§7. Elements of Morse Theory

Lecture 12
11/02-2021

- Not directly related to symplectic geom but extr. important on its own.
- Connections with many things inc s.g., ODE's, PDE's, everything

General setting & motivation

X some space: a manifold, loop space, path space

$f: X \longrightarrow \mathbb{R}$ a function (smooth)

Looking for critical pts of f

Does it have them? How many, etc?

Ex a) x_0 & $x_1 \in Q \leftarrow$ Rieman. manifold

From mechanics → diff geometry → Original Morse theory

$X = \{$path connecting $x_0, x_1\}$

$= \{\gamma : [0,1] \longrightarrow Q \mid \gamma(i) = x_i\}$

Fix a R.m. metric on Q

$\underset{p}{T^*Q} = \underset{v}{TQ}$

potential energy

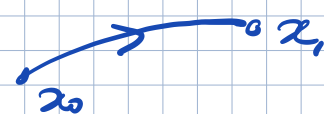
$H = \frac{1}{2}\langle p, p\rangle + V(q)$

$$f(\gamma) = \int_0^1 H(\dot{\gamma}, \gamma)\, dt$$

Least action principle (LAP)

$\mathrm{Crit}(f)$ = integral curves connecting x_0 & x_1 in time-1

E.g. $V = 0$: geodesics from x_0 to x_1

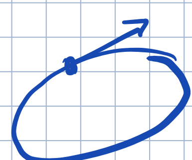


Relations to diff geometry: Hopf-Rinow, closed geodesics, Hadamard

b) X = loop space
$= \{\gamma : S^1 \longrightarrow Q\}$

f = the same

LAP: $\mathrm{Crit}(f)$ = periodic traj of φ_H^t of period 1 in T^*Q



E.g. closed geodesics

Etc: Everything of interest in physics is a crit pt of some functional

Calculus of variations

Finite dimensional setting

- $X = M$ a compact (or even closed) finited-dim. manifold
- $f : M \longrightarrow \mathbb{R}$ $\qquad (f|\partial X = \text{const})$

Q. How many crit pts does f have?

- Nontrivial even in simple cases.
- Assume M is closed:

$\max f$ & $\min f$ — critical values

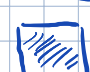
Anything else

- In general yes, unless $M \underset{}{\overset{\text{homeo}}{\simeq}} S^n$

But might be few

$\boxed{\text{Ex. } \mathrm{Crit} = \{\max, \min\} \Rightarrow M \underset{\text{homeo}}{\simeq} S^n}$

Ex. . Construct $f : \Sigma_{g \geq 1} \longrightarrow \mathbb{R}$ with exactly 3 critical pts

. sketch the levels for T^2:  /~

The third pt = Monkey saddle

- Situation changes when we impose a non-deg cond on f, satisfied generically ← explain

Definitions

$p \in M$ critical pt of $f: M \longrightarrow \mathbb{R}$

Def Hessian of f at p is the quadratic (or bilinear) form:

$$d^2 f: T_pM \times T_pM \longrightarrow \mathbb{R}$$
$$v, w \longrightarrow (L_{\tilde{v}} L_{\tilde{w}} f)(p)$$

↖ Ext of v & w to v.f.

Ex. • Show that $d^2 f$ is well defined
• symmetric
• In local coordinates $x_1, \dots, x_n$

$$d^2 f = \sum \frac{\partial^2 f}{\partial x_i \partial x_j}(p)\, x_i x_j$$
$$dx_i\, dx_j$$

Do some of it

Then assume that $d^2 f$ is non-deg

Morse index of p = index of $d^2 f$:

$$d^2 f = -(x_1^2 + \dots + x_k^2) + (x_{k+1}^2 + \dots + x_n^2)$$

Ex. p = max : index = n
p = min : index = 0

Def f is Morse if all its critical pts are non-deg

Note: Morse functions form an open and dense subset of $C^\infty(M)$

Thm (Morse Lemma) f is C^3

Near a non-deg critical pt p of a function f is diffeo to its Hessian $H = d^2 f$:

- $\exists \ \varphi : (U, p) \to (V, p)$ s.t.
  $f \circ \varphi = \varphi^* f = H + f(p)$
- In some coordinates $x_1, \dots, x_n$
  $f(x) = f(p) + \sum a_{ij} x_i x_j$

Rmk
- One of the normal form results
- Another example:
  $df_p \neq 0 \ \exists \ (x_1, \dots, x_n)$ s.t.
  $f(x) = f(p) + x_1$

  Ex prove directly

  More generally, the local norm form for submersions (also immersion)

- Similar questions for other objects: vector fields, maps, etc
- E.g. v v.f.

  $v(p) \neq 0 \Rightarrow \exists \ x_1, \dots, x_n$
  $v(x) = \frac{\partial}{\partial x_n}$

  What if $v(p) = 0$?

Pf: Moser's homotopy method...

Morse Homology

Lecture 13
10/08-2021
Ref: Audin-Damian
Banyaga-Hurtubise

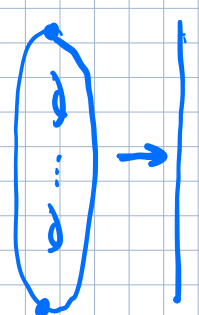
- $f: M \longrightarrow \mathbb{R}$ morse function
- $\mathrm{Crit}_k(f) \longleftarrow$ the collection of critical pts of index k
- Fix a ground ring $\mathbb{F}$: $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{Z}_2$, etc
- $CM_k(f)$ = free module over $\mathbb{F}$ generated by $\mathrm{Crit}_k(f)$

E.g. Height function on Σ_g

max $k=2$

saddles $k=1$

min $k=0$



$CM_0 = \mathbb{F}$

$CM_1 = \mathbb{F}^{2g}$

$CM_2 = \mathbb{F}$

$\boxed{\partial^2 = 0}$

Goal: turn CM_k into a complex

$$0 \to CM_n \xrightarrow{\partial} CM_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} CM_1 \xrightarrow{\partial} CM_0 \to 0$$

so that

Morse Homology

$$\boxed{\underbrace{H_*(CM_*(f), \partial)}_{=: HM_*(f)} \cong \underbrace{H_*(M; \mathbb{F})}_{\text{Homology of of } M}}$$

$\boxed{\begin{array}{l}\textbf{Cor (Morse inequalities)} \\ \quad \mathbb{F} \text{ a field} \\ \text{(a)} \quad \underbrace{\# \mathrm{Crit}_k(f)}_{c_k} \geq \underbrace{\dim H_k(M)}_{b_k} \\ \text{(b)} \quad c_k - c_{k-1} + c_{k-2} - \dots \pm c_0 \geq b_k - b_{k-1} + \dots \pm b_0 \end{array}}$

Pf (a) $c_k = \dim CM_k \geq b_k = \dim HM_k$

(b) $\leftarrow$ Ex $\leftarrow$ Purely algebraic statement

C_* a complex over $\mathbb{F}$

show that C_* can be decomposed as a sum of elementary complexes explain

check (b) for an elementary complex $\triangleleft$

Yet a different formulation. Set

$Q(t) = \sum c_k t^k$, $\quad P(t) = \sum b_k t^k$ $\leftarrow$ Poincaré Pol

Then coeff ≥ 0

$$Q(t) = P(t) + (1+t) R(t)$$

Ex. Prove this - alg, same method

Ex A Morse function on Σ_g has at least $2g+2$ critical pts.

Rmk. Morse inequalities can be further refined

Construction of the Morse differential ∂ : Preliminaries

- While $CM_*(f)$ is completely determined by f, ∂ depends on an extra str: a R. metric on M
- Fix a R. m. on M (has to be from a certain open and dense set of R.m.'s)

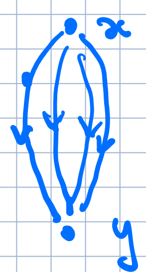
Consider the antigradient flow of f:

$$\dot{x} = -\nabla f(x) \quad : \quad \varphi_t$$

Set

$$\mathcal{M}(x, y) = \left\{ z \;\middle|\; \varphi_t(z) \begin{array}{l} \to x \quad t \to -\infty \\ \to y \quad t \to +\infty \end{array} \right\}$$

x, y crit

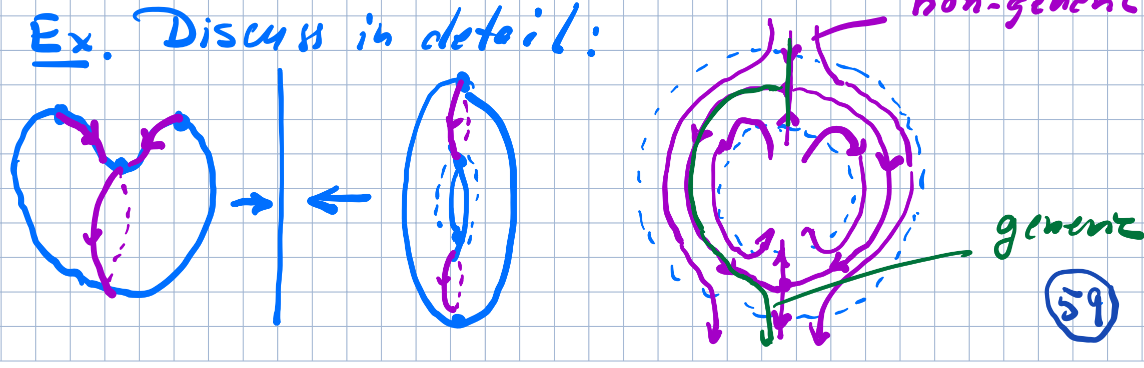


Denote the index of x by $\mu(x)$.

__Note__: "$\dim \mathcal{M}(x, y) \geq 1$" if $\neq \emptyset$

__Thm__ For a generic metric, $\mathcal{M}(x, y)$ is a smooth manifold of dimension $\mu(x) - \mu(y)$

__Ex.__ Discuss in detail:

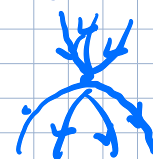


One way to prove the theorem:

$$W^u(x) = \{ z \mid \varphi^t(z) \to x,\ t \to -\infty \}$$

$$W^s(x) = \{ \_\_\_ \cdot \_\_\_ \cdot \ldots t \to +\infty \}$$

Stable, unstable manifolds



Morse Lemma: $\Rightarrow$ $W^u(x) \underset{\text{diffeo}}{\simeq} D^{\mu(x)}$

(Look at the examples) $\quad W^s(x) \simeq D^{n-\mu(x)}$

$$\mathcal{M}(x,y) = W^u(x) \cap W^s(y)$$

If $\quad W^u(x) \pitchfork W^s(y)$,

$\mathcal{M}(x,y)$ is smooth and

$$\begin{aligned} \dim \mathcal{M}(x,y) &= \dim W^u(x) + \dim W^s(y) - n \\ &= \mu(x) + n - \mu(y) - n \\ &= \mu(x) - \mu(y) \end{aligned}$$

How to achive transversality

Look at $(W^u(x) \cap \{f=c\}) \cap (W^s(y) \cap \{f=c\})$

$f(y) < \underset{\text{reg}}{c} < f(x)$, perturb the metric slightly above c to alter $\triangleleft$

Cor By thm, for a generic metric

$$\mu(y) \geq \mu(x) \Rightarrow \mathcal{M}(x,y) = \emptyset$$