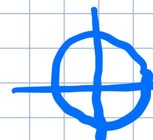
Lagr. submanifolds of $\mathbb{R}^{2n}$

Lecture 10

10/26-2021

Important question in sympl topology

Some simple observations

- $S^1 \hookrightarrow \mathbb{R}^2$ Lagr  $\Rightarrow$

$T^n = S^1 \times \dots \times S^1 \hookrightarrow \mathbb{R}^{2n}$

A lot of different (non-equivalent)

$\omega_{st} = d\lambda$, $\lambda = \frac{1}{2}\sum(p_i\, dq_i - q_i\, dp_i)$

$\lambda|_{T^n}$ closed $\quad [\lambda|_{T^n}] \in H^1(T^n; \mathbb{R})$ inv

- **Prop** $L \subset \mathbb{R}^{2n}$ closed Lagr $\Rightarrow \chi(L) = 0$

Pf. $N_L = T^*L \Rightarrow L \cdot L = \chi(L)$

But $\underset{\text{in } \mathbb{R}^{2n}}{L \cdot L = 0}$ e.g. because $[L] = 0$

or by deformation invariance

Cor. $\Sigma_{g \neq 1}$ does not admit Lagr. embeddings into $\mathbb{R}^4$

Liouville class

Rmk $\exists$ much more subtle results

E.g. $L \subset \mathbb{R}^{2n}$ Lagr $\Rightarrow H^1(L; \mathbb{R}) \neq 0$

In fact $[\lambda|_L] \neq 0$ (Gromov)

$\Rightarrow S^3$ does have Lagr. embedding into $\mathbb{R}^6$

Maslov class

$L \looparrowright \mathbb{R}^{2n}$ Lagr., immersed, closed

$\Rightarrow G: L \to \Lambda$ $\leftarrow$ Gauss map

$x \mapsto T_x L$

$\mu \in H^1(\Lambda; \mathbb{Z})$ Maslov

$\mu_L \in H^1(L; \mathbb{Z}) \leftarrow$ Maslov class of L.

$\mu_L = G^*\mu$

Ex. L orientable $\Rightarrow \mu_L$ is even

Fact (Gromov) $L \subset \mathbb{R}^{2n}$ embedded

$\Rightarrow \mu_L \neq 0 \Rightarrow H^1 \neq 0$

$\Rightarrow$ again S^3 does not have a Lagr. emb

Rmk. μ_L can also be defined for $L \subset T^*Q$ (but it can be 0)

§6 Contact manifolds

Contact str = odd-dim sister of sympl str

M^{2n+1} ← odd dimensional

Def • $\alpha \in \Omega^1(M)$ is contact if

$\alpha \wedge (d\alpha)^n \neq 0$ ← vol. forms $\boxed{\Rightarrow M^{2n+1} \text{ orientable}}$

$\Leftrightarrow$ $d\alpha|_{\ker\alpha}$ is non-deg

• $\xi = \ker\alpha$ is a contact str

strictly speaking: a codim-1 distr ξ is contact if locally $\xi = \ker\alpha$ (↑ contact)

can be made globally $\Leftrightarrow$ ξ is coorientable

Note: α contact $\Rightarrow$ $f\alpha$ contact, $f \neq 0$, same contact str. ξ

$(f\alpha) \wedge [d(f\alpha)]^n$

$= (f\alpha) \wedge [df \wedge \alpha + f\, d\alpha]^n$

$\overset{\alpha^2=0}{=} f^{n+1} \alpha \wedge (d\alpha)^n$

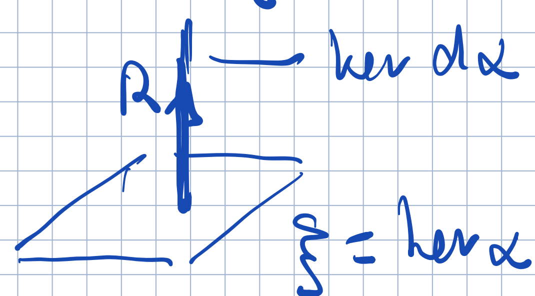
Cor M^{2n+1} admits a contact str ξ (not necessarily coorientable) and $n+1$ even $\Rightarrow$ M is orientable

Def - fact α contact form $\Rightarrow$
$\exists!$ v.f. R Reeb vf :

$$\alpha(R) = 1$$
$$i_R \, d\alpha = 0$$

In fact α contact

$$\Leftrightarrow \begin{cases} \ker(d\alpha) \ \text{1-dim} \\ \ker d\alpha \pitchfork \underbrace{\ker\alpha}_{\xi} \end{cases}$$



Reeb v.f. $\rightsquigarrow$ Reeb flow

Examples

Ex 1. $\mathbb{R}^{2n+1}$ (p, q, z)

$\alpha = dz + p\,dq$ or $dz + \frac{1}{2}(p\,dq - q\,dp)$

are contact : st. contact form on $\mathbb{R}^{2n+1}$

$R = \frac{\partial}{\partial z}$ The st. contact form or str. on $\mathbb{R}^{2n+1}$

Visualize $n=1$

More generally $M = \underset{z}{\mathbb{R}} \times W \xrightarrow{\pi}$ $\omega = d\lambda$
$\alpha = dz + \pi^* \lambda$ ← contact

Ex 2

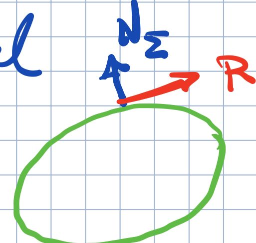
$\Sigma \subset \mathbb{R}^{2n}$, $\Sigma = \partial(\text{starshaped})$

e.g. convex

$\lambda = \frac{1}{2}(p\,dq - q\,dp)$

$\alpha = \lambda|_\Sigma$ is contact

Motivation coming from Ham dynamics



• $R = J N_\Sigma$ ← unit normal

• $\Sigma = \{H = \text{const}\}$ ← reg

Then $R = f X_H$ on Σ

Lecture 11

10/28-2021

More generally

$\Sigma^{2n-1} \subset (M^{2n}, \omega)$ symplectic

Def. Σ has contact type if
$\omega|_\Sigma$ has a contact primitive α:
$d\alpha = \omega|_\Sigma$, $\alpha \wedge (d\alpha)^{n-1} \neq 0$

Thm: $\Sigma = \{H = \text{const}\} \leftarrow$ regular
$\Rightarrow R = f X_H$ on Σ

Rmk. Not every closed hypersurface in $\mathbb{R}^{2n}$ has contact type

Ex - Weinstein: two spheres

Ex 3 $\Sigma \subset T^*Q$ fiberwise starshaped

$\lambda = p\,dq$ Liouville form

$\alpha = \lambda|_\Sigma$ is contact

Σ fiberwise convex: Finsler metric

Reeb flow = Finsler geodesic flow

Motivation! geometric optics

Ex 4 - Fact every closed orientable $\leftarrow$ necessary
3-manifold admits a contact structure

Existence of contact str. Discuss in more detail?

Contact topology $\leftarrow$ active area

Contact Darboux Thm and all that

Thm (Contact Darboux)

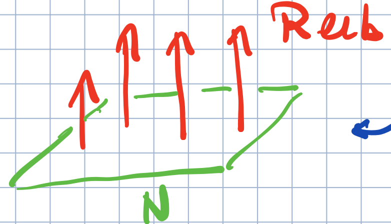
$\boxed{\text{Any two contact forms (fixed dim) are locally diffeomorphic}}$

Thm' (Contact Darboux)

Any contact form in dim $= 2n+1$ is locally diffeo to the standard contact form on $\mathbb{R}^{2n+1}$: $\exists$ coord p, q, z such that $\alpha = dz + p\,dq$

Two ways to prove:

1) As a consequence of symplectic Darboux – Ex



crossection

$d\alpha|_N$ = symplectic $= d(\underbrace{p\,dq}_{\lambda})$

$d\alpha|_N = d\lambda|_N \Rightarrow \alpha - \lambda = df$ ← can assume 0 at 0

"z = time of Reeb flow from N + f"

2) Use Moser's homotopy method directly

Rmk No global version for contact forms

$M, \alpha_t \leftarrow$ a family of contact forms
cannot expect α_s to be diffeo to each other

$\alpha_s \rightsquigarrow R_s \rightsquigarrow$ dynamics changes with s

Ex. $\underset{\{H=1\}}{\Sigma_t} \subset \mathbb{R}^{2n}$ a family of ellipsoids

$\{H=1\} \leftarrow$ quadratic Ham

$(\Sigma_t, \lambda|_\Sigma) \cong (S^{2n-1}_t, \alpha_t)$

$R_t = X_t \rightsquigarrow R_t$

we have seen that things depend on eigenvalues

Thm (Gray's Thm)

(M^{2n+1}, ξ_t) contact, M closed $\Rightarrow \varphi_t$ s.t.

$(\varphi_t)_* \xi_t = \xi_0$

What is actually proved

$M^{2n+1}, \xi_t = \ker \alpha_t$

$\Rightarrow \exists\ \varphi_t\ \&\ f_t > 0 : \varphi_t^*(f_t \alpha_t) = \alpha_0$

Pf: Moser's homotopy method

Rmk. Discuss symplectization

A glimpse of contact topology

ξ cooriented contact str

$\rightsquigarrow$ R_α Reeb : non-vanishing section of TM

$\rightsquigarrow$ The homotopy type of R is ind of α

A section of STM

$\rightsquigarrow$ A top inv. of ξ

homotopy type of ξ as cooriented distr.

E.g. $M = S^3$ $\quad STS^3 = S^3 \times S^2$

homotopy types of sections

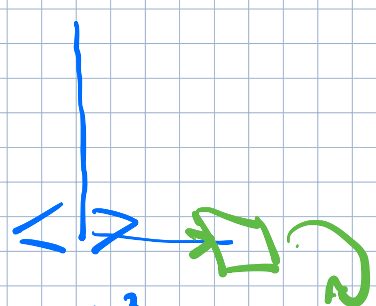
$\delta(\xi) \in [S^3, S^2] = \pi_3(S^2) = \mathbb{Z}$

Each of them can be realized by a contact str, and those contact str are not diffeo to each other

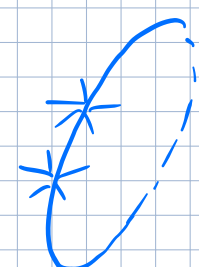
standard $\rightsquigarrow$ $0 = \delta(\xi_{st})$

But $\exists$ (exactly one) contact str ξ_{ot} with $\delta(\xi_{ot}) = 0 = \delta(\xi_{st})$

Describe



in $\mathbb{R}^3$



in S^3