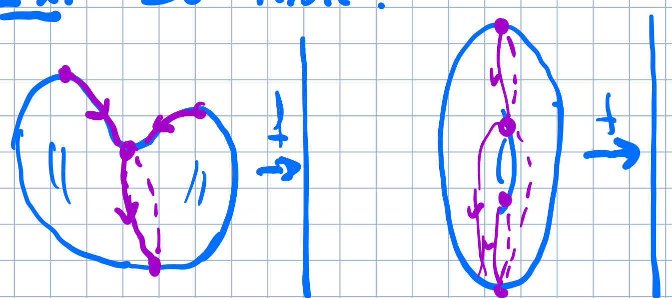
$$\mu(x) = \mu(y) + 1$$

- Over $\mathbb{Z}_2$, set

$$\mathbb{Z}_2 \ni m(x,y) = \# \hat{\mathcal{M}}(x,y) \mod 2$$

$$\boxed{\partial x = \sum_{\substack{y \\ \mu(x) = \mu(y)+1}} m(x,y)\, y} \quad (*)$$

Ex. Do these:



- **Over $\mathbb{Z}$** (and hence any ring)

Need to take into account orientations

Fix orientation of $T_x W^u(x)$ $\forall x$

$\Rightarrow$ coorientations of $T_x W^s(x)$

$\Rightarrow \begin{cases} \text{orientations of } W^u(x) \\ \text{coorientation of } W^s(x) \end{cases}$

$\Rightarrow$ orientations of

$$\mathcal{M}(x,y) = W^u(x) \cap W^s(x)$$

when $\mu(x) = \mu(y)+1$

$\mathcal{M}(x,y)$ = disj union of finite # of trajectories

Each trajectory γ is also oriented by the flow

$\Rightarrow$ Two orientations

$$\mathrm{sign}(\gamma) = \begin{cases} +1 & \text{orientations agree} \\ -1 & \text{—— . —— . disagree} \end{cases}$$

And

$$m(x,y) = \sum_{x \overset{\gamma}{\rightsquigarrow} y} \mathrm{sign}(\gamma)$$

$*$: $\boxed{\partial x = \sum m(x,y)\, y}$

Ex. Look at the torus example again.

Checking that $(CM(f), \partial)$ is a complex

$\boxed{\textbf{Thm}\quad \partial^2 = 0}$

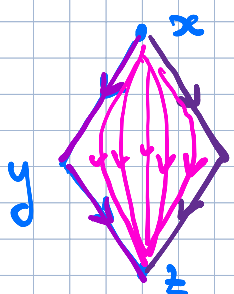
Pf. For the sake of simplicity over $\mathbb{Z}_2$

$$\partial^2 x = \partial \sum_y m(x,y)\, y$$

$\mu(x) = \mu(y)+1$
$\mu(y) = \mu(z)+1$

$$= \sum_y m(x,y) \sum_z m(y,z)\, z$$

$$= \sum_z \Big(\underbrace{\sum_y m(x,y)\, m(y,z)}_{\substack{\#\text{ of broken trajectories (one break)}\\ \text{from } x \text{ to } z \ (\text{mod } 2)}} \Big) \quad \text{mod } 2$$



But $\hat{M}(x,y)$ one-dim manifold
its compactification: S^1 or I (closed interval)

$\Rightarrow$ broken trajectories come in pairs

$\Rightarrow$ # is even

$$\Rightarrow \sum_y m(x,y)\, m(y,z) = 0 \mod 2$$

$$\Rightarrow \partial^2 = 0 \qquad \triangleleft$$

Set $HM_*(f) = H_*(CM(f), \partial)$; fixed coefficients

$$\boxed{\begin{array}{l}\underline{\text{Thm}}\ (\text{Morse theory})\\ \qquad HM_*(f) = H_*(M)\end{array}}$$

<u>Rmk</u> • As a consequence, R.h.s is independent of f

• We have already seen some consequences: Morse inequalities, etc

<u>Outline of the pf:</u>

"Classical" Morse theory

<u>Morse function f on M</u> $\rightsquigarrow$ <u>Cellular decomposition of M</u>

$Crit_k(f) \rightsquigarrow W^u(x) \leftarrow$ cells

 $Crit_*(f)$

Morse complex $CM_k(f)$ $\longleftrightarrow$ Cellular complex of M: $C(M)$

$\partial_M \quad = \quad \partial_{cw}$

$$\Rightarrow \quad H_*(CM(f), \partial_M) = \underbrace{H_*(C(M), \partial_{cw})}_{H_*(M)} \qquad \triangleleft$$

<u>Details</u>: Audin-Damian

Important:

we could have established the isomorphism

$$HM_*(f_0) \longleftrightarrow HM_*(f_1)$$

without going throug $H_*(M)$

Method

$$f_0 \overset{s}{\sim} f_1 : f_s \leftarrow \text{not necessarily Morse}$$

$$\Rightarrow \quad CM_*(f_0) \xrightarrow{\ncong} CM_*(f_1)$$

$$\Rightarrow \quad HM_*(f_0) \xrightarrow{\cong} HM_*(f_1)$$

an isom

Applications: A quick look

Lecture 15
11/16-2021

Two types:
- Lower bounds on $\# \mathrm{Crit}(f)$ ← Morse
- Calculation of $H_*(M)$
  Or more generally diff top of M

(1) Lower bounds: Morse inequalities

$$\# \mathrm{Crit}_k(f) \geq b_k = \dim H_k(M)$$

$$\# \mathrm{Crit}(f) \geq \Sigma b_k$$

Morse

E.g.
- f on $\mathbb{T}^n$ $\quad \# \mathrm{Crit}(f) \geq 2^n$
- f on $\mathbb{CP}^n$ or $\mathbb{RP}^n$
  $\# \mathrm{Crit}(f) \geq n+1$
- f on Σ_g $\quad \# \mathrm{Crit}(f) \geq 2+2g$

Etc.

(2) Calculations of $H_*(M)$ using Morse homology.

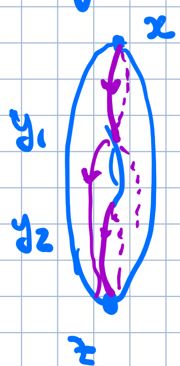
$$H_*(M) = H_*(HM_*(f), \partial_M)$$

very difficult to deal with in general

Works well when $\partial = 0$

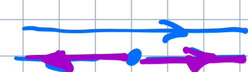
Examples (over $\mathbb{Z}$ or $\mathbb{F}$)

1) Σ_g on π^2



$$\partial x = y_1 + - y_1 + y_2 + - y_2 = 0$$

← work out unstable traj of y_1 & y_2 should come from x (Orientations



$$\partial y_1 = 0 = \partial y_2$$

$\Rightarrow$

$$H_*(\Sigma_g) = \begin{cases} \mathbb{F} & k=2 \\ \mathbb{F}^{2g} & k=1 \\ \mathbb{F} & k=0 \end{cases}$$

2) $\mathbb{CP}^n$ (over $\mathbb{Z}$ or $\mathbb{F}$)

$$\mathbb{CP}^n = \{ (z_0 : \dots : z_n) \mid \sum |z_j|^2 = 1 \}$$

$$f(z) = \sum \lambda_j |z_j|^2 \quad \text{or} \quad \frac{\sum \lambda_j |z_j|^2}{\sum |z_j|^2}$$

distinct

$$\lambda_0 < \lambda_1 < \dots < \lambda_n \leftarrow \text{convenient to not assume}$$

Ex. a) $\mathrm{Crit}(f) =$ "coordinate axes"

Work out in detail

$$= \{ (0, \dots, 0, \underset{j}{1}, 0, \dots, 0) = x_j \}$$

b) In coordinates

$$u_j = (u_0, \dots, u_{j-1}, 1, u_{j+1}, \dots, u_n)$$

near x_j the Hessian is

$$(\lambda_0 - \lambda_j)|u_0|^2 + (\lambda_1 - \lambda_j)|u_1|^2 + \dots \text{ skip } (\lambda_j - \lambda_j)$$

$\Rightarrow f$ is morse & $\mu(z_j) = 2j \leftarrow z_j$ is a complex #.

& $\partial = 0$

$$\Rightarrow H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{F} & 0 \le k = 2j \le 2n \\ 0 & \text{otherwise} \end{cases}$$

Some details

a) Lagrange multipliers

$$\begin{tikzcd} \mathbb{C}^{n+1} \supset S^{2n+1} \arrow[d, "S^1"'] \arrow[dr, "\tilde{f}"] & \\ \mathbb{CP}^n \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$S^{2n+1} = \{g = 1\}$

Crit pts:

$\nabla g = c \cdot \nabla \tilde{f}$

$$\left.\begin{aligned} \nabla g &= 2z = 2(z_1, \dots, z_n) \\ \nabla \tilde{f} &= 2(\lambda_1 z_1, \dots, \lambda_n z_n) \end{aligned}\right\} \Rightarrow \text{crit pts} = \text{coord axes}$$

b) Hessian say at $x_0 = (1, 0 \dots 0)$: $j = 0$

U_1 : $u_1 = \frac{z_1}{z_0}$, $u_2 = \frac{z_2}{z_0}$, ...

$$f(u) = \frac{\sum \lambda_k |z_k|^2}{\sum |z_k|^2}$$

Here $k = 0, \dots, n$

$$= \frac{\lambda_0 + \sum \lambda_k |u_k|^2}{1 + \sum |u_k|^2}$$

But here $k = 1, \dots n$

$$= (\lambda_0 + \sum \lambda_k |u_k|^2)(1 + \sum |u_k|^2)^{-1}$$

$$= \lambda_0 + \sum \lambda_k |u_k|^2 - \lambda_0 \sum |u_k|^2$$

$$= \lambda_0 + \sum_{k=1}^{n} (\lambda_k - \lambda_0) |u_k|^2$$

3) $\mathbb{RP}^n$ over $\mathbb{Z}_2$

Similarly $\mathbb{RP}^n = \{ (y_0 : \dots : y_n) \mid \Sigma |y_j|^2 = 1 \}$

$$f(y) = \Sigma \lambda_j |y_j|^2$$

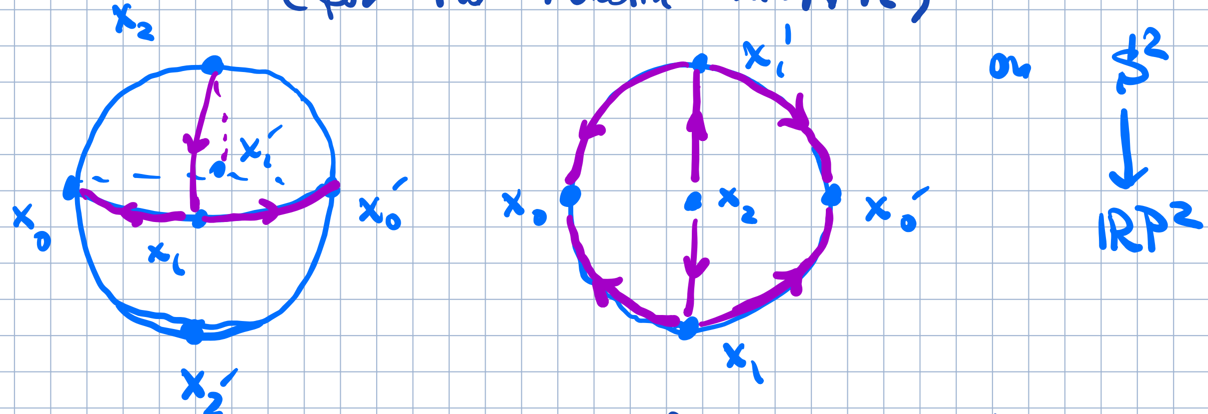
Ex. similarly

a) $x_j = (0, \dots, 0, \underset{j}{1}, 0, \dots, 0) \leftarrow$ Critical pts

b) Hessian : similar — same calculation

$$\Rightarrow \mu(x_j) = j$$

c) $\partial = 0$ over $\mathbb{Z}_2$: exactly two trajectories from x_{j+1} to x_j (for the round metric)



$$\Rightarrow \quad H_k(\mathbb{RP}^n; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

Rmk. Over $\mathbb{Z}$, harder — orientations

Other applications:

- General: Poincaré duality

$$CM_*(f) = CM_{m-*}(f)^* \qquad m = \dim$$

$$\Rightarrow \quad H_*(M) = H_{m-*}(M)^* = H_{m-*}(M)$$

- Diff Topology:
  - Handle body decomposition
  - 3-manifolds
  - classification of surfaces

The Poincaré conj & classification of manifolds

- More homology calculations
- Loop spaces & closed geodesics