Symplectic Geometry, Math 248
2021 F

Lecture 1
09/23 - 2021

→ Go through basic info

* No exams, no hw
* Problems stated in lectures

Up to them how much they take home

* OH: TBA

→ why is SG important? Two aspects:

* Math Language (like algebra or diff. manifolds) connecting disparate areas:
  - classical mechanics, quantum physics, diff geometry, string and mirror theory etc ... Keep coming up
* Genuinely deep results ...

In this class - always a choice - a bit of both ...

Math 248, Symplectic Geometry, Fall 2021

- **Lectures:** TTh 9:50 AM - 11:25 AM, McHenry Clrm 4130
- **Instructor:** Viktor Ginzburg; office: McHenry 4124
  email: ginzburg(at)ucsc.edu
- **Office Hours:** TBA or by appointment
- **Text:** There will be no "official" textbook in this course. Suggested reading:
  - *Introduction to Symplectic Topology* by Dusa McDuff and Dietmar Salamon,
  - *Lectures on Symplectic Geometry* by Ana Canas da Silva,
  - *Morse Theory and Floer Homology* by Michelle Audin and Mihai Damian
- **Tentative Syllabus:** The course will cover fundamentals from symplectic geometry and touch upon Morse theory with an eye on applications of modern symplectic topological techniques to Hamiltonian dynamics. We will begin with an (ideally, brief) discussion of basic concepts of symplectic geometry: symplectic manifolds, Hamiltonian diffeomorphisms and flows, Lagrangian submanifolds, the least action principle, etc. We will also introduce several classes of dynamical systems of interest, such as geodesic flows and twisted geodesic (or magnetic) flows, and formulate the main problems in dynamics (e.g., Arnold's and Weinstein's conjectures, i.e., the existence of fixed points and periodic orbits) studied by symplectic techniques. Then we turn to a very brief review of Morse theory. In contrast with previous iterations of this course, this time I plan to focus more on Lagrangian submanifolds -- one of the most fundamental objects in symplectic geometry. Time permitting, we will touch upon symplectic topological methods (e.g., Lagrangian and Hamiltonian Floer homology) and/or conclude the course with student presentations.

  It should be said that this is not a comprehensive course in symplectic geometry and many important concepts (mainly those concerning symmetries) will be entirely omitted or just briefly mentioned.

  COVID-19 Information: Please take care to comply with all university guidelines about masking in indoor settings, performing daily symptom and badge checks, testing as required by the campus vaccine policy, self-isolating in the event of exposure, and respecting others’ comfort with distancing. Please do not come to class if your badge is not green. If you are ill or suspect you may have been exposed to someone who is ill, or if you have symptoms that are in any way similar to those of COVID-19, please err on the side of caution and stay home until you are well or have tested negative after an exposure.

§1. Symplectic manifolds
– Def's and basic examples

Origins – Hamiltonian dynamics
to be discussed later

Def A real finite dim symplectic v.s.

(V, ω) ← skew-symmetric form

$\omega : V \times V \longrightarrow \mathbb{R} \qquad \omega(X, Y) = -\omega(Y, X)$

* non-degenerate

- $\forall X \neq 0 \, \exists Y : \quad \omega(X, Y) \neq 0$
- $e_1, \dots, e_m$ basis $\quad \omega = \sum \omega_{ij} \, e_i^* \wedge e_j^*$

$\det \omega_{ij} \neq 0$

$\Rightarrow \quad \dim V = \text{even} = 2n$:

Pf: $\det \omega = \det \omega^T = \det(-\omega) = (-1)^{\dim V} \det(\omega)$

- $\omega^\# : V \xrightarrow{\cong} V^*$

Ex $\exists$ basis $v_1, w_1, \dots, v_n, w_n$ "Darboux"

s.t. $\omega = \sum v_i^* \wedge w_i^*$

$$\text{Matrix}(\omega) = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & 0 \\ & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \\ 0 & & \ddots \end{pmatrix}$$

"Linear Darboux Theorem"

Def $(M^m, \omega) \leftarrow$ symplectic manifold

$\omega \leftarrow$ symplectic form

$\omega \in \Omega^2(M)$

- $d\omega = 0$
- ω non-deg: every T_pM is a s.v.s.
  - $\omega^\#: TM \xrightarrow{\cong} T^*M$
    $X \longmapsto i_X \omega$
  - $\omega = \sum \omega_{ij}\, dx_i \wedge dx_j \leftarrow$ locally
    $\det(\omega_{ij}) \neq 0$

Note $\Rightarrow$ $\dim M = \text{even} = 2n$

Non-deg $\Leftrightarrow$ $\omega^n \neq 0$

Examples

0. <u>(V, ω) symplectic v.s</u>

$$\cong (\mathbb{R}^{2n}, \omega_{st}) \; ; \quad \omega = \sum dp_i \wedge dq_i = \text{"}dp \wedge dq\text{"}$$

$(p_1, \dots, p_n, q_1, \dots, q_n)$

- obviously $d\omega = 0$

$\omega^n \neq 0 \Leftarrow dp_1 \wedge \dots \wedge dp_n \wedge dq_1 \dots$

Standard s.s. on $\mathbb{R}^{2n}$

1. <u>$\mathbb{T}^{2n} = \mathbb{R}^{2n} / \mathbb{Z}^{2n}$</u> same formula

$p_1, \dots, q_n$ mod 1

Or ω_{st} is transl inv $\Rightarrow$ descends to $\mathbb{T}^{2n}$

2. <u>M^2 orientable surface</u> $\supset$ orientability

<u>ω = area form</u> : $\omega \neq 0 \leftrightarrow$ non. deg

$d\omega = 0$ – $\dim M = 2$

Can be associated with a R. metric

3. <u>Kähler manifolds</u> skew

M, $\langle , \rangle_{\mathbb{C}} = (,) + i\omega(,)$ $\Uparrow$

complex r.s. symmetric

- ω non.deg $\Leftarrow \langle , \rangle_{\mathbb{C}}$ non.deg
- $d\omega = 0 \Longleftrightarrow$ Kähler

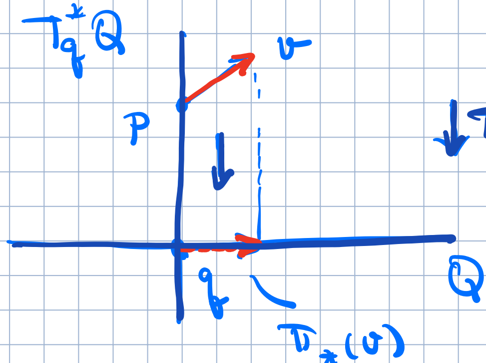
Hermitian

To be discussed in detail latter

4. Cotangent bundles { first time confusing

$M = T^*Q$ contest a canonical s.f.

Construction



$\pi : T^*Q \longrightarrow Q$

$$\lambda_p(v) = \underset{T^*_q Q}{\underset{\cap}{p}}(\underset{T_q Q}{\underset{\cap}{\pi_* v}})$$

in $T_p T^* Q$

$\Rightarrow$ Get $\lambda \in \Omega^1(T^*Q)$

By def: $\omega = d\lambda$. $(d^2 = 0 \Rightarrow d\omega = 0)$

Non-degeneracy – write λ in local coordinate

Local expression

$\begin{cases} q_1, \dots, q_n \leftarrow \text{local coord on } Q \\ p_1, \dots, p_n \leftarrow \text{"dual coord"} : \underbrace{T^*Q}_{\text{subset}} \longrightarrow \mathbb{R} \end{cases}$

$\alpha = p_1(\alpha)\, dq_1 + \dots + p_n(\alpha)\, dq_n$

$\Rightarrow$ coord on T^*Q (should be $q_i \circ \pi$ ...)

$\Rightarrow \boxed{\lambda = \sum p_i\, dq_i}$ (*)

Then $\omega = \sum dp_i \wedge dq_i$ as for $\mathbb{R}^{2n}$

$\Rightarrow$ non-degeneracy

$\boxed{\text{Rmk. } \mathbb{R}^{2n} = T^*\mathbb{R}^n}$

Pf of (*)

$$v = \sum a_i \frac{\partial}{\partial q_i} + \cancel{b_i \frac{\partial}{\partial p_i}}$$

killed by π_*

$\pi : (p,q) \longmapsto q$

$$\alpha = \sum p_i \, dq_i \longleftarrow \text{dd of } p_i\text{'s}$$

$$\pi_*(v) = \sum a_i \frac{\partial}{\partial q_i}$$

$$\underbrace{\lambda(v)}_{\alpha(\pi_* v)} = \sum p_i a_i = \sum p_i \, dq_i(v)$$

$\triangleleft$

5. Twisted cotangent bundle

$$\left(T^*Q, \ \omega = \underbrace{d\lambda}_{\text{standard}} + \pi^*\sigma\right)$$

$\sigma \in \Omega^2(Q)$

$d\sigma = 0$

$$d\omega = d(d\lambda + \pi^*\sigma) = 0$$

Non-deg: $\omega = \sum dp_i \wedge dq_i + \overbrace{\sum_{i,j} \sigma_{ij} \, dq_i \wedge dq_j}^{\sigma}$

$$\begin{array}{c|c} & \begin{array}{cc} p & q \end{array} \\ \begin{array}{c} p \\ q \end{array} & \left[\begin{array}{c|c} 0 & -I \\ \hline -I & \sigma \end{array}\right] \end{array} \longleftarrow \text{non-deg no matter what } \sigma \text{ is}$$

• More examples later

Non-examples

- To admit a s.f. M has to be orientable:

ω sympl $\Rightarrow$ $\underbrace{\omega^n \neq 0 \leftarrow \text{"volume form"}}_{\text{orientation}}$

- $\boxed{\omega \text{ symplectic}, \ M \text{ closed} \Rightarrow [\omega] \neq 0 \text{ in } H^2(M;\mathbb{R})}$ (**)

compact $\partial M = \emptyset$

Cor. $S^{2n \geq 4}$ does not admit a sympl. form

Pf. Assume not: $[\omega] = 0$: $\omega = d\lambda$

$$\int_M \omega^n = \int_M (d\lambda)^n = \int_M d\left(\lambda \wedge (d\lambda)^{n-1}\right)$$

$$= \int_{\partial M = \emptyset} \lambda \wedge d\lambda^{n-1} = 0$$

But $\omega^n \neq 0 \Rightarrow \int_M \omega^n \neq 0$ ← sign depends on the orientation

◁

- More to follow...

Lecture 8

09/28-2021

Symplectomorphisms

- $\varphi : (M_0, \omega_0) \longrightarrow (M_1, \omega_1)$ is symplectic if $\varphi^* \omega_1 = \omega_0$ $\Big\}$ $\overset{Ex}{\Longrightarrow}$ φ is an immersion, $\mathrm{rk}\, D\varphi = \dim M_0$

$\Rightarrow$: $\dim M_0 = \dim M_1 \Rightarrow \varphi$ is a local diffeo

- $\varphi : (M_0, \omega_0) \longrightarrow (M_1, \omega_1)$ is a symplectomorphism if φ is a diffeo & $\varphi^* \omega_1 = \omega_0$

- Group $\mathrm{Sympl}(M, \omega) < \mathrm{Diff}_{\omega^n}(M)$

symplectomorphism $\Rightarrow$ vol preserving

$$\varphi^* \omega_1^n = \omega_0^n$$

Discussion: Gromov's thm

symplectic vs volume preserving

§2 Darboux and Moser's thm's

Darboux: all symplect forms (of the same dim) are locally symplectomorphic

More rigorously

Thm (Darboux) (M^{2n}, ω) symplectic, $M \ni p$

$\exists$ nbd. $U \ni p$ and a diffeo
$\varphi : (U, x) \longrightarrow (B^{2n}, 0)$ s.t. $B^{2n} \subset \mathbb{R}^{2n}$

$\omega = \varphi^* \omega_{st}$

$\Updownarrow$

Thm' (Darboux) (M_0^{2n}, ω_0) & (M_1^{2n}, ω_1) sympl, $M_0 \ni x_0$, $M_1 \ni x_1$

$\Rightarrow$ $U_0 \ni x_0$ & $U_1 \ni x_1$ and a diffeo
$\varphi : (U_1, x_1) \longrightarrow (U_0, x_0)$ s.t.
$\omega_1 = \varphi^* \omega_0$

Rmk Contrast with diff geometry: sympl str's are more like diff str's !

Pf Moser's homotopy method – extremely important

- Result is local can assume

$$M^{2n} = \mathbb{R}^{2n}, \quad x = 0$$

- Linear Darboux theorem $\omega_0 = \omega$ on $T_0\mathbb{R}^{4} = \mathbb{R}^{2n}$ can be made standard by a lin transf.
- Consider $\omega_t = (1-t)\omega_0 + t\omega$
  - ω_t at 0 is $\omega_0 \Rightarrow$ sympl on a nbd of 0
  - $\omega_0 \overset{\omega_t}{\rightsquigarrow} \omega_1 = \omega$
- Looking for φ_t : nbd of $0 \longrightarrow$ nbd of 0

$$\omega_0 = \varphi_t^* \omega_t$$

Then φ_1 does the job

- φ_t is generated by the time dependent v.f. v_t : $\frac{d}{dt}\varphi_t(x) = v_t(\varphi_t(x))$

Discuss?

Digression: isotopies, etc → p. 11

Looking for $v_t \rightsquigarrow \varphi_t$

uniqueness and existence of solutions of ODE

- $\frac{d}{dt}$ of $\varphi_t^* \omega_t = 0$

$$\varphi_t^* L_{v_t}\omega_t + \varphi_t^* \frac{d}{dt}\omega_t = 0$$

Apply $(\varphi_t^*)^{-1}$: $\quad L_{v_t}\omega_t + \frac{d}{dt}\omega_t = 0 \quad (*)$

$$L_{v_t}\omega_t = \underbrace{\cancel{i_{v_t} d\omega_t}}_{d\omega=0} + d i_{v_t}\omega_t$$

$$(*) \iff d i_{v_t}\omega_t = -\frac{d}{dt}\omega_t = \underbrace{\omega_1 - \omega_0}_{d\lambda}$$

$\omega_t = (1-t)\omega_0 + t\omega_1$

Poincaré

$$\Leftarrow \quad i_{v_t}\omega_t = \lambda$$

$$\iff v_t = (\omega_t^{\#})^{-1}\lambda \leftarrow \text{Non-degeneracy}$$

<u>Nuance</u>: need to know that φ_t is defined for $t \in [0,1]$

$\iff$ solutions of $\dot{x} = v_t(x)$ with initial conditions near 0 exist for $[0,1]$

Does <u>not</u> follow automatically from existence & uniqueness

<u>From ODE's sufficient to have</u> $v_t(0) = 0 \quad \forall t$

$\Leftarrow \lambda|_0 = 0$

Modify λ: $\lambda \rightsquigarrow \lambda - \lambda_0$ ← linear extension

$\triangleleft$

<u>**Rmk.**</u> Stark contrast with Riemannian geometry!
- ω does not have local invariants
- R.m. (symmetric tensors) do: curvature

Extra discussion: time dependent v.f.

- v ind of time $\rightsquigarrow$ φ^t flow
  - $\varphi^{t_1+t_2} = \varphi^{t_1} \varphi^{t_2}$
  - $\varphi^0 = id$

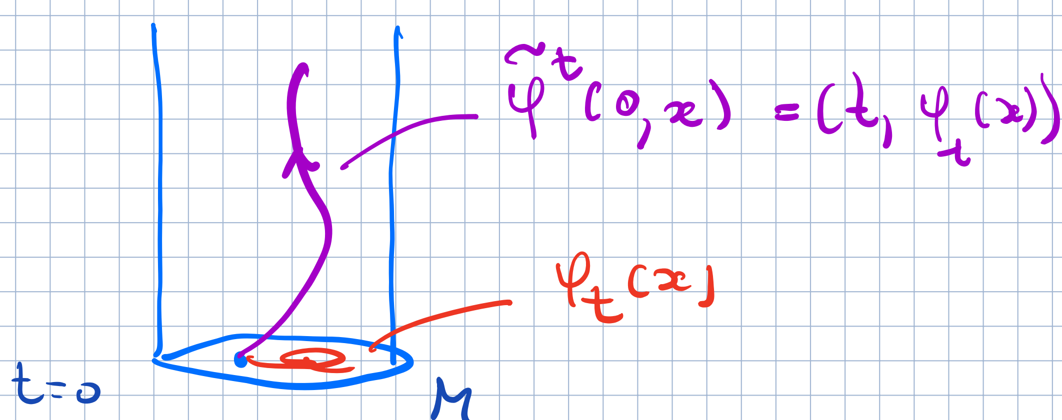
$t \mapsto \varphi^t(x)$ = sol of $\dot{x} = v(x)$
with $\varphi^0(x) = x$

- v_t depends on t $\rightsquigarrow$ φ_t isotopy
  $\varphi_0 = id$

Construction: pass to $\tilde{M} = \mathbb{R} \times M$
ind of time $\longrightarrow$ $\tilde{v} = \frac{\partial}{\partial t} + v_t$

flow $\tilde{\varphi}^t$ then $\tilde{\varphi}^t(0,x) = (t, \varphi_t(x))$

or $t \mapsto \varphi_t(x)$ is a sol of $\dot{x} = v_t(x)$
with initial condition x at $t=0$



Global Rigidity: Moser's thm's

Lecture 3
09/30-21

- Volume form = non-vanishing top deg form
  - E.g. ω symplectic $\Rightarrow$ ω^n vol. form
- η_1 & η_0 vol. form $\quad \eta_1 = \underset{\substack{\neq\\0}}{f}\,\eta_0$

  $f > 0$: η_1 & η_0 have the same sign

  $f < 0$ —— . —— . - opposite signs

- Existence of a vol. form $\Rightarrow$ orientability
- $\int_M : H^m(M) \xrightarrow{\cong} \mathbb{R}$ ← Discuss

Moser: total volume is the only inv of a volume form

Discuss: Local Moser = Darboux for vol forms

Easy by a direct pf

Go through

Thm (Moser) M closed (orientable)

η_1 & η_0 vol. forms and

$$\int_M \eta_1 = \int_M \eta_0 \quad (\Rightarrow \text{ same sign})$$

$\Rightarrow$ $\exists$ $\varphi : M \circlearrowleft$ diffeo: $\eta_1 = \varphi^* \eta_0$

Pf Set

$$\eta_t = (1-t)\eta_0 + t\eta_1 \leftarrow \text{all volume forms}$$

$$= \underbrace{(1-t+tf)}_{\neq 0}\eta_0 \qquad \Uparrow \text{ same sign}$$

Note $\int_M \eta_t = \text{const} \Leftrightarrow [\eta_t] = \text{const}$ Discuss

As before: looking for φ_t $\leftarrow$ generated by v_t

$$\varphi_t^* \eta_t = \eta_0$$

$$\frac{d}{dt}: \quad \varphi_t^* \mathcal{L}_{v_t} \eta_t + \varphi_t^* \frac{d}{dt}\eta_t = 0$$

$$\underbrace{\mathcal{L}_{v_t}\eta_t}_{d i_{v_t}\eta_t} = -\frac{d}{dt}\eta_t = \underbrace{\eta_0 - \eta_1}_{\int_M(\eta_0-\eta_1)=0} = d\lambda$$

$\Uparrow$ Discuss

$$i_{v_t}\eta_t = \lambda \in \Omega^{n-1}(M) \qquad (*)$$

Ex: linear alg

V^n v.s. $\quad \eta \in \Lambda^n V^* \neq 0$ vol. form

$$V \xrightarrow{\cong} \Lambda^{n-1}V^* \quad \text{isomorphism}$$
$$v \longmapsto i_v\eta$$

$\Rightarrow \exists!\ v_t$ solving $(*)$

M closed $\Rightarrow$ the flow exists for $t \in [0,1]$,

φ_1 does the job

$\triangleleft$

Rmk. Rigidity in general: deforming a str results in an equivalent str.

E.g. η_t family of vol. forms, M closed

$[\eta_t] = \text{const} \Rightarrow \exists\, \varphi_t : \varphi_t^*\eta_t = \eta_0$

Difficulty: Hodge theory $\longrightarrow$ Discuss

Similar rigidity for symplectic forms

But $\exists$ some complications:

(1) ω_1 & ω_0 symplectic on M^{2n}

$\not\Rightarrow$ $\omega_t = (1-t)\omega_0 + t\,\omega_1$ symplectic:
sum of non.deg matrices need not be non-deg

Thm (Moser)

- M^{2n}, $\omega_t \leftarrow$ family of sympl. forms
  $\uparrow$ closed
- $[\omega_t] =$ const !

$\Rightarrow \exists\ \varphi : M \circlearrowleft$ diffeo s.t.
$\varphi^* \omega_1 = \omega_0$

On the pf: look for $\varphi_t^* \omega_t = \omega_0$

$\Leftarrow$ • $d\, i_{v_t} \omega_t = -\dot{\omega}_t := -\frac{d}{dt}\omega_t$

- $[\omega_t] = $ const $\Rightarrow$ all $\dot{\omega}_t$ exact
  discuss: cycles & $\frac{d}{dt}[\cdot] = [\frac{d}{dt}\cdot]$
- As before $-\dot{\omega}_t = d\lambda_t$
  $i_{v_t}\omega_t = \lambda_t \Rightarrow \dots$ as before

(2) Need λ_t to be smooth (or cont) in t
Not obvious at all.
Hodge theory or de Rham = Čech

Nothing like that can possibly be true for R.M.

Cor (Local rigidity of sympl. forms)

M closed

ω_0 symplectic

Assume that • ω is sufficiently close to ω_0

• $[\omega] = [\omega_0]$

$\Rightarrow \; \exists \; \varphi : M \circlearrowleft$ s.t. $\omega = \varphi^* \omega_0$

Pf

$\omega_t = (1-t)\omega_0 + t\omega_1$

• symplectic $\Longleftarrow$ $\begin{cases} \forall \text{ any two } X, Y \\ \quad \|X\| = \|Y\| = 1 \\ \omega(X,Y) \approx \omega_0(X,Y) \end{cases}$

• $[\omega_t] = \text{const}$

Apply Global Moser thm for symplectic forms

$\triangleleft$