Lecture 8

10/19-2021

Digression:

complex & symplectic v.s's

complex str: V real v.s

$J: V \to V$, $J^2 = -I$

Then $(a+ib)v = a + bJv$

Complex linear maps: $A: V \circlearrowleft$: $JA = AJ$

Ex: $A: V \circlearrowleft$ complex linear

$\Rightarrow A$ is $\mathbb{R}$-linear

Prove that $\det_{\mathbb{R}} A_{\mathbb{R}} = |\det_{\mathbb{C}} A|^2$

Cor: • $GL(n, \mathbb{C}) \subset \underbrace{GL^+(2n, \mathbb{R})}_{\det > 0}$

orientation pres

• $U(n) \subset O(2n)$

Complexification: $V = W \otimes \mathbb{C}$

W real v.s. $\quad V = W_{\mathbb{C}} = V \oplus V \circlearrowleft J \Big\}$ complex

$J(x,y) = (-y, x)$

But it also has an extra str

conjugation: $(x,y) \mapsto (x, -y)$

$e_1, \dots, e_n$ basis in $W \Rightarrow$ also a basis in V

Functorial

- $A: W \to W \Rightarrow A_{\mathbb{C}}: V \to A$
  $\mathbb{R}$-linear $\quad$ $\mathbb{C}$-linear
  <u>same matrix</u>

- inner product $\Rightarrow$ Hermitian product
  on W $\quad$ on V

$\Rightarrow \quad O(n) \subset U(n)$ and $\boxed{O(n) = U(n) \cap GL(n, \mathbb{R})}$

Symplectic v.s. Hermitian

(V, ω) symplectic v.s.
J complex str.
ω & J are <u>compatible</u> if

Ex J always exists

1) $\omega(X, JX) \geq 0$
$> 0 \quad X \neq 0$

2) $J \in Sp(V, \omega)$

$\Leftrightarrow \langle X, Y \rangle = \omega(X, JY)$ is an inner product
and $J \in O(V, \langle , \rangle)$

$\Leftrightarrow \langle X, Y \rangle_{\mathbb{C}} = \langle X, Y \rangle + i\,\omega(X, Y)$
is a Hermitian inner product

- $L \subset V$ Lagrangian, J compatible with ω

  $\Rightarrow JL \cap L = \{0\}$

  $V = L \otimes \mathbb{C}$

  $A \in O(L) \Rightarrow A_{\mathbb{C}} \in U(V)$

minor generalization

$\Rightarrow U(n)$ acts transitively on Λ

$$\begin{array}{ccc} L & \overset{A}{\rightsquigarrow} & L' \\ \underbrace{e_1, \dots, e_n}_{\text{orthogonal}} & \rightsquigarrow & \underbrace{e_1', \dots, e_n'}_{\text{orthogonal}} \end{array}$$

Then $A_{\mathbb{C}} : V = L \otimes \mathbb{C} \longrightarrow L' \otimes \mathbb{C} = V$ is unitary

And $\mathrm{Stab}(L) \subset U(n)$ is $O(n)$
$= U(n) \cap GL(n, \mathbb{R})$

$\Rightarrow$

Ex.
- $\Lambda = U(n)/O(n) \leftarrow$ Explain
- $\pi_1(\Lambda) \xrightarrow{\simeq} \mathbb{Z} \leftarrow$ "Maslov class"
  $A \longmapsto \det_{\mathbb{C}}^2(A)$
- $H_1(\Lambda) \longrightarrow \mathbb{Z}$: $\mathrm{Maslov} \in H^1(\Lambda; \mathbb{Z})$

Ex. $\Lambda_1 = \mathbb{RP}^1 = S^1$
$\Lambda_2 = S^1 \times S^2 / \sim \leftarrow$ antipodal in both factors

Back to symplectic manifolds

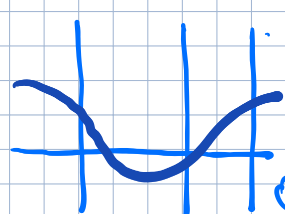
$(M^{2n}, \omega) \supset L$

Def. L Lagr (iso, coiso, sympl) if
$T_x L \subset T_x M$ is Lagr (iso ...) $\forall x \in L$

Ex. $\dim L = 1 \Rightarrow$ iso
$\operatorname{codim} L = 1$ (hypersurface) $\Rightarrow$ coiso

Focus on Lagr. submanifolds

Ex 1 • $M = T^*Q \longrightarrow Q$
$\alpha \in \Omega^1(Q) =$ section of T^*Q
$\Rightarrow \quad L_\alpha \subset T^*Q$



$d\alpha = 0 \Leftrightarrow L_\alpha$ Lagr

$\omega = d\lambda$

Pf $\alpha = \lambda|_{L_\alpha = Q}$

$d\lambda|_L = 0 \Leftrightarrow d\alpha = 0$ $\triangleleft$

Or if you wish

$$\psi : Q \longrightarrow T^*Q$$
$$x \longmapsto \alpha_x$$

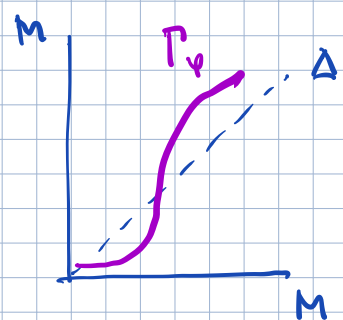
$\psi^* \lambda = \alpha$
$d\lambda|_{L_\alpha} = 0 \Leftrightarrow \psi^* d\lambda = 0$
$\Leftrightarrow d\alpha = 0$

Ex 2 $W = (M \times M, \underbrace{(-\omega, \omega)}_{\tilde{\omega}})$

$\varphi : M \longrightarrow M$

$L = \Gamma_\varphi \subset W$, the graph of φ

$L = \{(x, \varphi(x)) \mid x \in M\}$



Claim:

Γ_φ Lagr $\iff$ φ is symplectic

$\varphi^* \omega = \omega$

Pf Identify $\varphi : M \longrightarrow \Gamma_\varphi$

$x \longmapsto (x, \varphi(x))$

$$\tilde{\omega}|_{\Gamma_\varphi} = 0 \iff \varphi^* \tilde{\omega} = 0$$

$$\iff -\omega + \varphi^* \omega = 0$$

$$\iff \varphi^* \omega = \omega$$

$\triangleleft$

Lecture 9
10/21-2021

Thm (Weinstein's tubular nbd)

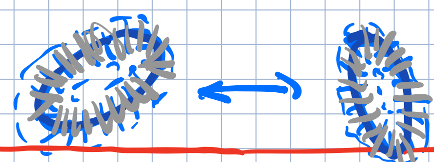
$$\begin{array}{l} L \xrightarrow{i} (M,\omega) \\ L \xrightarrow{i'} (M',\omega) \end{array}$$

L closed

two Lagr. embeddings

$\Rightarrow$ $\exists$ nbds $U \supset i(L)$ & $U' \supset i'(L)$

s.t. $(U, i(L))$ & $(U', i'(L))$ are symplecto...

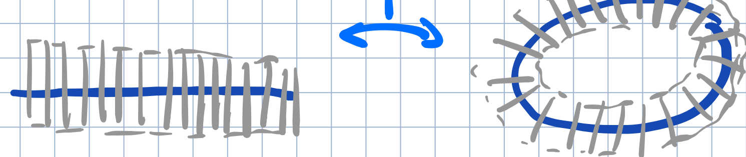


Thm' (. ... .)

closed $L \hookrightarrow M$ Lagr. embedding

$\Rightarrow$ nbd of L is symplectomorphic to a nbd $L \subset T^*L$

T^*L $\xleftrightarrow{\varphi}$ $L \subset M$



Discuss the tubular nbd thm?

On the pf: similar to Darboux

$L \subset M$ Lagr. closed

- Preliminary (lin. alg)

N_L normal budle: $N_L \oplus TL = T_L M$

can be chosen Lagr

$N_L \cong T^*L$

Now use ordinary tubular nbd thm

Discuss

to identify a nbd of $L \subset M$ with a nbd of L in T^*L

$\mathcal{U} = ($nbd of L in M $\cong$ nbd of L in $T^*L)$

$\Rightarrow$ $\mathcal{U} \subset$ nbd of L in T^*L

ω_0 & ω_1 two symplectic forms s.t. L is Lagr for both and by construction

ω_0 & ω_1 $\underbrace{\text{agree on } T_L(T^*L)}_{\Downarrow \text{ symplectic}}$ ⊛

standard

• Set $\omega_t = (1-t)\omega_0 + t\omega_1$

Run the homotopy method

X_t : $\quad i_{X_t}\omega_t = \lambda \qquad \underbrace{d\lambda = \omega_0 - \omega_1}_{\text{exists}}$

$H^2(\mathcal{U}) \cong H^2(L)$

$[\omega_0 - \omega_1] \leftrightarrow [\underset{\overset{\parallel}{0}}{\omega_0|_L} - \underset{\overset{\parallel}{0}}{\omega_1|_L}]$

Nuance:

Need $\lambda = 0$ at every pt of L

(To make sure φ_t is defined for $t \in [0,1]$)

Not only $\lambda|_L = 0$

<u>Ex</u>: Such λ exists by (*)

Not obvious $\triangleleft$