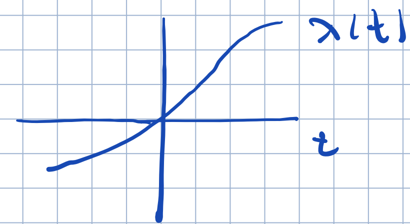
Time - change

Lecture 6
10/12 - 2021

$H : \mathbb{R} \times M \to \mathbb{R}$ Hamiltonian

$$\underbrace{\lambda : \mathbb{R} \to \mathbb{R}}_{\text{time-change}}$$



Does not have to be monotone but usually is

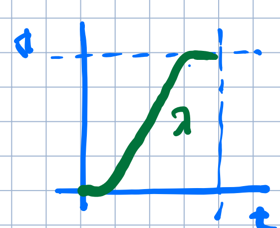
set $K_t(x) = \lambda'(t)\, H_{\lambda(t)}(x)$

Ex. Show that $\varphi^t_K = \varphi^{\lambda(t)}_H$

Ex. 1) $\lambda(t) = T \cdot t$: $K = T\, H_{Tt}$

$\Rightarrow \varphi^1_K = \varphi^T_H$: Looking at φ^T_H can always assume that $T = 1$

2)



$\varphi^1_K = \varphi^1_H$ but $K \equiv 0$ when $t \approx 0$ & $t \approx 1$

$\Rightarrow$ Looking at φ^K_H can always assume H is 1-periodic in time $H_{t+1} = H_t$

§4 Relevant Groups:
Ham v.s. Symp

Def. $\varphi_H = \varphi_H^1$ is called a Hamiltonian diffeo

Rmks
- Can assume that H is 1-periodic in t
- Can replace 1 by anything
- Hamiltonian $\Rightarrow$ Symplectic: $\varphi^* \omega = \omega$
- When M is not compact, need to assume smth about H at ∞. We'll usually assume that H is compactly supported $\Rightarrow$ supp φ is compact $\not\Leftarrow$

Prop The collection of Ham diffeos $Ham = Ham(M, \omega)$ is a gp.

Rmk Not obvious: H is not autonomous
$$(\varphi_H^t)^{-1} \neq \varphi_H^{-t}$$
and it's not clear why
$$\varphi_H^t \varphi_K^t \text{ is Hamiltonian}$$

Focus on the product:

Pf 1. Consider H_t, $t \in [0,1]$; K_{t-1}, $t \in [1,2]$ $\Big\}\ F_t$

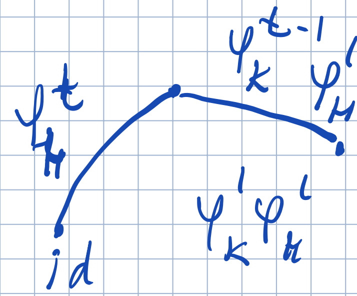
smooth in t when say

$H_t \equiv 0$ for $t \approx 1$; $K_t \equiv 0$ for $t \approx 0$ $\Big\}$ can be achieved by time chg

Then F_t generates

φ_H^t for $t \in [0,1]$

$\varphi_K^{t-1}\varphi_H^1$ for $t \in [1,2]$



So over $t \in [0,2]$ it generates $\varphi_K^1\varphi_H^1$

$\Rightarrow$ Ham(M,ω) is closed under the product $\lhd$

Ex: generate $(\varphi_H^1)^{-1}$

Pf 2 - Ex

- $\varphi_K^t\varphi_H^t$ is generated by $K_t + H_t \circ (\varphi_K^t)^{-1}$

$\Downarrow$

- $(\varphi_H^t)^{-1}$ — . — . — $-H_t \circ (\varphi_H^t)^{-1}$ $\lhd$

Now we have

usually strict (Examples) T^{2n}!

$$\boxed{\mathrm{Ham}(M,\omega) \subset \underbrace{\mathrm{Symp}_0(M,\omega)}_{\text{connected component of the id}}} \subset \begin{array}{c}\mathrm{Sympl}(M,\omega)\\ \cap \\ \mathrm{Diff}_{\omega^n}(M)\end{array}$$

should think of these as ∞-dim Lie groups

On the level of Lie algebras: vector fields

$$\mathrm{Ham} \quad \subset \quad \mathrm{Symp}_0$$

Lie algebras

$\wr$ Discuss in more detail

$$\boxed{\begin{array}{ccc} \text{Ham v.f.} & \subset & \text{Sympl. v.f.} \\ i_X\omega = \text{exact} & & i_X\omega \text{ closed} \iff L_X\omega = 0 \end{array}}$$

$$\begin{array}{ccc} \updownarrow \; X \mapsto i_X\omega & & \updownarrow \; X \mapsto i_X\omega \\ \text{exact 1-forms} & \subset & \text{closed 1-forms} \end{array}$$

$$\Rightarrow \boxed{\frac{\text{Sympl. v.f.}}{\text{Ham v.f}} \Longrightarrow H^1(M;\mathbb{R})}$$

$$\boxed{\underline{\text{Cor}} \quad H^1 = 0 \Rightarrow \text{Symp. v.f} = \text{Ham. v.f.}}$$

Ex. Shifts of $\mathbb{T}^2$

$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (x, y) "coordinates"

$\varphi: (x,y) \mapsto (x+a, y)$ $\omega = dx \wedge dy$

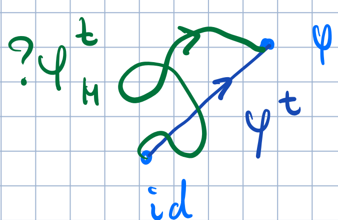
Generated by $X = a\frac{\partial}{\partial x}$, $\varphi = \varphi^1$

Symplectic but not Hamiltonian:

$i_X \omega = a\, dy$ closed but not exact

$\not\Rightarrow$ $\varphi \notin$ Ham

what if $\exists$ some other φ^t_H from id to φ?



E.g. $a = 1$, $\varphi = \mathrm{id}$ by $X = \frac{\partial}{\partial x} \neq 0$

But can take $H = 0$

In fact, in this case $\varphi \notin$ Ham

and $\mathrm{Symp}_0/\mathrm{Ham} = H^1(\mathbb{T}^2; \mathbb{R})/H^1(\mathbb{T}^2; \mathbb{Z})$

$= \mathbb{R}^2/\mathbb{Z}^2$

$= \mathbb{T}^2$

Non-obvious: flux, cf McDuff-Salamon

Note: $\text{Ham v.f.} = \underbrace{C^\infty(M)}_{\substack{\text{Lie algebra with} \\ \text{Poisson bracket}}} / \mathbb{R}$ ← in the center

$$\{H, K\} := \omega(X_H, X_K) = -dH(X_K)$$

Ex.

- Check the Jacobi id
- Prove that

$$H \longmapsto X_H$$

is a Lie alg homo: $\{H,K\} \mapsto [X_H, X_K]$

$$C^\infty(M) \longrightarrow \text{Ham. v.f.}$$

- For $\mathbb{R}^{2n}$

$$\left\{ \begin{matrix} \text{quadratic} \\ \text{forms} \end{matrix} \right\} \xrightarrow{\cong} sp(2n)$$

§5 Submanifolds of symplectic manifolds

Lecture 7

10/14-2021

Linear algebra

(V^{2n}, ω) symplectic v.s. : $\mathbb{R}^{2n} = \mathbb{C}^n$, $i = J$

$L \subset V$ linear subspace, $d = \dim L$

Def symplectic orthogonal

$$\boxed{L^\omega = \{X \in V \mid \omega(X, Y) = 0 \ \forall Y \in L\}}$$

Obvious properties

- $\dim L^\omega = 2n - d$
- $(L^\omega)^\omega = L$

Def
- L is isotropic if $L \subset L^\omega \iff \omega|_L = 0 \implies d \leq n$
- L is coisotropic if $L^\omega \subset L \implies d \geq n$
- L is Lagrangian if $L = L^\omega$ (coiso & iso) $\implies d = n$
- L is symplectic if $\omega|_L$ is non-deg $\iff L^\omega \cap L = 0$

most important

Ex.
- $\dim L = 1 \Rightarrow$ isotropic
- $\operatorname{codim} L = 1 \Rightarrow$ coisotropic
- $L \subset \mathbb{C}^n$ complex $\Rightarrow$ symplectic
  $JL = L \;\; \not\Leftarrow$
- L Lagr $\Rightarrow$ L is real: $JL \cap L = 0$

Prop Given $L \Rightarrow$

$\exists$ Darboux basis $e_1, f_1, \dots, e_n, f_n$ s.t.

- L isotropic : $L = \operatorname{span}(e_1, e_2, \dots, e_d)$
- L coiso : $L = \operatorname{span}(e_1, \dots, e_n, f_1, \dots, f_k)$
- L Lagr : $L = \operatorname{span}(e_1, \dots, e_n)$
- L sympl : $L = \operatorname{span}(e_1, f_1, \dots, e_k, f_k)$

Cor All Lagr. subspaces a conj. by $Sp(2n)$
(Likewise for other types with d fixed)

Rmk. $V = L \oplus L' \leftarrow$ Lagr

$$\left.\begin{aligned} \Rightarrow \quad L' &\cong L^* \\ X &\mapsto i_X \omega|_L \end{aligned}\right\} \Rightarrow V = T^*L = L \times L^*$$

Ex. L coisotropic $\supset L^\omega$

L/L^ω symplectic : ω_{red}

$$\omega_{red}(X, Y) = \omega(\underbrace{\tilde{X}, \tilde{Y}}_{\text{lifts}})$$

More generally : $L / L \cap L^\omega$ is symplectic

Lagr. Grassmannian

$\Lambda = \{ L \subset \mathbb{R}^{2n} \mid \text{Lagr} \}$

Λ manifold

$\mathbb{R}^{2n} = \mathbb{C}^n$

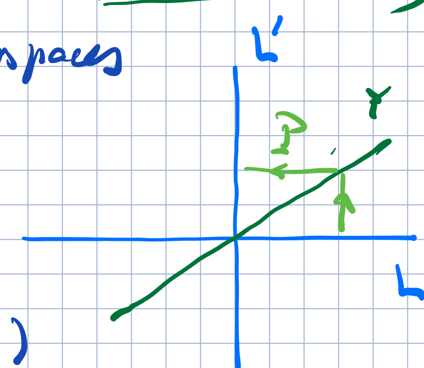
$\cup$

$\mathbb{R}^n$

$\boxed{\text{Discuss first general Grassmannians}}$

charts : $L \oplus L' = \mathbb{R}^{2n}$ - fixed

from some collection
e.g. coordinate subspaces



$\mathcal{U} = \mathcal{U}_{L,L'} = \{ Y \pitchfork L' \mid Y \text{ Lagr} \}$

$Y = \text{Graph}(P : L \to L' = L^*)$

$\{ P : L \to L^* \} \longleftrightarrow \{ \text{bilinear forms } \beta \text{ on } L \}$

$P \longleftrightarrow (x,y) \mapsto P(x)(y) = \beta$

$Y \text{ Lagr} \iff \beta \text{ is symmetric}$

$\mathcal{U}_{L,L'} \longleftrightarrow$ quadratic forms on L
(symmetric matrices)

$\Rightarrow \dim \Lambda = \dfrac{n(n+1)}{2}$