§ 3. Hamiltonian Dynamics: Definitions, Basic facts, Examples

Lecture 4

10/05-2021

(M^{2n}, ω) symplectic

- $H: \mathbb{R} \times M \longrightarrow \mathbb{R}$ — Hamiltonian
  t $\qquad H(t, \cdot) = H_t$
- Autonomous if ind of t: $H: M \longrightarrow \mathbb{R}$
- often 1-periodic in t, $S^1 = \mathbb{R}/\mathbb{Z}$
  $H: S^1 \times M \longrightarrow \mathbb{R}$ $\qquad H_{t+1} = H$

Def. • Hamiltonian v.f. generated by H:
$$i_{X_H} \omega = -dH \quad \exists! \, X_H - \text{non-deg}$$
• $X_H \rightsquigarrow$ "$\underbrace{\text{time-dependent flow}}_{\text{isotopy}}$"

time dependent

$\varphi_H^t \leftarrow$ Hamiltonian flow generated by H

Need not be defined for all t, and is not sometimes (collisions) but we will assume it is. (E.g. M is compact, etc)

Rmk. Doing dynamics, usually interested in • φ_H^t, $t \in \mathbb{R}$, H autonomous
or φ_H^{kT}, $k \in \mathbb{N}$, H time-dependent

Examples

Ex 1. $\mathbb{R}^{2n}$, $\omega_{st} = dp\wedge dq = \sum dp_i \wedge dq_i$

$$\begin{cases} \dot p = -\dfrac{\partial H}{\partial q} \\ \dot q = \dfrac{\partial H}{\partial p} \end{cases} \iff X_H = -\frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}$$

Checking $\quad i_{X_H}\omega = -\frac{\partial H}{\partial q}dq - \frac{\partial H}{\partial p}dp = -dH$

Subexample $\quad \mathbb{R}^{2n} = T^*\mathbb{R}^n = \mathbb{R}^n_q \times \mathbb{R}^n_p$

$H = \frac{1}{2m}\|p\|^2 + V(q)$ = kinetic + potential

$$\begin{cases} \dot p = -\dfrac{\partial V}{\partial q} \iff \boxed{m\ddot q = -\nabla V} \leftarrow \text{cons force} \\ \dot q = \dfrac{1}{m} p \iff p = m\dot q \leftarrow \text{momentum} \end{cases}$$

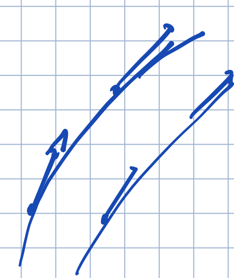
Newton's eq

2. **Cotangent bundle** $\quad M = T^*Q$, ω_{st}

Fix R.M. on Q $\quad TQ \longleftrightarrow T^*Q$
$\langle , \rangle$ $\quad v \longleftrightarrow \langle v, \cdot \rangle$

$H = \frac{1}{2}\langle , \rangle : T^*Q \longrightarrow \mathbb{R}$

X_H = geodesic spray
φ^t_H = geodesic flow
Describe.



Motion of a free particle on Q.

3. Twisted cotangent bundle

$$\begin{array}{ll} M = T^*Q \\ \quad \downarrow \pi \\ \quad Q \end{array}, \qquad \omega = \omega_{st} + \pi^*\sigma$$

magnetic field

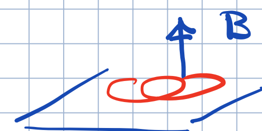
Q, σ, $d\sigma = 0$

$H = \frac{1}{2}\langle \, , \, \rangle$

The flow governs the motion of a charge on Q in magnetic field σ.

Subexample a) $Q = \mathbb{R}^2$, $\sigma = B \, dq_1 \wedge dq_2$

conf space $\to (q_1, q_2)$



unit charge, unit mass

$\vec{B} \perp (q_1, q_2)$ plane, charge in $\mathbb{R}^2 (x, y)$ $\qquad q_1, q_2 = (x, y)$

H.E. $\iff$ $\ddot{q} = B(q) \, J \dot{q}$; $\quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b) $Q = \mathbb{R}^3$, $\vec{B}$ = v.f. on $\mathbb{R}^3$ $\leftarrow$ magn. field

$\sigma = i_{\vec{B}} \, dq_1 \wedge dq_2 \wedge dq_3$

$d\sigma = 0 \iff \operatorname{div} B = 0$ $\Big\}$ one of the maxwell eq

H.E. $\iff$ $\ddot{q} = \dot{q} \times \vec{B}(q)$ $\leftarrow$ Lorentz force

(unit charge, unit mass)

Energy and ω-conservation

Let φ_H^t be the Ham flow of H_t.

Prop

essential

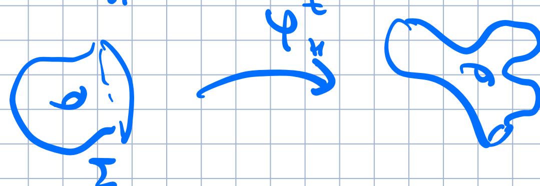
(a) Energy conservation

Assume that H is autonomous. Then

$$(\varphi_H^t)^*H = H \quad : \quad H(\varphi_H^t(p)) = \text{const} \quad \forall p$$

(b) "Integral invariant": φ_H^t is symplectic

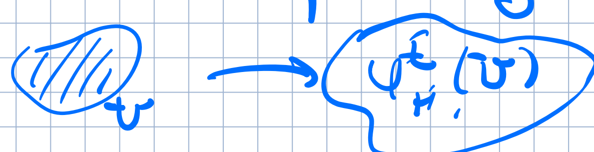
$$(\varphi_H^t)^*\omega = \omega$$



$\varphi_H^t(\Sigma) = \Sigma'$

$$\int_{\Sigma'}\omega = \int_{\Sigma}\omega$$

Cor. φ_H^t is vol. preserving: $(\varphi_H^t)^*\omega^n = \omega^n$



$$\int_U \omega^n = \int_{\varphi_H^t(U)} \omega^n$$

$\Rightarrow$ Huge restrictions on dynamics

Pf. (a) $\frac{d}{dt} H(\varphi_H^t(p)) = (L_{X_H} H)(\underbrace{\varphi_H^t(p)}_{x})$

$$= (i_{X_H} d\cancel{dH}^{\,0} + d\, i_{X_H} dH)(\ldots)$$

p. 11a

$$= d(-\underbrace{i_{X_H} i_{X_H}\omega}_{HE!}) = d\,\overbrace{\omega(X_H, X_H)} = 0$$

$i_{X_H}\omega = -dH$

(b) $\frac{d}{dt}(\varphi_H^t)^* \omega = (\varphi_H^t)^* L_{X_H} \omega$ see p 11a

$= (\varphi_H^t)^* \left(d i_{X_H} \omega + i_{X_H} \cancel{d\omega} \right)$

$= (\varphi_H^t)^* d\underbrace{(-dH)}_{HE} = 0$

$\triangleleft$

Ex. Prove the Prop using a direct calculation in Darboux coordinates.

Cor $\dim M = 2 \leftarrow$ surface

H autonomous

sol of ODE

$\Rightarrow$ "integral curves of φ_H^t (unparametrized) are" levels $\underbrace{H = \mathrm{const}}_{\text{"alg equation"}}$.

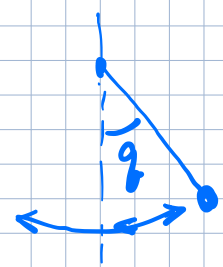
Rmk Newtonian mechanics:

- $\dot{q} = F(q)$ Energy consv $\Leftarrow \begin{cases} F \text{ is consv:} \\ F = -\nabla V \leftarrow \text{indep of } t \end{cases}$
- $\ddot{q} = F(q, \dot{q})$ as in Lorentz

  Energy conservation $\Leftarrow F \perp \dot{q}$

Continuing Examples

4. Investigating the pendulum



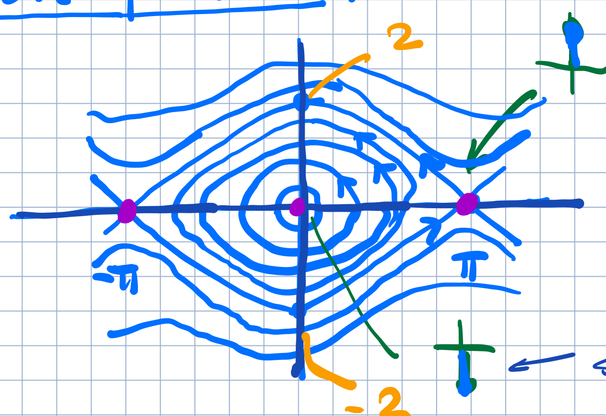
$M = T^*S^1 = \mathbb{R} \times S^1$
$\quad\quad\quad\quad\quad\quad p \quad q \leftarrow \mod 2\pi$

or $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$
$\quad\quad\quad\quad\quad p \quad q$

$H = \frac{1}{2}|p|^2 - \cos q + 1$

$= \underbrace{\frac{1}{2}(p^2 + q^2)}_{} + \cdots \leftarrow$ Harmonic oscillator

Phase portrait:



$\leftarrow$ unstable

$\leftarrow$ stable

$m = 1,\ k = 1$

$\ddot{q} = -\sin q$

Hooke's law:

$\ddot{q} = -q$

Gives the behavior of integral curves up to parametrization

Further details - Ex (Not easy)

(a) $q \in (0, \pi)$ $\quad H(0, q) = \cos q - 1 = h$

integral curve through $\underbrace{(q, 0)}_{x}$ is $\{H = h\}$

$T(h)$ its period

- Show that $T(h)$ monotone increasing function from $\underset{h=0}{2\pi}$ to $\underset{h=2}{\infty}$ as $h \to 2$

(Compare with the harmonic oscillator $H = \frac{1}{2}(p^2+q^2)$ & $T = \text{const}$)

- Find the Taylor exp of $T(h)$ at $h=0$

(b) Consider $D\varphi_H^T : T_x\mathbb{R}^2 \circlearrowleft$

Show that $D\varphi_H^T = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ $\lambda \neq 0$

(c) Find explicitly $\varphi_H^t(0,2)$ in elementary functions

Ex Hint to a): Area-period relation

$H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ proper

- $\{H \leq h\}$ connected
- h - regular

$A(h) = \text{area of } \{H \leq h\} = \int_{H \leq h} \omega$

$T(h) = \text{period of } \{H = h\}$

Show that $\frac{dA}{dh} = T(h)$

5. Positive-Def quadratic Ham

Lecture 5
10/07-2021

$\mathbb{R}^{2n} = \mathbb{C}^n \qquad z_j = p_j + i q_j \;, \; z = (z_1, \dots, z_n)$

$$H = \frac{1}{2} \sum \underset{\overset{\vee}{0}}{\lambda_j} (p_j^2 + q_j^2) = \frac{1}{2} \sum \lambda_j |z_j|^2$$

$\sim$ or just $\neq 0$

Governs n uncoupled oscillators with frequencies λ_j

$E = \{H = h\}$ is an ellipsoid

- Find X_H and show that $\varphi_H^t(z) = (e^{\lambda_1 i t} z_1, \dots, e^{\lambda_n i t} z_n)$
- "Coordinate axis" $(0, \dots, 0, z_j, 0, \dots 0) \cap E$ are periodic orbits of φ_H^t with $T_j = \dfrac{2\pi}{\lambda_j}$
- Are there other periodic orbits? (The answer depends on $(\lambda_1, \dots, \lambda_n)$.)

6. Linear HE

$M = \mathbb{R}^{2n}$ $\qquad \omega = \omega_{st} = dp \wedge dq$

$= \mathbb{C}^n$,

Matrix of ω:

$$J = \begin{pmatrix} \begin{array}{cc|} 0 & -1 \\ 1 & 0 \\ \hline \end{array} & & 0 \\ & \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & \\ 0 & & \ddots \end{pmatrix} \leftarrow \text{multiplication by } i \text{ in } \mathbb{C}^n = \mathbb{R}^{2n}$$

$H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ quadratic form

$H(x) = \frac{1}{2}\langle Ax, x\rangle$, $\quad A^T = A$

$A = \nabla H$

HE: $\qquad \dot{x} = J\nabla H(x) = JAx = X_H(x)$

$\varphi_H^t(x) = \exp(tJA)x$ $\qquad$ Discuss a bit Lie alg & Lie gps

$\exp : sp(2n) \longrightarrow Sp(2n)$

$X_H \longmapsto \varphi_H$

$sp(2n) =$ lin Ham v.f. $\qquad XJ + JX^T = 0$

$=$ quadratic forms on $\mathbb{R}^{2n}$

$X \longrightarrow H = -\frac{1}{2}\langle JXx, x\rangle$

Normal forms

$\boxed{\text{Discuss linear alg } Sp(2n) \quad sp(2n) \text{ vs } SL}$

- solving ODE's: $\dot{x} = Px$
  Bring A to a Jordan form
  to calculate $\exp(Pt)x$

$$\underbrace{PJP^T = J}_{P \in Sp(2n)}$$

- symplectic normal forms are more complicated

  Q symmetric

$$\underset{\substack{\cap \\ GL(n)}}{S} Q S^T \rightsquigarrow \mathrm{diag}(1, \dots, 1, 0 \dots 0)$$

$$\underset{\substack{\cap \\ O(n)}}{S} Q S^T \rightsquigarrow \mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

$$\underset{\substack{\cap \\ Sp(2n)}}{S} Q S^T \rightsquigarrow$$ much more complicated list

Not just $\sum \lambda_j |z_j|^2$

__Rmk.__ Unless $A > 0$, then it can be diagonalized