Symplectic Geometry, Math 248
2020W

→ Go through basic info

* No exams, no hw
* Problems stated in lectures }
  Up to them how much they take home
* OH: Th 9:50 - 11:25

→ Student Presentations: Volunteers?

0

Math 248, Symplectic Geometry, Winter 2020

- **Lectures:** TTh 1:30-3:05pm, McHenry ~~1279~~ 1270

- **Instructor:** Viktor Ginzburg; office: McHenry 4124
  email: ginzburg(at)ucsc.edu

- **Office Hours:** TBA or by appointment

- **Text:** There will be no "official" textbook in this course. Suggested reading:
  - *Introduction to Symplectic Topology* by Dusa McDuff and Dietmar Salamon,
  - *Lectures on Symplectic Geometry* by Ana Canas da Silva,
  - *Morse Theory and Floer Homology* by Michelle Audin and Mihai Damian

- **Tentative Syllabus:** The course will cover fundamental from symplectic geometry and Morse theory with an eye on applications of modern symplectic topological techniques to Hamiltonian dynamics. We will begin with an (ideally, brief) discussion of basic concepts of symplectic geometry: symplectic manifolds, Hamiltonian diffeomorphisms and flows, Lagrangian submanifolds, the least action principle, etc. We will also introduce several classes of dynamical systems of interest, such as geodesic flows and twisted geodesic (or magnetic) flows, and formulate the main problems in dynamics (e.g., Arnold's and Weinstein's conjectures, i.e., the existence of fixed points and periodic orbits) studied by symplectic techniques. Then we turn to a review of Morse theory with applications to Hamiltonian circle-actions and homology calculations. Time permitting, we will touch upon symplectic topological methods (e.g., Hamiltonian Floer homology) and/or conclude the course with student presentations.

  It should be said that this is not a comprehensive course in symplectic geometry and many important concepts (mainly those concerning symmetries) will be entirely omitted or just briefly mentioned.

§1. Symplectic manifolds

– Def'ns and basic examples

Origins – Hamiltonian dynamics
to be discussed later

Def A real finite dim symplectic v.s. (V,ω)

ω ← skew-symmetric form

$\omega : V \times V \longrightarrow \mathbb{R} \qquad \omega(X,Y) = -\omega(Y,X)$

* non-degenerate
  - $\forall X \neq 0 \ \exists Y : \quad \omega(X,Y) \neq 0$
  - $e_1, \dots, e_m$ basis $\quad \omega = \sum \omega_{ij} e_i^* \wedge e_j^*$

    $\det \omega_{ij} \neq 0$

$\Rightarrow \quad \dim V = \text{even} = 2n$: $\quad \det \omega = \det \omega^T$

Pf $\qquad = \det(-\omega)$

$\qquad = (-1)^{\dim V} \det(\omega)$

- $\omega^\# : V \xrightarrow{\cong} V^*$

Ex $\exists$ basis $v_1, w_1, \dots, v_n, w_n$ "Darboux"

s.t. $\quad \omega = \sum v_i^* \wedge w_i^*$

"Linear Darboux Theorem"

$$\text{Matrix}(\omega) = \begin{vmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & 0 \\ & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \\ 0 & & \ddots \end{vmatrix}$$

Def (M^m, ω) ← symplectic manifold

ω ← symplectic form

$\omega \in \Omega^2(M)$

- $d\omega = 0$
- ω non-deg: every T_pM is a s.v.s.
  - $\omega^\#: TM \xrightarrow{\cong} T^*M$

    $X \longmapsto i_X\omega$
  - $\omega = \sum \omega_{ij}\, dx_i \wedge dx_j$ ← locally

    $\det(\omega_{ij}) \neq 0$

Note $\Rightarrow$ $\dim M = \text{even} = 2n$

Non-deg $\Leftrightarrow$ $\omega^n \neq 0$

Examples

0. (V,ω) symplectic v.s

$\cong (\mathbb{R}^{2n}, \omega_{st})$; $\omega = \sum dp_i \wedge dq_i$ = "$dp \wedge dq$"

$(p_1, \dots, p_n, q_1, \dots, q_n)$

- obviously $d\omega = 0$

$\omega^n \neq 0 \Leftarrow dp_1 \wedge \dots \wedge dp_n \wedge dq_1 \dots$

Standard s.s. on $\mathbb{R}^{2n}$

1. $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ same formula

$p_1, \dots, q_n$ mod 1

Or ω_{st} is transl inv $\Rightarrow$ descends to $\mathbb{T}^{2n}$

2. M^2 orientable surface $\Rightarrow$ orientability

ω = area form ; $\omega \neq 0 \leftrightarrow$ non. deg

$d\omega = 0$ - $\dim M = 2$

Can be associated with a R. metric

3. Kähler manifolds

M , $\langle , \rangle_{\mathbb{C}} = (,) + i\omega(,)$ skew

complex

Hermitian

1.s. symmetric

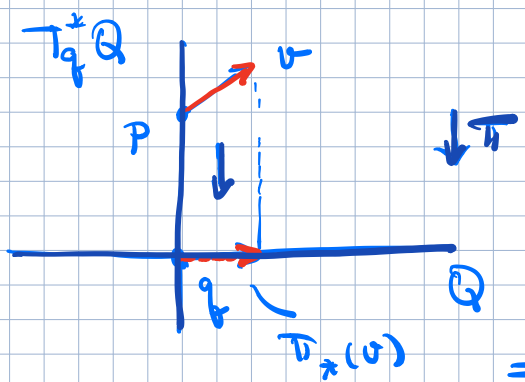
- ω non.deg $\Leftarrow$ $\langle , \rangle_{\mathbb{C}}$ non.deg
- $d\omega = 0 \Longleftrightarrow$ Kahler

To be discussed in detail latter

4. Cotangent bundles { first time confusing

$M = T^*Q$ carries a canonical s.f.

Construction



$\pi : T^*Q \to Q$

$$\underbrace{\lambda_p(v)}_{\text{in } T_p T^*Q} = \underbrace{p}_{\in T_q^*Q}(\underbrace{\pi_* v}_{\in T_qQ})$$

$\Rightarrow$ Get $\lambda \in \Omega^1(T^*Q)$

By def: $\omega = d\lambda$. $(d^2 = 0 \Rightarrow d\omega = 0)$

Non-degeneracy – write λ in local coordinates

Local expression

$q_1, \dots, q_n \leftarrow$ local coord on Q

$p_1, \dots, p_n \leftarrow$ "dual coord" : $\underbrace{T^*Q}_{\text{subset}} \to \mathbb{R}$

$\alpha = p_1(\alpha)\, dq_1 + \dots + p_n(\alpha)\, dq_n$

$\rightarrow$ coord on T^*Q (should be $q_i \circ \pi$ ...)

$\Rightarrow \boxed{\lambda = \sum p_i \, dq_i}$ $(*)$

Then $\omega = \sum dp_i \wedge dq_i \leftarrow$ as for $\mathbb{R}^{2n}$

$\Rightarrow$ non-degeneracy

$\boxed{\text{Rmk: } \mathbb{R}^{2n} = T^*\mathbb{R}^n}$

Pf of (*)

killed by π_*

$\pi : (p,q) \longmapsto q$

$$v = \sum a_i \frac{\partial}{\partial q_i} + \cancel{b_i \frac{\partial}{\partial p_i}}$$

$$\alpha = \sum p_i \, dq_i \longleftarrow \text{ind of } p_i\text{'s}$$

$$\pi_*(v) = \sum a_i \frac{\partial}{\partial q_i}$$

$$\underbrace{\lambda(v)}_{\alpha(\pi_* v)} = \sum p_i a_i = \sum p_i \, dq_i(v)$$

$\triangleleft$

5. Twisted cotangent bundle

$$\big(T^*Q,\ \omega = \underbrace{d\lambda}_{\text{standard}} + \pi^*\sigma\big)$$

$\sigma \in \Omega^2(Q)$

$d\sigma = 0$

$$d\omega = d(d\lambda + \pi^*\sigma) = 0$$

Non-deg: $\omega = \sum dp_i \wedge dq_i + \overbrace{\sum_{i,j} \sigma_{ij} \, dq_i \wedge dq_j}^{\sigma}$

$$\begin{array}{c|c} & \begin{array}{cc} p & q \end{array} \\ \begin{array}{c} p \\ q \end{array} & \left[\begin{array}{c|c} 0 & -I \\ \hline -I & \sigma \end{array}\right] \end{array} \longleftarrow \text{non-deg no matter what } \sigma \text{ is}$$

- More examples later

Non-examples

- To admit a s.f. M has to be orientable:

  ω sympl $\Rightarrow$ $\underbrace{\omega^n \neq 0 \leftarrow \text{"volume form"}}_{\text{orientation}}$

- $\boxed{\omega \text{ symplectic, } M \text{ closed} \Rightarrow [\omega] \neq 0 \text{ in } H^2(M;\mathbb{R})}$ (**)

  closed: compact, $\partial M = \emptyset$

Cor. $S^{2n \geq 4}$ does not admit a sympl. form

Pf. Assume not: $[\omega] = 0$: $\omega = d\lambda$

$$\int_M \omega^n = \int_M (d\lambda)^n = \int_M d\left(\lambda \wedge (d\lambda)^{n-1}\right)$$

$$= \int_{\partial M \neq 0} \lambda \wedge d\lambda^{n-1} = 0$$

But $\omega^n \neq 0 \Rightarrow \int_M \omega^n \neq 0$ $\leftarrow$ sign depends on the orientation $\triangleleft$

- More to follow...

§2 Darboux and Moser's thm's

<u>Darboux</u>: all symplect forms (of the same dim) are locally equivalent

More rigorously

<u>Thm</u> (Darboux) (M^{2n}, ω) symplectic, $p \in M$

$\exists$ nbd. $U \ni p$ and a diffeo $\varphi : (U, x) \longrightarrow (B^{2n}, 0)$ s.t. $B^{2n} \subset \mathbb{R}^{2n}$

$\omega = \varphi^* \omega_{st}$

$\Updownarrow$

<u>Thm</u>' (Darboux) (M_0^{2n}, ω_0) & (M_1^{2n}, ω_1) sympl, $x_0 \in M_0$, $x_1 \in M_1$

$\Rightarrow$ $U_0 \ni x_0$ & $U_1 \ni x_1$ and a diffeo $\varphi : (U_1, x_1) \longrightarrow (U_0, x_0)$ s.t.

$\omega_1 = \varphi^* \omega_0$

Pf Moser's homotopy method
- extremely important

- Result is local can assume
  $M^{2n} = \mathbb{R}^{2n}, \quad x = 0$
- Linear Darboux theorem $\omega_0 = \omega$ on $T_0\mathbb{R}^{4} = \mathbb{R}^{2n}$ can be made standard by a lin transf.
- Consider $\omega_t = (1-t)\omega_0 + t\omega$
  - ω_t at 0 is $\omega_0 \Rightarrow$ sympl on a nbd of 0
  - $\omega_0 \overset{\omega_t}{\rightsquigarrow} \omega_1 = \omega$
- Looking for φ_t^{t}: nbd of $0 \longrightarrow$ nbd of 0
  $\omega_0 = \varphi_t^* \omega_t$

Then φ_1 does the job

- φ_t is generated by the time dependent v.f. v_t: $\frac{d}{dt}\varphi_t(x) = v_t(\varphi_t(x))$

Discuss?

Looking for $v_t \rightsquigarrow \varphi_t$

uniqueness and existence of solutions of ODE

- $\frac{d}{dt}$ of $\varphi_t^* \omega_t = 0$

$$\varphi_t^* L_{v_t}\omega_t + \varphi_t^* \frac{d}{dt}\omega_t = 0$$

Apply $(\varphi_t^*)^{-1}$: $\quad L_{v_t}\omega_t + \frac{d}{dt}\omega_t = 0 \quad (*)$

$$L_{v_t}\omega_t = i_{v_t}\underbrace{\cancel{d\omega_t}}_{d\omega=0}^{0} + d\, i_{v_t}\omega_t$$

$$(*) \iff d\, i_{v_t}\omega_t = -\frac{d}{dt}\omega_t = \underbrace{\omega_1 - \omega_0}_{d\lambda}$$

$\omega_t = (1-t)\omega_0 + t\omega_1$ Poincaré

$$\Leftarrow \quad i_{v_t}\omega_t = \lambda$$

$$\iff v_t = (\omega_t^{\#})^{-1}\lambda \leftarrow \text{Non-degeneracy}$$

<u>Nuance</u>: need to know that φ_t is defined for $t \in [0,1]$

$\iff$ solutions of $\dot{x} = v_t(x)$ with initial condition near 0 exist for $[0,1]$

Does <u>not</u> follow automatically from existence & uniqueness

<u>From ODE's sufficient to have</u> $v_t(0) = 0 \quad \forall t$

$\Leftarrow \lambda|_0 = 0$

Modify λ: $\lambda \rightsquigarrow \lambda - \lambda_0$ ← linear extension

$\triangleleft$

<u>Rmk.</u> Stark contrast with Riemannian geometry: • ω does not have local invariants

• R.m. (symmetric tensors) do: curvature

Extra discussion: time dependent v.f.

- v ind of time $\leadsto$ φ^t flow
  - $\varphi^{t_1+t_2} = \varphi^{t_1}\varphi^{t_2}$
  - $\varphi^0 = id$

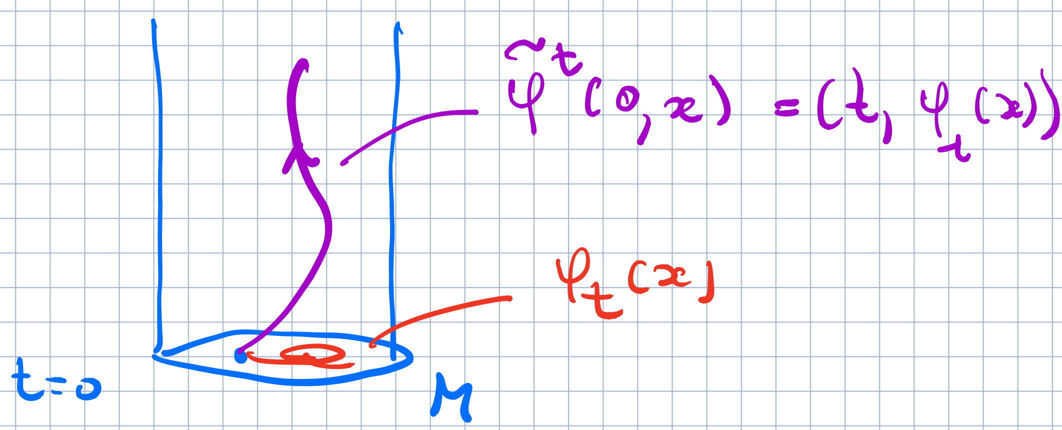
$t \mapsto \varphi^t(x)$ = sol of $\dot{x} = v(x)$
with $\varphi^0(x) = x$

- v_t depends on t $\leadsto$ φ_t isotopy
  $\varphi_0 = id$

Construction: pass to $\tilde{M} = \mathbb{R} \times M$
ind of time $\longrightarrow$ $\tilde{v} = \frac{\partial}{\partial t} + v_t$

flow $\tilde{\varphi}^t$ then $\tilde{\varphi}^t(0,x) = (t, \varphi_t(x))$

or $t \mapsto \varphi_t(x)$ is a sol of $\dot{x} = v_t(x)$
with initial condition x at $t=0$



Global Rigidity: Moser's thm's

- Volume form = non-vanishing top deg form
  - E.g. ω symplectic $\Rightarrow$ ω^n vol. form
- η_1 & η_0 vol. form $\quad \eta_1 = \underset{\neq 0}{f}\,\eta_0$

  $f>0$: η_1 & η_0 have the same sign

  $f<0$ —— . —— . . opposite signs
- Existence of a vol. form $\Rightarrow$ orientability
- $\int_M : H^m(M) \xrightarrow{\cong} \mathbb{R}$ ← Discuss

<u>Moser</u>: total volume is the only inv of a volume form

<u>Thm</u> (<u>Moser</u>) M closed (orientable)

η_1 & η_0 vol. forms and

$$\int_M \eta_1 = \int_M \eta_0 \quad (\Rightarrow \text{same sign})$$

$\Rightarrow$ $\exists$ $\varphi : M \circlearrowleft$ diffeo: $\eta_1 = \varphi^* \eta_0$

<u>Pf</u> Set

$$\eta_t = (1-t)\eta_0 + t\eta_1 \leftarrow \text{all volume forms}$$

$$= \underbrace{(1-t+tf)}_{\neq 0}\eta_0 \qquad \Uparrow \text{ same sign}$$

Discuss

Note $\int_M \eta_t = \text{const} \iff [\eta_t] = \text{const}$

As before: looking for $\varphi_t \leftarrow$ generated by v_t

$$\varphi_t^* \eta_t = \eta_0$$

$\frac{d}{dt}$: $\varphi_t^* \mathcal{L}_{v_t} \eta_t + \varphi_t^* \frac{d}{dt}\eta_t = 0$

$$\underbrace{\mathcal{L}_{v_t} \eta_t}_{} = -\frac{d}{dt}\eta_t = \underbrace{\eta_0 - \eta_1}_{} = d\lambda$$

$\mathrm{div}_{v_t} \eta_t$ $\qquad\qquad$ $\Uparrow$ $\int_M (\eta_0 - \eta_1) = 0$ Discuss

$\Uparrow$

$$i_{v_t} \eta_t = \lambda \in \Omega^{n-1}(M) \qquad (*)$$

Ex: linear alg

V^n v.s. $\qquad \eta \in \Lambda^n V^* \neq 0$ vol. form

$$V \xrightarrow{\cong} \Lambda^{n-1} V^* \qquad \text{isomorphism}$$
$$v \longmapsto i_v \eta$$

$\Rightarrow \exists!\ v_t$ solving $(*)$

M closed $\Rightarrow$ the flow exists for $t \in [0,1]$, φ_1 does the job

$\lhd$

<u>Rmk.</u> Rigidity in general: deforming a str results in an equivalent str.

<u>E.g.</u> η_t family of vol. forms, M closed

$[\eta_t] = \text{const} \Rightarrow \exists\ \varphi_t : \varphi_t^* \eta_t = \eta_0$

Difficulty: Hodge theory

Similar rigidity for symplectic forms
But $\exists$ some complications:

(1) ω_1 & ω_0 symplectic on M^{2n}

$\not\Rightarrow$ $\omega_t = (1-t)\omega_0 + t\,\omega_1$ symplectic:
sum of non.deg matrices need not be non-deg

<u>Thm</u> (Moser)

- M^{2n}, $\omega_t \leftarrow$ family of sympl. forms
  (M closed)
- $[\omega_t]$ = const !

Nothing like that can possibly be true for R.M.

$\Rightarrow$ $\exists$ $\varphi: M \circlearrowleft$ diffeo s.t.
$\varphi^*\omega_1 = \omega_0$

<u>On the pf:</u> look for $\varphi_t^*\omega_t = \omega_0$

$\Leftarrow$
- $d\, i_{v_t}\omega_t = -\dot\omega_t := -\frac{d}{dt}\omega_t$
- $[\omega_t]$ = const $\Rightarrow$ all $\dot\omega_t$ exact $\leftarrow$ discuss: cycles & $\frac{d}{dt}[\cdot] = [\frac{d}{dt}\cdot]$
- As before $-\dot\omega_t = d\lambda_t$

$i_{v_t}\omega_t = \lambda_t \Rightarrow \ldots$ as before

(2) Need λ_t to be smooth (or cont) in t
Not obvious at all.
Hodge theory or de Rham = Čech

§ 3. Hamiltonian Dynamics: Definitions, Basic facts, Examples

(M^{2n}, ω) symplectic

- $H : \mathbb{R} \times M \longrightarrow \mathbb{R}$ — Hamiltonian
  (t) $\quad H(t, \cdot) = H_t$
- Autonomous if ind of t: $H : M \longrightarrow \mathbb{R}$
- often 1-periodic in t, $S^1 = \mathbb{R}/\mathbb{Z}$
  $H : S^1 \times M \longrightarrow \mathbb{R} \qquad H_{t+1} = H$

Def. • Hamiltonian v.f. generated by H:

$$i_{X_H} \omega = - dH \qquad \exists!\, X_H - \text{non-deg}$$

• $X_H \rightsquigarrow$ "$\underbrace{\text{time-dependent flow}}_{\text{isotopy}}$"

time dependent

$\varphi_H^t \leftarrow$ Hamiltonian flow generated by H

Need not be defined for all t, and is not sometimes (collisions) but we will assume it-s. (E.g. M is compact, etc)

Rmk. Doing dynamics, usually interested in • φ_H^t, $t \in \mathbb{R}$, H autonomous
or φ_H^{kT}, $k \in \mathbb{N}$, H time-dependent

Examples

Ex 1. $\mathbb{R}^{2n}$, $\omega_{st} = dp\wedge dq = \sum dp_i \wedge dq_i$

$$\begin{cases} \dot p = -\dfrac{\partial H}{\partial q} \\ \dot q = \dfrac{\partial H}{\partial p} \end{cases} \iff X_H = -\frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}$$

Checking $i_{X_H}\omega = -\frac{\partial H}{\partial q}dq - \frac{\partial H}{\partial p}dp = -dH$

Subexample $\mathbb{R}^{2n} = T^*\mathbb{R}^n = \underset{q}{\mathbb{R}^n} \times \underset{p}{\mathbb{R}^n}$

$H = \frac{1}{2m}\|p\|^2 + V(q) = \text{kinetic} + \text{potential}$

$$\begin{cases} \dot p = -\dfrac{\partial V}{\partial q} \iff \boxed{m\ddot q = -\nabla V} \leftarrow \text{cons force} \\ \dot q = \dfrac{1}{m}p \iff p = m\dot q \leftarrow \text{momentum} \end{cases}$$

Newton's eq

2. **Cotangent bundle** $M = T^*Q$, ω_{st}

Fix R.M. on Q $\langle , \rangle$ $TQ \longleftrightarrow T^*Q$

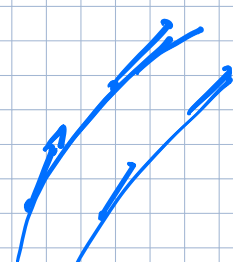
$v \longleftrightarrow \langle v, \cdot \rangle$

$H = \frac{1}{2}\langle , \rangle : T^*Q \longrightarrow \mathbb{R}$

X_H = geodesic spray

φ_H^t = geodesic flow

Describe.



Motion of a free particle on Q.

3. Twisted cotangent bundle

$$M = T^*Q \ , \quad \omega = \omega_{st} + \pi^*\sigma$$
$$\downarrow \pi$$
$$Q \ , \ \sigma \ , \quad d\sigma = 0$$

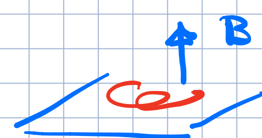
magnetic field

$$H = \tfrac{1}{2}\langle\ ,\ \rangle$$

The flow governs the motion of a charge on Q in magnetic field σ.

Subexample a) $Q = \mathbb{R}^2$, $\quad \sigma = B\, dq_1 \wedge dq_2$

conf space $\longrightarrow (q_1, q_2)$



$q_1, q_2 = (x, y)$

unit charge, unit mass

$\vec{B} \perp (q_1, q_2)$ plane, charge in $\mathbb{R}^2(x,y)$

$$\text{H.E.} \iff \ddot{q} = B(q)\, J\dot{q} \ ; \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

b) $Q = \mathbb{R}^3$, $\quad \vec{B} = $ v.f. on $\mathbb{R}^3 \leftarrow$ magn. field

$$\sigma = i_{\vec{B}}\, dq_1 \wedge dq_2 \wedge dq_3$$

$$d\sigma = 0 \iff \operatorname{div} B = 0 \ \Big\} \text{ one of the maxwell eq}$$

$$\text{H.E.} \iff \ddot{q} = \dot{q} \times \vec{B}(q)$$

Lorentz force

(unit charge, unit mass)

Energy and ω-conservation

Let φ^t_H be the Ham flow of H_t.

Prop

(a) Energy conservation

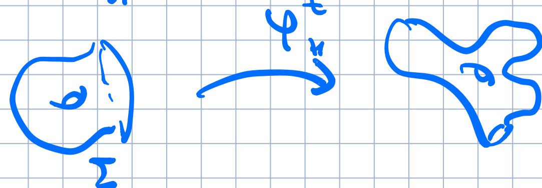
essential

Assume that H is autonomous. Then

$$(\varphi^t_H)^* H = H \quad : \quad H(\varphi^t_H(p)) = \text{const} \ \forall p$$

(b) "Integral invariant": φ^t_H is symplectic

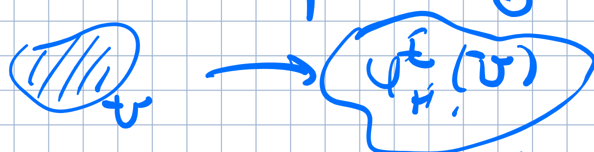
$$(\varphi^t_H)^* \omega = \omega$$



$\varphi^t_H(\Sigma) = \Sigma'$

$$\int_{\Sigma'} \omega = \int_{\Sigma} \omega$$

Cor. φ^t_H is vol. preserving: $(\varphi^t_H)^* \omega^n = \omega^n$



$$\int_U \omega^n = \int_{\varphi^t_H(U)} \omega^n$$

$\Rightarrow$ Huge restrictions on dynamics

Pf. (a) $\frac{d}{dt} H(\varphi^t_H(p)) = (L_{X_H} H)(\underbrace{\varphi^t_H(p)}_{x})$

$$= (i_{X_H} \cancel{dd}H + d\, i_{X_H} dH)(\ldots)$$

$$= d(\underbrace{-i_{X_H} i_{X_H} \omega}_{\text{HE!}}) = d\,\overbrace{\omega(X_H, X_H)} = 0$$

$i_{X_H} \omega = -dH$

(b) $$\frac{d}{dt}(\varphi_H^t)^*\omega = (\varphi_H^t)^* L_{X_H}\omega$$

$$= (\varphi_H^t)^*\left(d\, i_{X_H}\omega + \cancel{i_{X_H} d\omega}\right)$$

$$= (\varphi_H^t)^* \underbrace{d(-dH)}_{HE} = 0$$

$\lhd$

Ex. Prove the Prop using a direct calculation in Darboux coordinates.

Cor $\dim M = 2 \leftarrow$ surface
H autonomous

sol of ODE

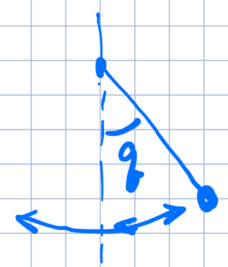
$\Rightarrow$ integral curves of φ_H^t (unparametrized) "are" levels $\underbrace{H = \text{const}}_{\text{"alg equation"}}$.

Rmk Newtonian mechanics:

- $\ddot{q} = F(q)$ Energy conserv $\Leftarrow \begin{cases} F \text{ is conserv: } \\ F = -\nabla V,\ \text{indep of } t \end{cases}$
- $\ddot{q} = F(q, \dot{q})$ as in Lorentz
  Energy conservation $\Leftarrow F \perp \dot{q}$

Continuing Examples

4. Investigating the pendulum



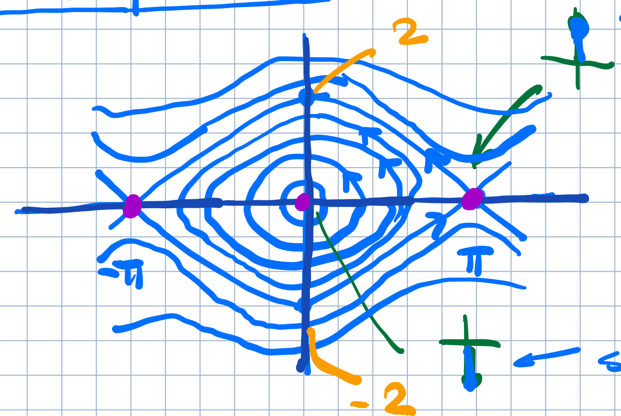
$$M = T^*S^1 = \underset{p}{\mathbb{R}} \times \underset{q \leftarrow \text{mod } 2\pi}{S^1}$$

$$\text{or} \quad \mathbb{R}^2 = \underset{p}{\mathbb{R}} \times \underset{q}{\mathbb{R}}$$

$$H = \frac{1}{2}|p|^2 - \cos q + 1$$

$$= \underbrace{\frac{1}{2}(p^2 + q^2)}_{\text{Harmonic oscillator}} + \dots$$

Phase portrait:



unstable

stable

$m = 1, \; k = 1$

$\ddot{q} = -\sin q$

Hooke's law: $\ddot{q} = -q$

Given the behavior of integral curves up to parametrization

Further details - Ex (Not easy)

(a) $q \in (0, \pi)$ $\quad H(0, q) = \cos q - 1 = h$

integral curve through $\underbrace{(q, 0)}_{x}$ is $\{H = h\}$

$T(h)$ its period

- Show that $T(h)$ monotone increasing function from $\underset{h=0}{2\pi}$ to $\underset{h=2}{\infty}$ as $h \to 2$

(Compare with the harmonic oscillator
$H = \frac{1}{2}(p^2+q^2)$ & $T = \text{const}$)

- Find the Taylor exp of $T(h)$ at $h=0$

(b) Consider $D\varphi_H^T : T_x\mathbb{R}^2 \circlearrowleft$
Show that $D\varphi_H^T = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ $\lambda \neq 0$

(c) Find explicitly $\varphi_H^t(0,2)$
in elementary functions

Ex <u>Hint to a): Area-period relation</u>
$H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ proper
- $\{H \leq h\}$ connected
- h - regular

$A(h) =$ area of $\{H \leq h\} = \int_{H \leq h} \omega$

$T(h) =$ period of $\{H = h\}$

Show that $\frac{dA}{dh} = T(h)$

5. Positive-Def quadratic Ham

$\mathbb{R}^{2n} = \mathbb{C}^n \qquad z_j = p_j + i q_j \ , \ z = (z_1, \dots, z_n)$

$$H = \frac{1}{2} \sum \underset{\substack{\vee \\ 0}}{\lambda_j} (p_j^2 + q_j^2) = \frac{1}{2} \sum \lambda_j |z_j|^2$$

or just $\neq 0$

Governs n uncoupled oscillators with frequencies λ_j

$E = \{H = h\}$ is an ellipsoid

- Find X_H and show that $\varphi_H^t(z) = (e^{\lambda_1 i t} z_1, \dots, e^{\lambda_n i t} z_n)$
- "Coordinate axis" $(0, \dots, 0, z_j, 0, \dots 0) \cap E$ are periodic orbits of φ_H^t with $T_j = \frac{2\pi}{\lambda_j}$
- Are there other periodic orbits? (The answer depends on $(\lambda_1, \dots, \lambda_n)$.)

6. Linear HE

$M = \mathbb{R}^{2n}$ $\quad \omega = \omega_{st} = dp \wedge dq$
$= \mathbb{C}^n$, $\quad$ Matrix of ω:

$$J = \begin{pmatrix} \begin{array}{cc|c} 0 & -1 & \\ 1 & 0 & 0 \\ \hline & & \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} \\ 0 & & \ddots \end{array} \end{pmatrix} \leftarrow \text{multiplication by } i \text{ in } \mathbb{C}^n = \mathbb{R}^{2n}$$

$H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ quadratic form

$H(x) = \frac{1}{2}\langle Ax, x\rangle$, $\quad A^T = A$

$A = \nabla H$

HE: $\quad \dot{x} = J\nabla H(x) = JAx = X_H(x)$

$$\varphi_H^t(x) = \exp(tJA)x$$

$$\exp: \ sp(2n) \longrightarrow Sp(2n)$$
$$X_H \longmapsto \varphi_H$$

$sp(2n) =$ lin Ham v.f.
$=$ quadratic forms on $\mathbb{R}^{2n}$
$X \longrightarrow H = -\frac{1}{2}\langle JXx, x\rangle$

Normal forms

- solving ODE's: $\dot{x} = Px$
  Bring A to a Jordan form to calculate $\exp(Pt)x$

- symplectic normal forms are more complicated

  A symmetric

$$\underset{GL(n)}{\underset{\cap}{S}}AS^T \rightsquigarrow \mathrm{diag}(1,\dots,1,0\dots0)$$

$$\underset{O(n)}{\underset{\cap}{S}}AS^T \rightsquigarrow \mathrm{diag}(\lambda_1,\dots,\lambda_n)$$

$$\underset{Sp(2n)}{\underset{\cap}{S}}AS^T \rightsquigarrow \text{much more complicated list}$$

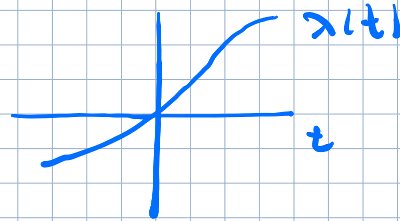
Not just $\sum \lambda_j |z_j|^2$

Rmk. Unless $A > 0$, then it can be diagonalized

Time - change

$H : \mathbb{R} \times M \to \mathbb{R}$ Hamiltonian

$\underbrace{\lambda : \mathbb{R} \to \mathbb{R}}_{\text{time-change}}$



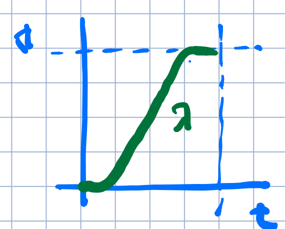
Does not have to be monotone but usually is

set $K_t(x) = \lambda'(t)\, H_{\lambda(t)}(x)$

Ex. show that $\varphi^t_K = \varphi^{\lambda(t)}_H$

Ex. 1) $\lambda(t) = T \cdot t$: $K = T\, H_{Tt}$

$\Rightarrow \varphi^1_K = \varphi^T_H$: Looking at φ^T_H can always assume that $T=1$

2) 

$\varphi^1_K = \varphi^1_H$ but $K \equiv 0$ when $t \approx 0$ & $t \approx 1$

$\Rightarrow$ Looking at φ^K_H can always assume H is 1-periodic in time $H_{t+1} = H_t$

§4 Relevant Groups: Ham v.s. Symp

Def. $\varphi_H = \varphi_H^1$ is called a Hamiltonian diffeo

Rmks
- Can assume that H is 1-periodic in t
- Can replace 1 by anything
- Hamiltonian $\Rightarrow$ Symplectic: $\varphi^* \omega = \omega$
- When M is not compact, need to assume smth about H at ∞. We'll usually assume that H is compactly supported $\Rightarrow$ supp φ is compact $\not\Leftarrow$

Prop The collection of Ham diffeos $Ham = Ham(M, \omega)$ is a gp.

Rmk Not obvious: H is not autonomous
$$(\varphi_H^t)^{-1} \neq \varphi_H^{-t}$$
and it's not clear why
$$\varphi_H^t \varphi_K^t \text{ is Hamiltonian}$$

Focus on the product:

Pf 1. Consider $\left.\begin{array}{l} H_t, \ t\in[0,1] \\ K_{t-1}, \ t\in[1,2] \end{array}\right\} F_t$

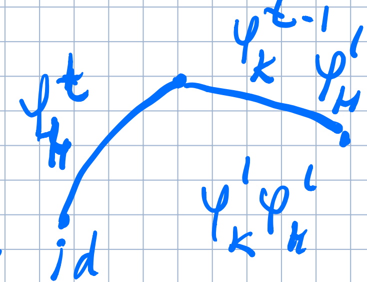
smooth in t when say

$\left.\begin{array}{lll} H_t \equiv 0 & \text{for} & t\approx 1 \\ K_t \equiv 0 & \text{for} & t \approx 0 \end{array}\right\}$ can be achieved by time chg

Then F_t generates

φ_H^t for $t\in[0,1]$

$\varphi_K^{t-1}\varphi_H^1$ for $t\in[1,2]$



So over $t\in[0,2]$ it generates $\varphi_K^1\varphi_H^1$

$\Rightarrow$ $\mathrm{Ham}(M,\omega)$ is closed under the product

Ex: generate $(\varphi_H^1)^{-1}$ $\lhd$

Pf 2 - Ex

- $\varphi_K^t\varphi_H^t$ is generated by $K_t + H_t\circ(\varphi_K^t)^{-1}$

$\Downarrow$

- $(\varphi_H^t)^{-1}$ — . — . — $-H_t\circ(\varphi_H^t)^{-1}$ $\lhd$

Now we have

$$\boxed{\mathrm{Ham}(M,\omega) \subset \underbrace{\mathrm{Symp}_0(M,\omega)}_{\text{connected component of the id}}} \subset \mathrm{Sympl}(M,\omega)$$

$$\circledast : \mathcal{H}_{\omega^n}(M) \supset \mathrm{Sympl}(M,\omega)$$

should think of these as ∞-dim Lie gps.

On the level of Lie algebras: vector fields

$$\begin{array}{ccc} \mathrm{Ham} & \subset & \mathrm{Symp}_0 \\ \Downarrow & & \end{array}$$

Lie algebras

Ham v.f.	$\subset$	Sympl. v.f.
$i_X\omega = \text{exact}$		$i_X\omega$ closed $\Leftrightarrow L_X\omega = 0$
$\updownarrow \; X \mapsto i_X\omega$		$\updownarrow \; X \mapsto i_X\omega$
exact 1-forms	$\subset$	closed 1-forms

$$\Rightarrow \boxed{\frac{\text{Sympl. v.f.}}{\text{Ham v.f.}} \Longrightarrow H^1(M;\mathbb{R})}$$

$\boxed{\underline{\text{Cor}}\quad H^1 = 0 \Rightarrow \text{Symp. v.f} = \text{Ham. v.f.}}$

Note: Ham v.f. $= \underbrace{C^\infty(M)}/\mathbb{R}$ ← in the center

Lie algebra with Poisson bracket

$$\{H,K\} := \omega(X_H, X_K) = -dH(X_K)$$

Ex.
- Check the Jacobi id
- Prove that

$$H \longmapsto X_H$$

is a Lie alg homo: $\{H,K\} \mapsto [X_H, X_K]$

$$C^\infty(M) \longrightarrow \text{Ham. v.f.}$$

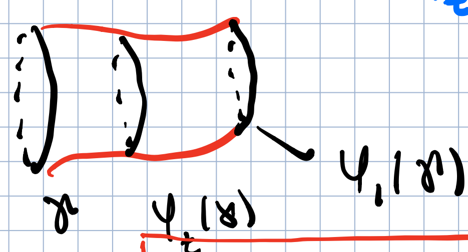
- For $\mathbb{R}^{2n}$

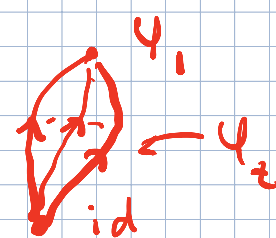
$$\left\{\begin{matrix}\text{quadratic}\\ \text{forms}\end{matrix}\right\} \xrightarrow{\cong} sp(2n)$$

Flux

M closed

- ψ_t = symplectic isotopy: a path in Symp starting at id
- $\gamma: \underset{\theta}{S^1} \to M$ a loop
- $\Pi: [0,1] \times S^1 \to M$
  $(t, \theta) \mapsto \psi_t(\gamma(\theta))$





- Def $\boxed{\widetilde{\text{Flux}}(\gamma) = \int_\Pi \omega}$

Prop

(a) $\widetilde{\text{Flux}}(\gamma)$ depends only on the homology class of γ & homotopy class of ψ_t with free end pts

(b) ψ_t is Hamiltonian $(\forall t)$
$\Rightarrow \widetilde{\text{Flux}}(\gamma) = 0 \quad \forall \gamma$

By (a) can think

$$\widetilde{\mathrm{Flux}} : H_1(M) \longrightarrow \mathbb{R} \quad \text{with } \varphi_t \text{ fixed}$$

or $\widetilde{\mathrm{Flux}} \in H^1(M;\mathbb{R})$

<u>Alternative description</u>

φ_t generated by $X_t = \frac{d}{dt}\varphi_t$

φ_t symplectic $\iff \alpha_t = i_{X_t}\omega$ is closed

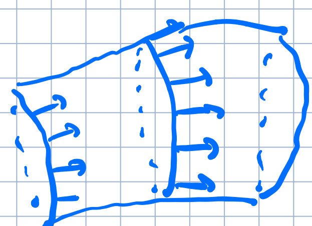
$$\iff L_{X_t}\omega = 0$$

Then

$$\boxed{\widetilde{\mathrm{Flux}}(\gamma) = \int_0^1 \int_{\varphi_t(\gamma)} \alpha_t \, dt = \int_0^1 \int_\gamma \varphi_t^* \alpha_t \, dt}$$

<u>Ex.</u> Prove this

$$\boxed{\widetilde{\mathrm{Flux}} = \int_0^1 [\alpha_t]\, dt}$$



Now b) is clear

φ_t Hamiltonian $\forall t$

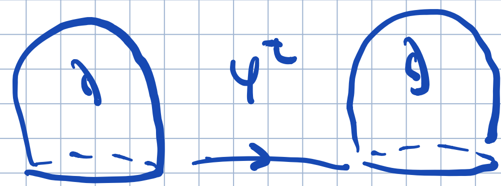
$\iff \alpha_t$ is exact $= -dH_t$

$$\Rightarrow \int_{\varphi_t(\gamma)} \alpha_t = \int_{\text{loop}} -dH_t = 0$$

or from this

Pf of a): homology class $[\gamma] = 0$:

$\gamma = \partial\Sigma$ (strictly speaking $\gamma = \sigma|_{\partial\Sigma}$, $\sigma : \Sigma \to M$)



$$[0,1]\times\Sigma \xrightarrow{\Phi} M$$
$$(t, \xi) \quad\quad \varphi_t(\sigma(\xi))$$

$$\Phi|_{\partial([0,1]\times\Sigma)} = \sigma \sqcup \varphi_1\circ\sigma \sqcup \Pi$$

$$d\,\Phi^*\omega = \Phi^* d\omega = 0$$

$$0 \overset{\text{Stokes}}{=} \int_{[0,1]\times\Sigma} d\Phi^*\omega = \underbrace{\int_\Sigma \sigma^*\omega - \int_\Sigma \sigma^*\varphi_1^*\omega}_{0} + \int_\Pi \omega$$

$$\varphi^*\omega = \omega$$

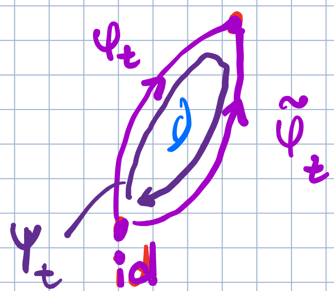
$$\Rightarrow \quad \gamma = \partial\Sigma \Rightarrow \widetilde{\mathrm{Flux}}(\gamma) = 0$$

Similarly for the homotopy of φ_t

To summarize universal covering (Recall?) $\triangleleft$

$$\boxed{\begin{array}{l} \widetilde{\mathrm{Flux}} : \widetilde{\mathrm{Symp}}_0 \times H_1(M;\mathbb{Z}) \to \mathbb{R} \quad \text{on} \\ \widetilde{\mathrm{Flux}} : \widetilde{\mathrm{Symp}}_0 \longrightarrow H^1(M;\mathbb{R}) \end{array}}$$

In general Flux does not descend to Symp_0



not homotopic

May have

$$\widetilde{\mathrm{Flux}}(\varphi_t) \neq \widetilde{\mathrm{Flux}}(\tilde{\varphi}_t)$$

Consider the loop $\psi_t = \text{"}\tilde{\varphi}_{1-t} \# \varphi_t\text{"}$

$$\Delta_{\psi_t}(\gamma) = \left(\widetilde{\mathrm{Flux}}(\varphi_t) - \widetilde{\mathrm{Flux}}(\tilde{\varphi}_t)\right)(\gamma)$$

$$= \int_{\Psi} \omega$$

$$\Psi : \overbrace{S^1 \times S^1}^{T^2} \longrightarrow M$$

$$(t, \theta) \longmapsto \psi_t(\gamma(\theta))$$

Δ_{ψ_t} depends only on the homotopy class of the loop ψ_t $\Rightarrow$

$$\Delta : \pi_1(\mathrm{Symp}) \longrightarrow H^1(M; \mathbb{R}) : \text{group homo}$$

$$\Gamma = \operatorname{im} \Delta \subset H^1(M; \mathbb{R})$$

<u>Now</u> we have

$$\boxed{\begin{array}{l} \mathrm{Flux} : \mathrm{Symp}_0 \longrightarrow H^1(M; \mathbb{R}) / \Gamma \\ \mathrm{Flux}(\mathrm{Ham}) = 0 \Leftarrow \text{Prop} \end{array}}$$

$$\boxed{\begin{array}{l}\textbf{Thm (Banyaga)} \text{ elementary but non-trivial} \\ \varphi \in Symp \text{ is Hamiltonian} \iff Flux(\varphi) = 0\end{array}}$$

$$\boxed{\begin{array}{l}\textbf{Cor} \quad H^1(M;\mathbb{R}) = 0 \Rightarrow \\ \qquad Ham = Symp_0\end{array}}$$

$\leftarrow$ clear anyway

$i_{X_t}\omega = \alpha_t \leftarrow$ closed

$\alpha_t = -dH_t$

Rmk. Compare: $H^1 = 0 \Rightarrow$ Symp v.f. = Ham. v.f.

Rmk. (Flux conjecture)

Γ is a discr. subgroup of H^1

Work McDuff, Lalonde, Polterovich

ultimately: Kaoru Ono, 2006

Ex. Assume that ω is integral

$[\omega] \in H^2(M;\mathbb{Z})$

or rational $[\omega] \in H^2(M;\mathbb{Q})$

Then Γ is discr (easily)

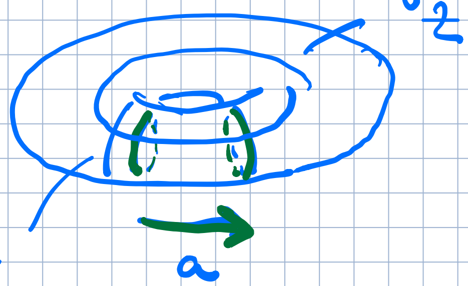
(In essence $\int_{\Psi \leftarrow \text{torus}} \omega \in \mathbb{Z}$ or $\lambda\mathbb{Z}$

Ex. Shifts of $\mathbb{T}^2$

$$\mathbb{T}^2 = S^1 \times S^1 \xrightarrow{\varphi} S^1 \times S^1 \quad \text{shift.}$$

(x, y) mod 1 $\qquad (x+a, y)$

Q. Is φ exact?



$\omega = dx \wedge dy$ integral

$$\varphi_t(x,y) = (x+ta, y)$$

$$\widetilde{\text{Flux}}(\varphi_t, \gamma) = \begin{cases} a, & \gamma = \gamma_1 \\ 0 & \gamma = \gamma_2 \end{cases}$$

$\text{Flux}(\varphi) = a \mod 1 \leftarrow$ area of $\mathbb{T}^2$

$$\Delta : \pi_1(\text{Symp}) \longrightarrow H^1(\mathbb{T}^2; \mathbb{R})$$
$$\cup$$
$$H^1(\mathbb{T}^2; \mathbb{Z}) = \mathbb{Z}^2$$

- because $[\omega] \in H^2(\mathbb{T}^2; \mathbb{Z})$ $\longrightarrow$
- easy to see: onto

φ is Hamiltonian $\iff a \in \mathbb{Z}$

$\iff \varphi = \mathrm{id}$

(But not φ_t)

More on Ham: Calabi homomorphism

M "open" non-compact, $\partial M = 0$

$Ham_0(M)$: Hamiltonian diffeos generated by $H_t \equiv 0$ at ∞ (compactly supported)

Not the same as φ_t compactly supp

$$\boxed{Cal(\varphi) = \int_0^1 \int_M H_t \omega^n dt}$$

Prop Assume that ω is exact $\omega = d\lambda$. Then $Cal(\varphi)$ is well-defined (independent of φ^t)

$\varphi^* \overline{\mathbb{R}}^{\circ}_{\circ}$

On the pf (McDuff-Salamon)

Step 1 : φ Ham $\Rightarrow$ $\varphi^*\lambda - \lambda = dF$ ← compactly supported

Step 2 : $Cal(\varphi) = \frac{-1}{n+1} \int F \omega^n$ ← depends only on φ

$\triangleleft$

Calabi Homomorphism:

$Cal : Ham_0 \longrightarrow \mathbb{R}$ is a group homo

$\Rightarrow [Ham_0, Ham_0] \subset Ker(Cal)$

↑ $\mathbb{R}$ is abelian

Thm (Banyaga) elementary but nontrivial

- M closed : $\mathrm{Ham}(M)$ is simple

$\mathrm{Ham} = [\mathrm{Ham}, \mathrm{Ham}]$

- M exact : The only normal subgroup of Ham_0 is $[\mathrm{Ham}_0, \mathrm{Ham}_0] = \mathrm{Ker}(\mathrm{Cal})$

§5 Submanifolds of symplectic manifolds

Linear algebra

(V^{2n}, ω) symplectic v.s. : $\mathbb{R}^{2n} = \mathbb{C}^n$, $i = J$

$L \subset V$ linear subspace, $d = \dim L$

Def symplectic orthogonal

$$\boxed{L^\omega = \{X \in V \mid \omega(X, Y) = 0 \ \forall Y \in L\}}$$

Obvious properties

- $\dim L^\omega = 2n - d$
- $(L^\omega)^\omega = L$

Def
- L is isotropic if $L \subset L^\omega \iff \omega|_L = 0 \quad \Rightarrow \quad d \leq n$
- L is coisotropic if $L^\omega \subset L \quad \Rightarrow \quad d \geq n$
- L is Lagrangian if $L = L^\omega$ (coiso & iso) $\quad \Rightarrow \quad d = n$
- L is symplectic if $\omega|_L$ is non-deg $\iff L^\omega \cap L = 0$

most important

Ex. • $\dim L = 1 \Rightarrow$ isotropic
• $\operatorname{codim} L = 1 \Rightarrow$ coisotropic
• $L \subset \mathbb{C}^n$ complex $\Rightarrow$ symplectic
$JL = L$ $\not\Leftarrow$

• L Lagr $\Rightarrow$ L is real: $JL \cap L = 0$

Prop Given $L \Rightarrow$

$\exists$ Darboux basis $e_1, f_1, \dots, e_n, f_n$ s.t.

• L isotropic : $L = \operatorname{span}(e_1, e_2, \dots, e_d)$
• L coiso : $L = \operatorname{span}(e_1, \dots, e_n, f_1, \dots, f_k)$
• L Lagr : $L = \operatorname{span}(e_1, \dots, e_n)$
• L sympl : $L = \operatorname{span}(e_1, f_1, \dots, e_k, f_k)$

Cor All Lagr. subspaces a conj. by $Sp(2n)$
(Likewise for other types with d fixed)

Rmk. $V = L \oplus L'$ $\leftarrow$ Lagr

$$\left.\begin{array}{l} \Rightarrow L' \cong L^* \\ \quad x \mapsto i_x \omega|_L \end{array}\right\} \Rightarrow V = T^*L = L \times L^*$$

Ex. L coisotropic $\supset L^\omega$
L/L^ω symplectic : ω_{red}

$\omega_{red}(X, Y) = \omega(\tilde{X}, \tilde{Y})$ lifts

More generally : $L / L \cap L^\omega$
is symplectic

Lagr. Grassmannian

$\Lambda = \{ L \subset \mathbb{R}^{2n} \mid L$ a n-dim Lagr. subspace $\}$ (written as: $\Lambda = \{ L \subset \mathbb{R}^{2n} \mid L\ a\ n \ \ \ g \}$)

$\mathbb{R}^{2n} = \mathbb{C}^n$
$\cup$
$\mathbb{R}^n$

Charts : $L \oplus L' = \mathbb{R}^{2n}$ - fixed
from some collection
e.g. coordinate subspaces

$\mathcal{U} = \mathcal{U}_{L,L'} = \{ Y \pitchfork L' \mid Y \text{ Lagr} \}$

$Y = \text{Graph}(P : L \to L' = L^*)$

$\{ P : L \to L^* \} \leftrightarrow \{$bilinear forms β on $L\}$

$P \leftrightarrow (x,y) \mapsto P(x)(y) = \beta$

Y Lagr $\Leftrightarrow$ β is symmetric

$\mathcal{U}_{L,L'} \leftrightarrow$ quadratic forms on L
(symmetric matrices)

$\Rightarrow \dim \Lambda = \dfrac{n(n+1)}{2}$

Ex. • $\Lambda = U(n)/O(n) \leftarrow$ Explain

• $\pi_1(\Lambda) \xrightarrow{\simeq} \mathbb{Z} \leftarrow$ "Maslov class"
$A \longmapsto \det^2_{\mathbb{C}}(A)$

• $H_1(\Lambda) \to \mathbb{Z}$: Maslov $\in H^1(\Lambda; \mathbb{Z})$

37

Details

- $U(n)$ - action on Λ

$$\begin{array}{ccc} L & & L' & \in \Lambda \\ e_1, \dots, e_n & \xrightarrow[A]{} & e_1', \dots, e_n' & \leftarrow \text{view as complex base} \end{array}$$

- A extends to $\mathbb{C}$-linear map $\mathbb{C}^n \circlearrowleft A_{\mathbb{C}}$ sending L to L'
- A orthogonal ($\Leftrightarrow$ bases orthonormal) $\Leftrightarrow$ $A_{\mathbb{C}} \in U(n)$

In particular $L = L'$: $O(n) \hookrightarrow U(n)$

- $\mathrm{stab}(L) = O(L) = O(n)$

$\Rightarrow$ $\Lambda = U(n)/O(n)$

Ex $\Lambda_1 = \mathbb{R}P^1$ $\quad n=1$

$\Lambda_2 = S^2 \times S^1/\sim$ $\quad n=2$

antipodal in both factors

Back to symplectic manifolds

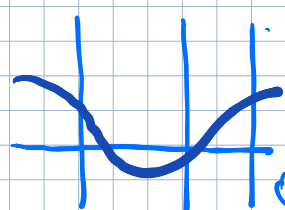
$(M^{2n}, \omega) \supset L$

Def. L Lagr (iso, coiso, sympl) if
$T_xL \subset T_xM$ is Lagr (iso ...) $\forall x \in L$

Ex. $\dim L = 1 \Rightarrow$ iso
$\operatorname{codim} L = 1$ (hypersurface) $\Rightarrow$ coiso

Focus on Lagr. submanifolds

Ex • $M = T^*Q \longrightarrow Q$
$\alpha \in \Omega^1(Q) =$ section of T^*Q
$\Rightarrow \quad L_\alpha \subset T^*Q$

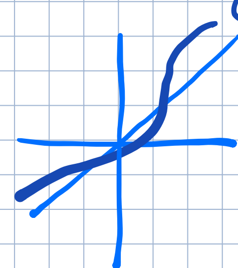


$d\alpha = 0 \Leftrightarrow L_\alpha$ Lagr

Pf $\alpha = \lambda|_{L_\alpha = Q}$

$\omega = d\lambda$ $\quad d\lambda|_L = 0 \Leftrightarrow d\alpha = 0$ $\triangleleft$

E.g. $Q \subset T^*Q$ is Lagr.



• $\varphi : M \circlearrowleft$

$(M \times M, \tilde{\omega} = \omega \oplus -\omega)$

$Gr(\varphi)$ Lagr $\Leftrightarrow \varphi$ is symplectic

E.g. $\Delta \subset M \times M$ is Lagr

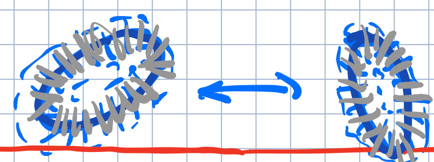
Thm (Weinstein's tubular nbd)

$L \xrightarrow{i} (M,\omega)$, $\quad L \xrightarrow{i'} (M',\omega')$ $\quad$ <u>L closed</u>

two Lagr. embeddings

$\Rightarrow$ $\exists$ nbds $U \supset i(L)$ & $U' \supset i'(L)$

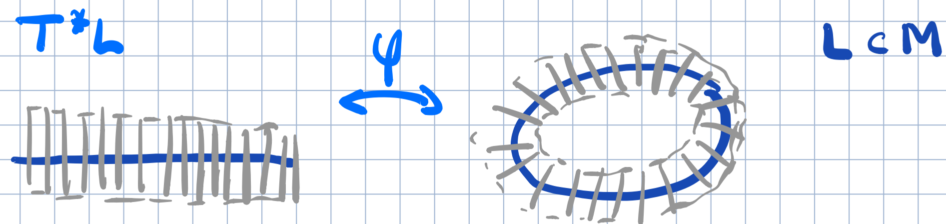
s.t. $(U, i(L))$ & $(U', i'(L))$ are symplecto...



Thm' (. ... . - . - . - - - . .)

closed $L \hookrightarrow M$ Lagr. embedding

$\Rightarrow$ nbd of L is symplectomorphic to a nbd $L \subset T^*L$



On the pf: similar to Darboux

$L \subset M$ Lagr. closed

- Preliminary (lin. alg)

N_L normal bdle: $N_L \oplus TL = T_L M$

can be chosen Lagr

$N_L \cong T^*L$

Now use ordinary tubular nbd to identify a nbd of $L \subset M$ with a nbd of L in T^*L

$\Rightarrow$ U = nbd of L in T^*L

ω_0 & ω_1 two symplectic forms

s.t. L is Lagr. for both

and by constr ω_0 & ω_1 $\underbrace{\text{agree on } T_L(T^*L)}_{(*)}$

standard

- Set $\omega_t = (1-t)\omega_0 + t\omega_1$ $\Downarrow$ symplectic

Run the homotopy method

X_t: $\quad i_{X_t}\omega_t = \lambda \qquad d\lambda = \omega_0 - \omega_1$

Need $\lambda = 0$ at every pt of L

(To make sure φ_t is defined for $t \in [0,1]$)

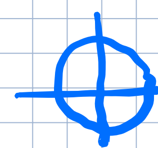
Not only $\lambda|_L = 0$

Ex: Such λ exists by $(*)$ $\triangleleft$

Lagr. submanifolds of $\mathbb{R}^{2n}$

Important question in sympl topology

Some simple observations

- $S^1 \hookrightarrow \mathbb{R}^2$ Lagr  $\Rightarrow$

$T^n = S^1 \times \dots \times S^1 \hookrightarrow \mathbb{R}^{2n}$

A lot of different (non-equivalent)

$\omega_{st} = d\lambda$, $\lambda = \frac{1}{2} \sum (p_i dq_i - q_i dp_i)$

$\lambda|_{T^n}$ closed $[\lambda|_{T^n}] \in H^1(T^n; \mathbb{R})$ inv

- **Prop** $L \subset \mathbb{R}^{2n}$ closed Lagr
  $\Rightarrow$ $\chi(L) = 0$

Pf. $N_L = T^*L \Rightarrow L \cdot L = \chi(L)$

But $L \cdot L = 0$ in $\mathbb{R}^{2n}$ e.g. because $[L] = 0$

or by deformation invariance

Cor. $\Sigma_{g \neq 1}$ does not admit Lagr. embeddings into $\mathbb{R}^4$

Liouville class

Rmk $\exists$ much more subtle results

E.g. $L \subset \mathbb{R}^{2n}$ Lagr $\Rightarrow$ $H^1(L; \mathbb{R}) \neq 0$

In fact $[\lambda|_L] \neq 0$ (Gromov)

$\Rightarrow$ S^3 does not have Lagr. embedding into $\mathbb{R}^6$

Maslov class

$L \looparrowright \mathbb{R}^{2n}$ Lagr, immersed, closed

$\Rightarrow$ $G: L \to \Lambda$ $\leftarrow$ Gauss map

$x \mapsto T_x L$

$\mu \in H^1(\Lambda; \mathbb{Z})$ Maslov

$\mu_L \in H^1(L; \mathbb{Z})$ $\leftarrow$ Maslov class of L.

$\shortparallel$

$G^*\mu$

Ex. L orientable $\Rightarrow$ μ_L is even

Fact (Gromov) $L \subset \mathbb{R}^{2n}$ embedded

$\Rightarrow \mu_L \neq 0 \Rightarrow H^1 \neq 0$

$\Rightarrow$ again S^3 does not have a Lagr. emb

Rmk. μ_L can also be defined for $L \subset T^*Q$ (but it can be 0)

§6 Contact manifolds

Contact str = odd-dim sister of sympl str

M^{2n+1} ← odd dimensional

Def • $\alpha \in \Omega^1(M)$ is contact if

$$\alpha \wedge (d\alpha)^n \neq 0 \quad \leftarrow \text{vol. forms}$$

$\boxed{\Rightarrow M^{2n+1} \text{ orientable}}$

$\Leftrightarrow \quad d\alpha\big|_{\ker\alpha}$ is non-deg

• $\xi = \ker\alpha$ is a contact str

strictly speaking: a codim-1 distr ξ is contact if locally $\xi = \ker\alpha$ (α contact)

can be made globally $\Leftrightarrow$ ξ is coorientd

Note: α contact $\Rightarrow$ $f\alpha$ contact, $f \neq 0$; same contact str. ξ

$$(f\alpha) \wedge [d(f\alpha)]^n$$
$$= (f\alpha) \wedge [df \wedge \alpha + f\, d\alpha]^n$$
$$\underset{\alpha^2=0}{=} f^{n+1} \alpha \wedge (d\alpha)^n$$

Cor M^{2n+1} admits a contact str ξ (not necessarily coorientable) and $n+1$ even $\Rightarrow$ M is orientable

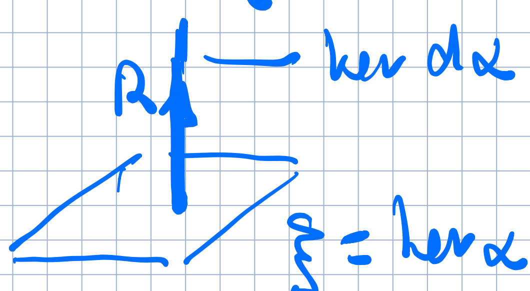
Def - fact α contact form $\Rightarrow$ $\exists!$ v.f. R Reeb v.f :

$$\alpha(R) = 1$$

$$i_R \, d\alpha = 0$$

In fact α contact

$$\Leftrightarrow \begin{cases} \ker(d\alpha) \ \ 1\text{-dim} \\ \ker d\alpha \pitchfork \underbrace{\ker \alpha}_{\xi} \end{cases}$$



Reeb v.f. $\rightsquigarrow$ Reeb flow

Examples

Ex 1. $\mathbb{R}^{2n+1}$ (p, q, z)

$\alpha = dz + p\,dq$ or $dz + \frac{1}{2}(p\,dq - q\,dp)$

are contact : st. contact form on $\mathbb{R}^{2n+1}$

$R = \frac{\partial}{\partial z}$ The st. contact form or str. on $\mathbb{R}^{2n+1}$

Visualize $n = 1$

More generally $M = \underset{z}{\mathbb{R}} \times W$ $\quad \overset{\pi}{\longrightarrow}$

$\alpha = dz + \pi^* \lambda \leftarrow$ contact $\quad \omega = d\lambda$

Ex 2

$\Sigma \subset \mathbb{R}^{2n}$, $\Sigma = \partial(\text{starshaped})$

e.g. convex

$\lambda = \frac{1}{2}(p\,dq - q\,dp)$

$\alpha = \lambda|_{\Sigma}$ is contact

- $R = J N_{\Sigma} \leftarrow$ ~~normal~~ unit normal

- $\Sigma = \{H = \text{const}\} \leftarrow$ reg

Then $R = f X_H$ on Σ

Move generally

$\Sigma^{2n-1} \subset (M^{2n}, \omega)$ symplectic

Def. Σ has contact type if $\omega|_\Sigma$ has a contact primitive α:

$d\alpha = \omega|_\Sigma$, $\alpha \wedge (d\alpha)^{n-1} \neq 0$

Thm: $\Sigma = \{H = \text{const}\} \leftarrow$ regular

$\Rightarrow R = f X_H$ on Σ

Rmk. Not every closed hypersurface in $\mathbb{R}^{2n}$ has contact type

Ex - Weinstein : two spheres

Ex 3 $\Sigma \subset T^*Q$ fiberwise starshaped

$\lambda = p\,dq$ Liouville form

$\alpha = \lambda|_\Sigma$ is contact

Σ fiberwise convex : Finsler metric

Reeb flow = Finsler geodesic flow

Ex 4 - Fact every closed orientable 3-manifold admits a contact structure

necessary

Existence of contact str.

Contact topology $\nwarrow$ active area

Contact Darboux Thm and all that

Thm (contact Darboux)

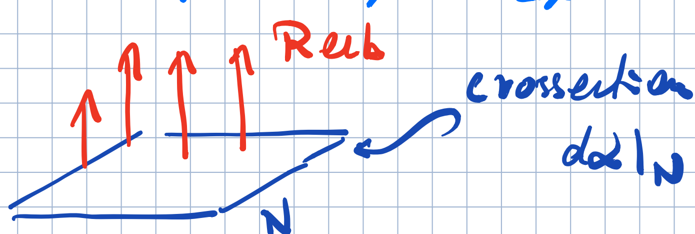
$\boxed{\text{Any two contact forms (fixed dim) are locally diffeomorphic}}$

Thm' (contact Darboux)

$\boxed{\text{Any contact form in } \dim = 2n+1 \text{ is locally diffeo to the standard contact form on } \mathbb{R}^{2n+1}: \exists \text{ coord } p, q, z \text{ such that } \alpha = dz + p\,dq}$

Two ways to prove:

1) As a consequence of symplectic Darboux - Ex



$d\alpha|_N$ = symplectic $= d(\underbrace{p\,dq}_{\lambda})$

$d\alpha|_N = d\lambda|_N \Rightarrow \alpha - \lambda = df$ ← can assume 0 at 0

"z = time of Reeb flow from $N + f$"

2) Use Moser's homotopy method directly

Rmk <u>No global version for contact forms</u>

$M, \alpha_t \leftarrow$ a family of contact forms

cannot expect α_s to be diffeo to each other

$\alpha_s \rightsquigarrow R_s \rightsquigarrow$ dynamics changes with s

Ex. $\underset{\{H=1\}}{\Sigma_t} \subset \mathbb{R}^{2n}$ a family of ellipsoids

$\{H=1\} \leftarrow$ quadratic Ham

$(\Sigma_t, \lambda|_\Sigma) \cong (S^{2n-1}_t, \alpha_t)$

$R_t = X_t \rightsquigarrow R_t$

we have seen that things depend on eigenvalues

$$\boxed{\begin{array}{l}\textbf{Thm (Gray's Thm)}\\ \underset{\text{closed}}{(M^{2n+1}}, \xi_t) \text{ contact} \Rightarrow \varphi_t \text{ s.t.}\\ \qquad (\varphi_t)_* \xi_t = \xi_0\end{array}}$$

What is actually proved

$M^{2n+1}, \xi_t = \ker\alpha\_$

$\Rightarrow \exists \; \varphi_t \; \& \; f_t > 0 : \; \varphi_t^*(f_t \alpha_t) = \alpha_0$

Pf: Moser's homotopy method

Rmk. <u>Discuss symplectization</u>

A glimpse of contact topology

ξ cooriented contact str

$\rightsquigarrow$ R_α Reeb : non-vanishing section of TM

$\rightsquigarrow$ The homotopy type of R is ind of α

A section of STM

homotopy type of ξ as cooriented distr.

$\rightsquigarrow$ A top inv. of ξ

E.g. $M = S^3$ $\quad STS^3 = S^3 \times S^2$

homotopy types of sections

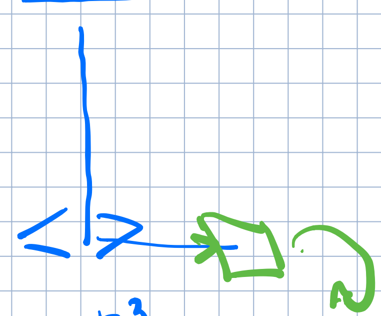
$\delta(\xi) \in [S^3, S^2] = \pi_3(S^2) = \mathbb{Z}$

Each of them can be realized by a contact str, and those contact str are not diffeo to each other

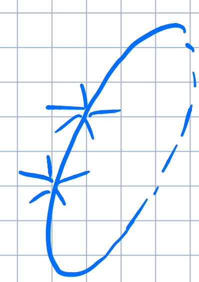
standard $\rightsquigarrow$ $0 = \delta(\xi_{st})$

But $\exists$ (exactly one) contact str ξ_{ot} with $\delta(\xi_{ot}) = 0 = \delta(\xi_{st})$

Describe



in $\mathbb{R}^3$



in S^3

§7. Morse Theory

- Not directly related to symplectic geom but extr. important on its own.
- Connections with many things inc s.g., ODE's, PDE's, everything

General setting & motivation

X some space: a manifold, loop space, path space

$f : X \to \mathbb{R}$ a function (smooth)

Looking for critical pts of f

Does it have them? How many, etc?

Ex a) x_0 & $x_1 \in Q$

From mechanics
diff geometry
Original Morse theory

$$X = \{\text{path connecting } x_0, x_1\}$$
$$= \{\gamma : [0,1] \to Q \mid \gamma(i) = x_i\}$$

Fix a R.m. metric on Q

$$\underset{p}{T^*Q} = \underset{v}{TQ}$$

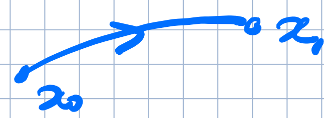
$$H = \frac{1}{2}\langle p, p\rangle + V(q)$$ potential energy

$$f(\gamma) = \int_0^1 H(\dot{\gamma}, \gamma)\, dt$$

Least action principle (LAP)

Crit(f) = integral curves connecting x_0 & x_1 in time-1

E.g. $V = 0$: geodesics from x_0 to x_1

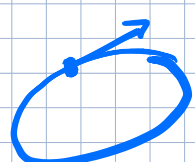


b) X = loop space
$= \{ \gamma : S^1 \longrightarrow Q \}$

f = the same

Relations to diff geometry:
Hopf-Rinow,
closed geodesics,
Hadamard

LAP: Crit(f) = periodic traj of φ_H^t of period 1 in T^*Q



E.g. closed geodesics

Etc: Everything of interest in physics is a crit pt of some functional

Calculus of variations

Finite dimensional setting

$X = M$ a compact (or even closed) finite-dim. manifold

$f : M \longrightarrow \mathbb{R}$ $\qquad (f|\partial X = \text{const})$

Q. How many crit pts does f have?

- Nontrivial even in simple cases.
- Assume M is closed:

max f & min f — critical values

Anything else

- In general Yes, unless $M \overset{\text{homeo}}{\simeq} S^n$

But might be few

Ex. Crit = {max, min} $\Rightarrow M \underset{\text{homeo}}{\simeq} S^n$

Ex. . Construct $f : \Sigma_{g \geq 1} \longrightarrow \mathbb{R}$ with exactly 3 critical pts

. sketch the levels for T^2:  /~

The third pt = Monkey Saddle

- Situation changes when we impose a non-deg cond on f, satisfied generically ← explain

Definitions

$p \in M$ critical pt of $f : M \to \mathbb{R}$

Def Hessian of f at p is the quadratic (or bilinear) form

$$d^2 f : T_pM \times T_pM \longrightarrow \mathbb{R}$$
$$v\,,\; w \longrightarrow (L_{\tilde v} L_{\tilde w} f)(p)$$

$\tilde{\ }$ Ext of v & w to v.f.

Ex. • Show that $d^2 f$ is well defined

• symmetric

• In local coordinates $x_1, \dots, x_n$

$$d^2 f = \sum \frac{\partial^2 f}{\partial x_i \partial x_j}(p)\, x_i x_j$$

$dx_i dx_j$

Do some of it

Then assume that $d^2 f$ is non-deg

Morse index of p = index of $d^2 f$:

$$d^2 f = -(x_1^2 + \dots + x_k^2) + (x_{k+1}^2 + \dots + x_n^2)$$

Ex. p = max : index = n

p = min : index = 0

Def f is Morse if all its critical pts are non-deg

Note: Morse functions form an open and dense subset of $C^\infty(M)$

Thm (**Morse Lemma**) f is C^3

Near a non-deg critical pt p of a function f is diffeo to its Hessian $H = d^2 f$:

- $\exists$ $\varphi : (V, p) \to (V, p)$ s.t. $f \circ \varphi = \varphi^* f = H + f(p)$
- In some coordinates $x_1, \dots, x_n$: $f(x) = f(p) + \sum a_{ij} x_i x_j$

Rmk
- One of the normal form results
- Another example:

  $df_p \neq 0$ $\exists$ $(x_1, \dots, x_n)$ s.t.
  $$f(x) = f(p) + x_1$$
  Ex: prove directly

  More generally, the local norm form for submersions (also immersion)

- Similar questions for other objects: vector fields, maps, etc
- E.g. v v.f.
  $$v(p) \neq 0 \Rightarrow \exists\ x_1, \dots, x_n$$
  $$v(x) = \frac{\partial}{\partial x_n}$$
  What if $v(p) = 0$?

Preliminaries

$$\boxed{\begin{array}{l}\textbf{Lemma (Hadamard)} \\ f \text{ is } C^n(\mathbb{R}^n), \quad f(0)=0 \\ \exists\ g_i \in C^{r-1}(\mathbb{R}^n) \text{ s.t.} \\ \quad f(x) = \sum x_i\, g_i(x) \quad \& \quad g(0) = \frac{\partial f}{\partial x_i}(0)\end{array}}$$

can be a nbhd of 0

Rmk: non-obvious — discuss

Pf 1) $n=1$ FTC

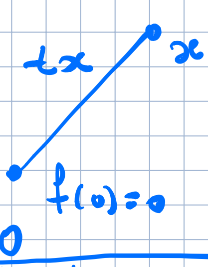
$$f(x) = \int_0^1 \frac{d}{dt} f(tx)\,dt - \overset{0}{\overset{\shortparallel}{f(0)}}$$

$$= \int_0^1 x\, f'(tx)\,dt$$

$$= x \underbrace{\int_0^1 f'(tx)\,dt}_{g \in C^{n-1}}$$

$$\boxed{\begin{array}{l}\text{Clearly} \\ f'(0) = g(0)\end{array}}$$

2) Any n:



$$f(x) = \int_0^1 \frac{d}{dt} f(tx)\,dt$$

$$= \int_0^1 \sum x_i \frac{\partial f}{\partial x_i}(tx)\,dt$$

$$= \sum x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx)\,dt}_{g_i \in C^{r-1}} \qquad \triangleleft$$

$$\boxed{\begin{array}{l}\text{Check} \\ g_i(0) = \frac{\partial f}{\partial x_i}(0)\end{array}}$$

Cor $d^2 f = 0$ at 0

$\Rightarrow$ $\exists\ g_{ij} \in C^{r-2}$ s.t.

$f(x) = \sum x_i x_j\, g_{ij}(x)$, $g_{ij}(0) = \dfrac{\partial^2 f}{\partial x_i \partial x_j}(0)$

and $g_{ij} = g_{ji}$, $r \geq 2$

Pf. Apply Hadamard twice.

Pf of Morse Lemma: Moser's homotopy method

- Local question $\Rightarrow$ work in $\mathbb{R}^n$, $p=0$
- $f(p)=0$ coordinates $y_1,\dots,y_n$
- $f(y) = \underbrace{\sum a_{ij} y_i y_j}_{H} + R$ ← higher order terms

$f_t = H + tR$, $t \in [0,1]$

Looking for φ_t generated by v_t s.t.

$$\varphi_t^* f_t = f_0 \; (= H) \qquad v_t(0)=0$$

$$\varphi_t^* \underbrace{L_{v_t} f_t}_{i_{v_t} df_t} + \varphi_t^* \underbrace{\frac{d}{dt} f_t}_{R} = 0$$

(*) $\boxed{i_{v_t} df_t = -R}$ $\qquad v_t(0)=0$

Rephrasing using Hadamard

- $f(x) = \sum x_i x_j g_{ij}$ $\quad \swarrow \dfrac{\partial^2 f}{\partial x_i x_j}(0)$

  $f_t(x) = \sum x_i x_j \underbrace{(t g_{ij} + (1-t) a_{ij})}$

  non-deg for $\longrightarrow$ g^t_{ij} : $g^t_{ij}(0) = a_{ij}\ \forall t$

  $x \approx 0$ for $t \in [0,1]$ $\quad \longleftarrow$ symmetric

- $df_t = \sum x_j g^t_{ij}\, dx_i + \sum x_i g^t_{ij}\, dx_j + \sum x_i x_j \dfrac{\partial g^t_{ij}}{\partial x_k}\, dx_k$

  $$= \sum_i x_i \left[\sum_j \underbrace{\left(2 g^t_{ij} + \sum_q x_q \frac{\partial g^t_{iq}}{\partial x_j}\right)}_{h^t_{ij}} dx_j \right]$$

  $h^t_{ij} \longleftarrow$ non-deg near 0 for all $t \in [0,1]$

  $$= \sum_i x_i \sum_j h^t_{ij}\, dx_j$$

  depends on t

- $v_t = \sum v_j \dfrac{\partial}{\partial x_j}$

$$\boxed{\imath_{v_t} df_t = \sum_i x_i \sum_j h^t_{ij}\, v_j}$$

$$-\dot{R} = \sum_{i,j} x_i x_j (\dot{a}_{ij} - \dot{g}_{ij})$$

$$-R =: \sum x_i R_i \,, \quad R_i(0) = 0$$

$$(*) \Leftrightarrow \sum_i x_i \sum_j h^t_{ij} v_j = \sum x_i R_i$$

$$\Leftarrow \sum_j \underbrace{h^t_{ij}}_{\tilde{H}} v_j = R_i \quad \forall i$$

$\tilde{H} \leftarrow$ "modified Hessian

$$\vec{R} = (R_1, \ldots, R_n)$$

$$\vec{v} = (v_1, \ldots, v_n)$$

$$(*) \Leftarrow \tilde{H}\vec{v} = \vec{R}$$ $\leftarrow$ has a unique sol smooth in x & t

$$\vec{R}(0) = 0 \Rightarrow \vec{v}(0) = 0$$

$\Rightarrow$ Flow exists for $t \in [0,1]$

$\triangleleft$

Morse Homology

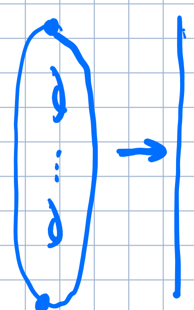
- $f: M \longrightarrow \mathbb{R}$ morse function
- $\mathrm{Crit}_k(f) \longleftarrow$ the collection of critical pts of index k
- Fix a ground ring $\mathbb{F}$: $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{Z}_2$, etc
- $CM_k(f)$ = free module over $\mathbb{F}$ generated by $\mathrm{Crit}_k(f)$

E.g. Height function on Σ_g

max $k=2$

saddles $k=1$

min $k=0$



$CM_0 = \mathbb{F}$
$CM_1 = \mathbb{F}^{2g}$
$CM_2 = \mathbb{F}$

$\boxed{\partial^2 = 0}$

Goal: turn CM_k into a complex

$$0 \to CM_n \xrightarrow{\partial} CM_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} CM_1 \xrightarrow{\partial} CM_0 \longrightarrow 0$$

so that

$$\boxed{\underbrace{H_*(CM_*(f), \partial)}_{=:\ HM_*(f)} \cong \underbrace{H_*(M; \mathbb{F})}_{\text{Homology of of } M}}$$

Morse Homology

Cor (Morse inequalities)

$\mathbb{F}$ a field

(a) $\underbrace{\# \mathrm{Crit}_k(f)}_{c_k} \geq \underbrace{\dim H_k(M)}_{b_k}$

(b) $c_k - c_{k-1} + c_{k-2} - \ldots \pm c_0 \geq b_k - b_{k-1} + \ldots \pm b_0$

Pf (a) $c_k = \dim CM_k \geq b_k = \dim HM_k$

(b) $\leftarrow$ Ex $\leftarrow$ Purely algebraic statement

C_* a complex over $\mathbb{F}$

show that C_* can be decomposed as a sum of elementary complexes $\leftarrow$ explain

check (b) for an elementary complex $\triangleleft$

Yet a different formulation. Set

$Q(t) = \sum c_k t^k$, $\quad P(t) = \sum b_k t^k \leftarrow$ Poincaré Pol

Then

$$Q(t) = P(t) + (1+t)R(t)$$

coeff ≥ 0

Ex. Prove this - alg, same method

Ex A Morse function on Σ_g has at least $2g+2$ critical pts.

Rmk. Morse inequalities can be further discussed.

Construction of the Morse differential ∂: Preliminaries

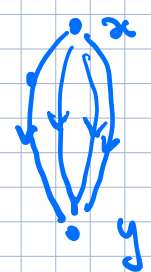
- While $CM_*(f)$ is completely determined by f, ∂ depends on an extra str: a R. metric on M
- Fix a R. m. on M (has to be from a certain open and dense set of R.m.'s)

Consider the antigradient flow of f:

$$\dot{x} = -\nabla f(x) \quad : \quad \varphi_t$$

Set $\mathcal{M}(x, y) = \left\{ z \;\middle|\; \varphi_t(z) \begin{array}{l} \to x \quad t \to -\infty \\ \to y \quad t \to +\infty \end{array} \right\}$

$\underset{\text{crit}}{x, y}$

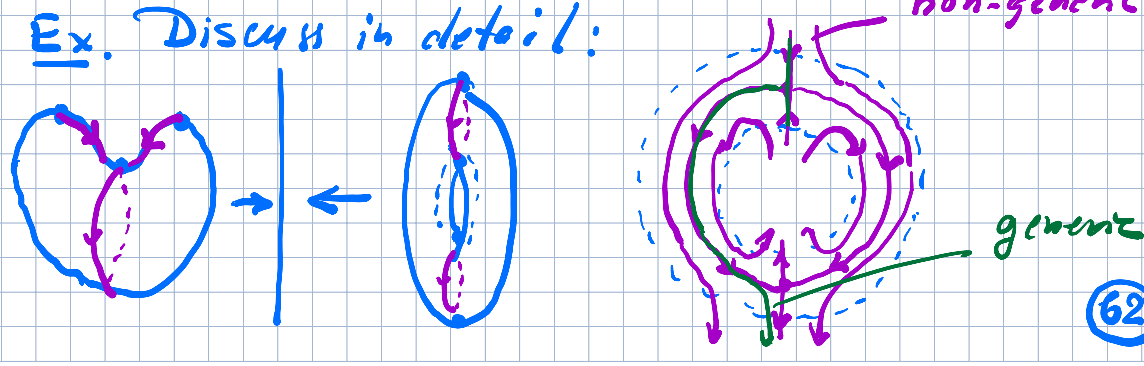


Denote the index of x by $\mu(x)$.

Note: "$\dim \mathcal{M}(x, y) \geq 1$" if $\neq \emptyset$

Thm For a generic metric, $\mathcal{M}(x, y)$ is a smooth manifold of dimension $\mu(x) - \mu(y)$

Ex. Discuss in detail:

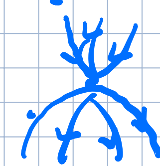


<u>One way to prove the theorem:</u>

$$W^u(x) = \{ z \mid \varphi^t(z) \longrightarrow x,\ t \to -\infty \}$$

$$W^s(x) = \{ \_\_\_ \ . \ \_\_\_ \ . \ \ldots\ t \to +\infty \}$$

stable, unstable manifolds



<u>Morse Lemma</u>: $\Rightarrow$ $W^u(x) \underset{\text{diffeo}}{\simeq} D^{\mu(x)}$

(Look at the examples) $\qquad W^s(x) \simeq D^{n-\mu(x)}$

$$\mathcal{M}(x,y) = W^u(x) \cap W^s(y)$$

If $\quad W^u(x) \pitchfork W^s(y)$,

$\mathcal{M}(x,y)$ is smooth and

$$\begin{aligned} \dim \mathcal{M}(x,y) &= \dim W^u(x) + \dim W^s(y) - n \\ &= \mu(x) + n - \mu(y) - n \\ &= \mu(x) - \mu(y) \end{aligned}$$

How to achive transversality

Look at $\big(W^u(x) \cap \{f=c\}\big) \cap \big(W^s(y) \cap \{f=c\}\big)$

$f(y) < \underset{\text{reg}}{c} < f(x)$, perturb the metric slightly above c to alter

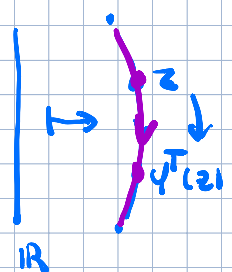
$\lhd$

<u>Cor</u> By thm, for a generic metric

$$\mu(y) \geq \mu(x) \Rightarrow \mathcal{M}(x,y) = \emptyset$$

More modern & different perspective

$\mathcal{M}(x,y)$ = the space of parametrized trajectories $t \mapsto \varphi^t(z)$ from x to y



trajectory $\longleftrightarrow$ initial condition

$\varphi^t(z) \longleftrightarrow z = \varphi^0(z)$

Time shift: $t \mapsto \varphi^t(z) \qquad z \mapsto \varphi^T(z)$

$$\wr$$

$t \mapsto \varphi^{t+T}(z)$

$\Rightarrow$ free $\mathbb{R}$-action on $\mathcal{M}(x,y)$, $x \neq y$

Space of unparametrized trajectories

$$\hat{\mathcal{M}}(x,y) = \mathcal{M}(x,y)/\mathbb{R}$$

Cor $\hat{\mathcal{M}}(x,y)$ is a smooth manifold of dim $\mu(x) - \mu(y) - 1$

E.g. $\mu(x) = \mu(y) + 1 \Rightarrow \hat{\mathcal{M}}$ is discr

Note
- $\mathcal{M}$ & $\hat{\mathcal{M}}$ are usually non-compact
- Geometrically, $\hat{\mathcal{M}}$ can be identified with $\mathcal{M} \cap \{f = c\}$

$f(y) < c < f(x)$

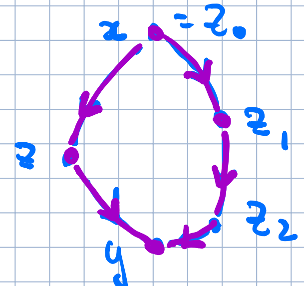
$\uparrow$

regular

(For a generic metric)

Thm $\hat{\mathcal{M}}(x,y)$ has a compactification formed by broken trajectories

Make it precise



Such trajectories $x = z_0 \rightsquigarrow z_1 \rightsquigarrow \dots \rightsquigarrow z_k = y$ form a compact manifold with corners.

Rmk $f(x) > f(z_1) > \dots > f(y)$

$\mu(x) > \mu(z_1) > \dots > \mu(y)$

Cor. $\mu(x) = \mu(y) + 1$

$\Rightarrow$ $\hat{\mathcal{M}}$ = compact $\Rightarrow$ finite collection of pts

$\Leftrightarrow$ $\exists$ finite many traj from x to y (for a generic metric)

Definition of ∂

Fix a generic metric so that all the things hold

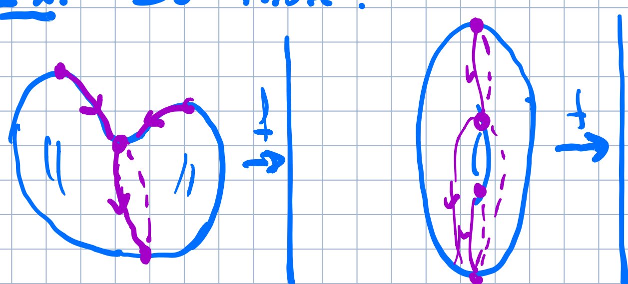
$$\mu(x) = \mu(y) + 1$$

- Over $\mathbb{Z}_2$, set

$$\mathbb{Z}_2 \ni m(x,y) = \#\hat{\mathcal{M}}(x,y) \mod 2$$

$$\boxed{\partial x = \sum_{\substack{y \\ \mu(x) = \mu(y)+1}} m(x,y)\, y} \qquad (*)$$

Ex. Do these:



- **Over $\mathbb{Z}$** (and hence any ring)

  Need to take into account orientations

Fix orientations of $T_x W^u(x)$ $\forall x$

$\Rightarrow$ coorientations of $T_x W^s(x)$

$\Rightarrow \begin{cases} \text{orientations of } W^u(x) \\ \text{coorientation of } W^s(x) \end{cases}$

$\Rightarrow$ orientations of

$$\mathcal{M}(x,y) = W^u(x) \cap W^s(x)$$

when $\mu(x) = \mu(y)+1$

$\mathcal{M}(x,y)$ = disj union of finite # of trajectories

Each trajectory γ is also oriented by the flow

$\Rightarrow$ Two orientations

$$\text{sign}(\gamma) = \begin{cases} +1 & \text{orientations agree} \\ -1 & \text{—— . —— . disagree} \end{cases}$$

And

$$m(x,y) = \sum_{x \overset{\gamma}{\rightsquigarrow} y} \text{sign}(\gamma)$$

$*$: $$\boxed{\partial x = \sum m(x,y)\, y}$$

<u>Ex.</u> Look at the torus example again.

Checking that $(CM(f), \partial)$ is a complex

$$\boxed{\textbf{Thm} \quad \partial^2 = 0}$$

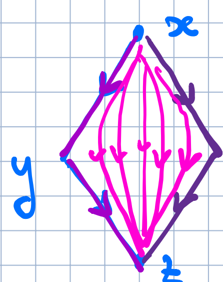
Pf. For the sake of simplicity over $\mathbb{Z}_2$

$$\partial^2 x = \partial \sum_y m(x,y)\, y$$

$\mu(x) = \mu(y)+1$
$\mu(y) = \mu(z)+1$

$$= \sum_y m(x,y) \sum_z m(y,z)\, z$$

$$= \sum_z \Big(\underbrace{\sum_y m(x,y)\, m(y,z)}_{\substack{\#\text{ of broken trajectories (one break)} \\ \text{from } x \text{ to } z \ (\mathrm{mod}\ 2)}} \Big) \ \mathrm{mod}\ 2$$



But $\hat{M}(x,y)$ one-dim manifold
its compactification: S^1 or I
closed interval

$\Rightarrow$ broken trajectories come in pairs

$\Rightarrow$ # is even

$\Rightarrow \sum_y m(x,y)\, m(y,z) = 0 \mod 2$

$\Rightarrow \partial^2 = 0$ $\triangleleft$

Set $HM_*(f) = H_*(CM(f), \partial)$; fixed coefficients

$$\boxed{\textbf{Thm} \text{ (Morse theory)} \quad HM_*(f) = H_*(M)}$$

<u>Rmk</u> • As a consequence, l.h.s is independent of f

• We have already seen some consequences: Morse inequalities, etc

<u>Outline of the pf:</u>

"Classical" Morse theory

<u>Morse function f on M</u> $\rightsquigarrow$ <u>Cellular decomposition of M</u>

$\mathrm{Crit}_k(f) \rightsquigarrow W^u(x) \leftarrow$ cells

$\mathrm{Crit}_*(f)$

Morse complex $CM_k(f)$ $\leftrightsquigarrow$ Cellular complex of M: $C(M)$

$\partial_n = \partial_{cw}$

$$\Rightarrow H_*(CM(f), \partial_n) = \underbrace{H_*(C(M), \partial_{cw})}_{H_*(M)} \qquad \triangleleft$$

<u>Details:</u> Audin-Damian

Applications: A quick look

Two types:
- Lower bounds on $\# \mathrm{Crit}(f)$ ← Morse
- Calculation of $H_*(M)$

(1) Lower bounds: Morse inequalities

$$\# \mathrm{Crit}_k(f) \geq b_k = \dim H_k(M)$$

$$\# \mathrm{Crit}(f) \geq \sum b_k$$

Morse ↗

E.g.
- f on T^n $\quad \# \mathrm{Crit}(f) \geq 2^n$
- f on $\mathbb{CP}^n$ or $\mathbb{RP}^n$ $\quad \# \mathrm{Crit}(f) \geq n+1$
- f on Σ_g $\quad \# \mathrm{Crit}(f) \geq 2+2g$

Etc.

(2) Calculations of $H_*(M)$ using Morse homology.

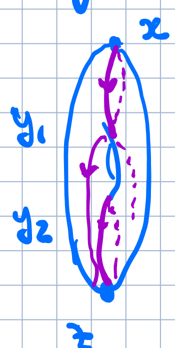
$$H_*(M) = H_*(HM_*(f), \partial_m)$$

very difficult to deal with in general

Works well when $\partial = 0$

Examples (over $\mathbb{Z}$ or $\mathbb{F}$)

1) Σ_g on $\mathbb{T}^2$



$$\partial x = y_1 + - y_1 + y_2 + - y_2 = 0$$

work out unstable traj of y_1 & y_2 should come from x (Orientations

$$\partial y_1 = 0 = \partial y_2$$

$\Rightarrow$

$$H_*(\Sigma_g) = \begin{cases} \mathbb{F} & k=2 \\ \mathbb{F}^{2g} & k=1 \\ \mathbb{F} & k=0 \end{cases}$$

2) $\mathbb{CP}^n$ (over $\mathbb{Z}$ or $\mathbb{F}$)

$$\mathbb{CP}^n = \{ (z_0 : \dots : z_n) \mid \sum |z_j|^2 = 1 \}$$

$$f(z) = \sum \lambda_j |z_j|^2$$

$$\lambda_0 < \lambda_1 < \dots < \lambda_n$$

Ex. a) Crit(f) = "coordinate axes"

$$= \{ (0, \dots, 0, \underset{j}{1}, 0, \dots, 0) = x_j \}$$

b) In coordinates

$$u = (u_0, \dots, u_{j-1}, 1, u_{j+1}, \dots, u_n\}$$

near x_j the Hessian is

$$(\lambda_0 - \lambda_j)|u_0|^2 + (\lambda_1 - \lambda_j)|u_1|^2 + \dots \text{ skip } (\lambda_j - \lambda_j)$$

$\Rightarrow$ f is morse & $\mu(z_j) = 2j \leftarrow z_j$ is a complex #.
& $\partial = 0$

$$\Rightarrow \quad H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{F} & 0 \le k = 2j \le 2n \\ 0 & \text{otherwise} \end{cases}$$

3) <u>$\mathbb{RP}^n$ over $\mathbb{Z}_2$</u>

Similarly $\mathbb{RP}^n = \{ (y_0 : \dots : y_n) \mid \sum |y_j|^2 = 1 \}$

$$f(y) = \sum \lambda_j |y_j|^2$$

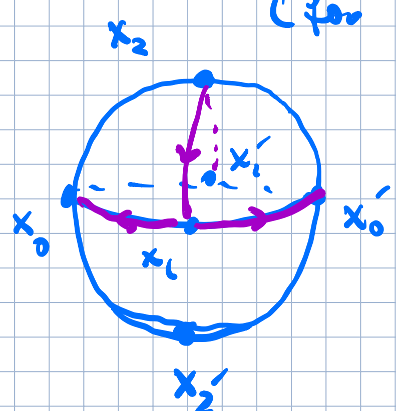
<u>Ex.</u> similarly

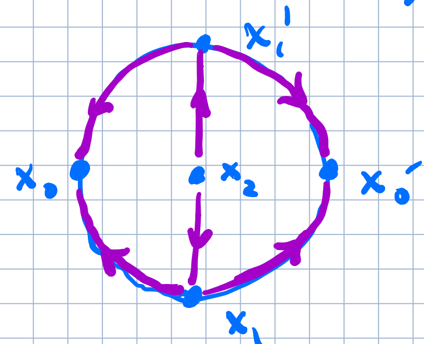
a) $x_j = (0, \dots, 0, \underset{j}{1}, 0, \dots, 0) \leftarrow$ critical pts

b) Hessian: similar — same calculation

$$\Rightarrow \quad \mu(x_j) = j$$

c) $\partial = 0$ over $\mathbb{Z}_2$: exactly two trajectories from x_{j+1} to x_j (for the round metric)





on S^2

$\downarrow$

$\mathbb{RP}^2$

$$\Rightarrow \quad H_k(\mathbb{RP}^n ; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

<u>Rmk.</u> Over $\mathbb{Z}$, harder — orientations

<u>More examples later</u>

Lusternik-Schnirelmann Theory -

- A Glance at - Modern version

Drop non-deg condition: we have already seen that the lower bound changes drastically:

$\exists\ f : \Sigma_g \xrightarrow{C^\infty} \mathbb{R}$ with $\# \mathrm{Crit}(f) = 3$

Fix a field $\mathbb{F}$ ← can be $\mathbb{Z}$

Def Cup-length of M (depends on $\mathbb{F}$) is $\max \left\{ k \;\middle|\; \begin{array}{l} \exists\ \alpha_1, \dots, \alpha_k \mid \alpha_i \in H^{*>0}(M) \\ \alpha_1 \cup \dots \cup \alpha_k \neq 0 \end{array} \right\}$

Ex.
- $CL(S^n) = 1$
- $CL(\Sigma_g) = 2$
- $H^*(M) \neq 0$ for $0 < * < n$

  $\Rightarrow CL(M) \geq 2$
- $CL(T^n) = n$ over any $\mathbb{F}$
- $CL(\mathbb{R}P^n) = n$ over $\mathbb{Z}_2$ but not $\mathbb{Q}$
- $CL(\mathbb{C}P^n) = n$ over any $\mathbb{F}$
- M^{2n} closed sympl $\Rightarrow CL(M) \geq n$: $[\omega]^n \neq 0$ over $\mathbb{R}$

$$\boxed{\begin{array}{l}\textbf{Thm (LS)} \quad f: \underset{\text{closed}}{M} \to \mathbb{R} \\ \Rightarrow \quad \# \operatorname{Crit}(f) \geq CL(M) + 1\end{array}}$$

$\Uparrow$

$$\boxed{\begin{array}{l}\textbf{Thm (LS)} \quad f: \underset{\text{closed}}{M} \to \mathbb{R} \quad \text{s.t.} \\ \quad \operatorname{Crit}(f) \text{ is finite} \quad (\Leftrightarrow \operatorname{Crit}(f) \text{ isolated}) \\ \Rightarrow \# \text{ of critical values} \geq CL(M) + 1\end{array}}$$

Outline of the pf

- Using PD pass to the intersection product on homology: more visual
- $\alpha \in H_*(M; \mathbb{F})$ ← can be $\mathbb{Z}$

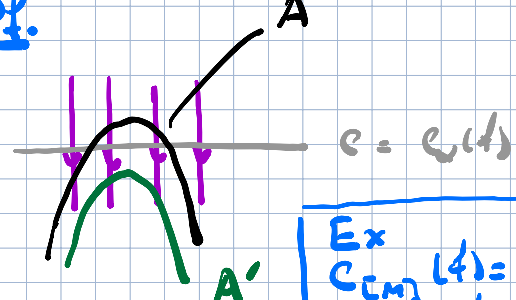
$$\boxed{c_\alpha(f) = \inf_{[A]=\alpha} \max_A f}$$

Gap: what is a cycle?

Def: $\alpha = 0 \Rightarrow c_0(f) = -\infty$

$$\boxed{\begin{array}{l}\textbf{Claim 1} \text{ (Min/Max Principle)} \\ \qquad c_\alpha(f) \text{ is a critical value of } f\end{array}}$$

Pf.



Assume not. Push A by the $-\nabla f$ to A' so that $\max f < c$

$A' \rightarrow\leftarrow$

$$\boxed{\begin{array}{l}\text{Ex} \\ c_{[M]}(f) = \max f \\ c_{[pt]}(f) = \min f\end{array}}$$

• Pretty much by construction

(*) $\underbrace{c_{\alpha\cap\beta}(f)}_{\inf_{[C]=\alpha\cap\beta} \max_C f} \leq \max\left(c_\alpha(f), c_\beta(f)\right)$ Next page

$$\inf_{[C]=\alpha\cap\beta} \max_C f \leq \inf_{A,B} \max_{A\cap B} f \leq \inf_A \max_A f$$

$$\inf_{A,B} \max_{A\cap B} f \leq \inf_B \max_B f$$

• $\underline{\text{Claim 2}}$ (without pf): Crit(f) isolated

$\underbrace{|\beta|<n}_{\text{essential: dont want } \beta=[M]} \Rightarrow c_{\alpha\cap\beta}(f) < c_\alpha(f)$

Next page

Now: $\alpha_1 \cap \alpha_2 \cap \alpha_3 \cap \ldots \cap \alpha_{k=cl(M)} \neq 0$

At least $CL(M)+1$ critical values:

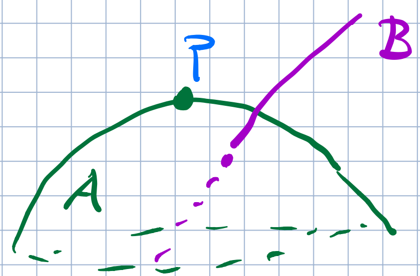
$$\underbrace{c_{[M]}(f)}_{\max} > c_{\alpha_1}(f) > c_{\alpha_1\cap\alpha_2}(f) > \ldots > c_{\alpha_1\cap\ldots\cap\alpha_k}(f)$$

$\triangleleft$

Pf of Claim 2 - outline

- Assume for the sake of simplicity

$$\max_A = c_\alpha(f) \quad \text{attained at} \quad p \in A \cap \mathrm{Crit}(f)$$



Generically: move B away from p

Important: p is an isolated pt. $\triangleleft$

Pf of (*)

$$\begin{aligned} c_{\alpha\cap\beta}(f) &= \inf_{[C]=\alpha\cap\beta} \max_C f \\ &\le \inf_{A \pitchfork B} \max_{A\cap B} f \\ &\le \inf_A \max_A f \\ &= c_\alpha(f) \end{aligned}$$

$\triangleleft$

76a

Origins:

Thm (LS, 1929) For any R.m. on S^2 $\exists$ at least 3 closed **simple** geodesics

no self-intersections

Idea of the pf (Complete pf: Ballmann, 1978)

- $\Lambda = \left\{ \begin{array}{l} \text{space of embedded unoriented} \\ S^1 \hookrightarrow S^2 \end{array} \right\}$

$\mathbb{R}P^2 = \{\text{equators}\} \hookrightarrow \Lambda$ $\quad = \left\{ \begin{array}{l} \text{planes in } \mathbb{R}^3 \\ \text{through } 0 \end{array} \right\}$

homotopy equivalence (non-obvious)

$\Rightarrow \quad CL(\Lambda) = CL(\mathbb{R}P^2) = 2$

- $L : \Lambda \longrightarrow \mathbb{R}$
  $\gamma \longmapsto \text{length of } (\gamma)$

Do LS theory for L on Λ $\leftarrow$ Hard analysis

$\Rightarrow \# \text{ Crit pts} \geq \underbrace{CL(\Lambda) + 1}_{3} = 2 + 1$

Rmk Important that L is symmetric
"$L(\vec{\gamma}) = L(\overleftarrow{\gamma})$"

$\Lambda' =$ oriented circles : $S^2 \hookrightarrow \Lambda'$

$CL(\Lambda') = 1 \Rightarrow$ lower bound $= \tilde{2}$

Katok's example ...

Rmks

More classical approach

$$\# \text{Crit values} \geq \mathrm{cat}(M) = \min\{k \mid M = \underbrace{X_1 \cup \dots \cup X_k}_{\text{all contractible in } M}\} \geq CL(M)+1$$

E.g. $\mathrm{cat}(S^n) = 2$

$\mathrm{cat}(T^2) = 3$

All inequalities are far from sharp.

77a

§8. Hamiltonian S^1-actions

Linearization of compact gp actions

General stuff.

G compact gp action on M

$\mathrm{Fix}(G) = M^G =$ fixed pt set

$p \in M^G \Rightarrow$ action on M gives rise to a linear action (aka representation) on T_pM

Thm (Linearization)

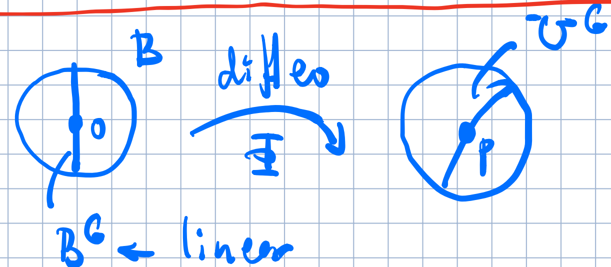
$\exists$ an open ball $\underset{\underset{0}{\psi}}{B} \subset T_pM$

an open nbd $\underset{\underset{p}{\psi}}{U} \subset M$

and an equivariant (commutes with G) diffeo $(B, 0) \xrightarrow{\Phi} (U, p)$

Cor. M^G is a smooth submanifold

Pf



Pf of Thm

Fact: M admits a G-invariant metric

$\Downarrow$

Thm $\quad \exp : T_pM \longrightarrow M$ is G-equiv

$\qquad B \longrightarrow U$

left invariant metric

Discuss

$B \subset T_pM \qquad\qquad U \subset M$

action of $g \in G$ on M

Pf $\langle\, ,\rangle_0$ any metric

set $\displaystyle \langle\, ,\rangle = \int_G g_* \langle\, ,\rangle_0 \, d\mu(g)$

$$\langle X, Y\rangle := \int_G \langle g_* X, g_* Y \rangle_0 \, d\mu(g)$$

push forward

Need: $\underbrace{\langle \underset{G}{\underset{\in}{h}}{}_* X, h_* Y\rangle}_{} = \langle X, Y\rangle$

$$\int_G \langle \overbrace{h_* g_*}^{g'} X, \overbrace{h_* g_*}^{g'} Y\rangle \, d\mu(g)$$

$$= \int \langle g'_* X, g'_* Y\rangle \, \underbrace{h^{-1}_* \, d\mu(g')}_{d\mu(g') \text{ left inv}}$$

$$= \langle X, Y\rangle$$

$\triangleleft$

Rmk. A very general method – averaging

Another application: G compact admits a bi-inv. metric, etc

Assume now that M is symplectic and the G-action is symplectic

Thm (Equivariant Darboux)
In the previous constr Φ can be taken symplectic:
$\exists$ an open ball $B \subset T_pM$, $B \ni 0$
an open nbd $U \subset M$, $U \ni p$
and an equivariant (commutes with G) symplecto $(B, 0) \xrightarrow{\Phi} (U, p)$
$$\omega_p = \Phi^* \omega$$

Pf. The previous method does not work
But Moser's homotopy method does: every step is G-equivariant (with some minimal care)

$\triangleleft$

Assume now that $G = S^1$

$\Rightarrow$ symplectic circle action on M

$\hookrightarrow$ a flow φ^t s.t. $\varphi^T = id$ for some T

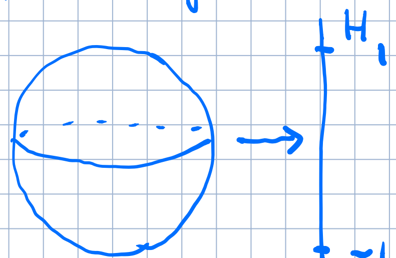
$(\varphi^t)^* \omega = \omega$ Can set $T = 1$.

Usually will assume Hamiltonian, generated by $H : M \longrightarrow \mathbb{R}$

Ex. 1) $M = \mathbb{R}^{2n} = \mathbb{C}^n$

$$H = \sum \lambda_j |z_j|^2, \qquad \lambda_j \in \pi \mathbb{Z}$$

2) Height function on S^2



$\omega = dH \wedge d\theta$

(perhaps some const) but probably not

Area $= 4\pi$

3) $\mathbb{T}^2 = S^1 \times S^1$ $(x, y) \overset{\varphi^t}{\longmapsto} (x + t\theta, y)$

symplectic but not Hamiltonian

Few manifolds admit compact group actions.

E.g. • S^2 is the only surface with a Ham circle action. Pf - soon

• Few manifolds admit gp actions:
Ex. show that $\Sigma_{g \geq 2}$ has no S^1-actions.

Equivariant Darboux $\Rightarrow$

$$\begin{array}{ccc} T_pM & \longrightarrow & U \ni p \\ D\varphi^t \circlearrowright S^1 & & \circlearrowright \\ \underbrace{d^2H}_{Q} \leftarrow \text{quadratic} & & H \end{array}$$

Q generates $\Phi^t = D\varphi^t$ $\leftarrow$ linear sympl
on $T_pM = \mathbb{R}^{2n}$ $\quad$ $\underline{S^1\text{-action}}$ (Ham)

$\Downarrow$ $\leftarrow$ not totally trivial: split the repr into irreducible ones

Ex $\quad Q = \sum \lambda_j |z_j|^2, \quad \lambda_j \in \pi\mathbb{Z}$
as in Ex 1.

$\underline{\text{Cor. 1}}$ Assume p isolated ($\Leftrightarrow \lambda_j \neq 0$)
$\Rightarrow \mu(p)$ is even

$\Downarrow$ $\leftarrow$ Morse Theory

$\underline{\text{Thm}}$ $\quad \varphi_H^t$ Ham S^1-action with isolated fixed pts
$\Rightarrow H$ is Morse, all $\mu(p)$ even
$\Rightarrow \partial_M = 0$ over $\mathbb{Z}$

$\Downarrow$

$\underline{\text{Cor 2}}$ $\quad b_k := \dim H_k(M;\mathbb{Z}) = \#\,\mathrm{Crit}_k(H)$

Some examples seem to follow

Meanwhile: $\underline{\text{Cor 1}} \Rightarrow H_{odd}(M) = 0$
and $\pi_1(M) = 0$

Symplectic reduction for Ham S^1-actions

- $H : M \longrightarrow \mathbb{R}$ a Ham generating S^1-action
- $\{H = 0\} = \Sigma \longleftarrow$ reg. level

  (0: can be anything)

$\Rightarrow$ S^1-action on Σ is locally free:

$X_H \neq 0$

Pf $X_H = 0 \underset{HE}{\Rightarrow} dH = 0 \Rightarrow$ not a reg value

- Assume furthermore that this action is free $\Rightarrow$ Σ / S^1 is a manifold

Thm (Symplectic reduction for S^1-actions)

$\exists !$ sympl str ω_{red} on Σ / S^1 s.t.

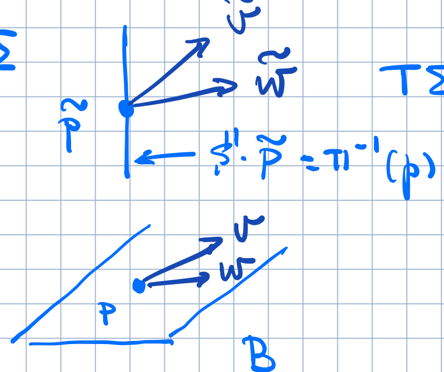
$$\pi^* \omega_{red} = \omega|_\Sigma \ ; \ \pi : \Sigma \longrightarrow \Sigma / S^1 = B$$

c.f. V sympl. v.s.

$L \subset V$ coisotropic e.g. a hypersurface

$\Rightarrow$ L / L^ω is symplectic

Pf * Existence



Recall

$$T\Sigma^{\omega} = \underbrace{\ker \omega|_{\Sigma} = \mathrm{span}\langle X_H \rangle}_{\text{Tangent to } S^1\text{-orbit}} \in T\Sigma$$

$\pi(\tilde{p}) = p$

$\pi_* \tilde{v} = v$

$\pi_* \tilde{w} = w$

Set $\omega_{red}(v, w) = \omega(\tilde{v}, \tilde{w})$

- ind of the lifts $\tilde{v}, \tilde{w}$
  $\Leftarrow$ any two lifts differ by a vector in $\ker \omega|_\Sigma$

- ind of $\tilde{p}$ over $p \Leftarrow \omega$ is S^1-invariant

This proves existence

* Uniqueness : π submersion
  $\Rightarrow \pi^* : \Omega^*(B) \longrightarrow \Omega^*(\Sigma)$ mono
  $\Rightarrow \pi^* \omega_{red} = \omega|_\Sigma$ determines ω_{red} uniquely

* Non-degeneracy : clear

* Closed : $d\omega_{red} = 0 \Leftrightarrow \underbrace{\pi^* d\omega_{red}}_{} = 0$

  $= d\pi^* \omega_{red} = d\,\omega|_\Sigma = 0$

$\triangleleft$

Cor. $\mathbb{C}P^n$ has a sympl. str: Fubini-Study

Pf $H = \sum_{j=0}^{n} |z_j|^2$ on $\mathbb{R}^{2(n+1)} = \mathbb{C}^{n+1}$

$\Sigma = S^{2n-1}$, S^1-action: Hopf action

$$\Sigma \xrightarrow{\pi} \mathbb{C}P^n \leftarrow \text{Hopf fibration}$$

$\triangleleft$

Generalization(s)

Hamiltonian $T^k = \underbrace{S^1_1 \times \ldots \times S^1_k}_{k}$-action on M:

- H_i generates S^1_i-action; $\underbrace{\Phi = (H_1, \ldots, H_k)}_{\text{"moment map"}}$

  $\Sigma = \Phi^{-1}(0)$ ← reg. value

  $\Rightarrow T^k$ action on Σ is locally free

- Assume that this action is free

  $\Rightarrow \Sigma / T^k =: B$ is smooth

- In any event $T\Sigma^\omega = T(T^k\text{-orbits})$

  $= \operatorname{span}\langle X_{H_1}, \ldots, X_{H_k} \rangle \subset T\Sigma$

Thm-Ex $\exists!$ sympl. str. ω_{red} on B s.t.

$\pi^* \omega_{red} = \omega|_\Sigma$; $\pi : \Sigma \to B$

Pf.: the same $\triangleleft$

Has important generalization to other gp actions
Marsden-Weinstein reduction

Ex $\{H_i, H_j\} = 0$

Digression: Complex str on $\mathbb{C}P^n$

- Almost complex str: $J : TM \circlearrowleft$; $J^2 = -I$
  Makes every T_pM into a complex v.s.
  and TM into a complex v.b.

- Complex str on M: cover by coord charts φ_j
  with values in $\mathbb{C}^n$ with hol transition maps:
  $\varphi_j \varphi_k^{-1}$: open in $\mathbb{C}^n \longrightarrow$ open in $\mathbb{C}^n$ is hol :

  $D(\varphi_j \varphi_k^{-1})$ is complex linear

  $\Rightarrow$ a well-defined almost complex str
  on M (carried over from $\mathbb{C}^n$)

"Thm" Almost complex str on M
with some integrability condition "$N_J = 0$"
$\Longleftrightarrow$ complex str on M

$\uparrow$ Newlander - Nirenberg thm

$$\underbrace{N_J(X,Y)}_{\text{Nijenhuis tensor}} := [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

Cor dim $M = 2$, J is automatically integrable.
Not immediately obvious, but follows
from the formal properties of N_J.

Let now ω be a symplectic str:
ω & J a compatible if
$g(v,w) := \omega(v, Jw)$ is a R.m.

$\Leftrightarrow$ $g + i\omega$ is (almost) Hermitian
i.e. Hermitian on each (T_pM, J_p)

Thm A sympl str. admits a compatible almost complex str and the latter is unique up to homotopy

$\Uparrow$

Linear algebra : (V, ω) sympl. v.s. $\mathbb{R}^{2n} = \mathbb{C}^n$
The space of all linear J's comp. with ω is contractible : $Sp(2n)/U(n)$
McDuff - Salamon : details

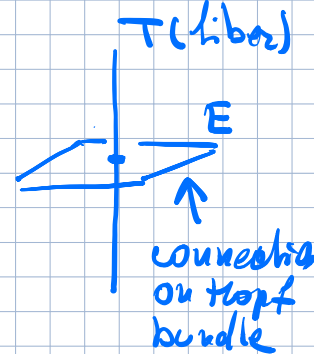
Back to $\mathbb{CP}^n$

$\mathbb{CP}^n$: complex charts $\Rightarrow$ complex str

How to see it from symplectic reduction

$$\mathbb{CP}^n = S^{2n+1}/S^1 \qquad \Sigma$$

$$S^{2n+1} \subset \mathbb{C}^{2(n+1)}, \omega, J, g$$



$$E \subset TS^{2n+1} \subset T\mathbb{C}^{n+1}$$

↑ maximal complex subspace

$$E \oplus \underbrace{T(\text{orbit})}_{JV} = T\Sigma$$

$JV \leftarrow$ normal vector field

$$\pi \downarrow \quad T\mathbb{CP}^n$$

$$(E, J) \xrightarrow{\pi_*} T\mathbb{CP}^n$$

is a complex linear isom

By constr J on $\mathbb{CP}^n$ is compatible with $\omega_{red} = \omega_{FS}$ (Because ω & J compatible on $\mathbb{C}^n$)

Cor A smooth alg variety, aka complex submanifold of $\mathbb{CP}^n$ is automatically symplectic with the induced sympl. str.

Pf : complex subspace of $\mathbb{C}^n$ $\Rightarrow$ symplectic ◁

Symplectic structures on coadjoint orbits

More examples

G a Lie gp

$\mathfrak{g}$ = Lie algebra

$\mathfrak{g}^*$ = dual space

G acts on $\mathfrak{g}$: adjoint action

$$Ad_g : \begin{array}{l} G \longrightarrow G \\ h \longmapsto g h g^{-1} \end{array} \quad ; \quad Ad_g(e) = e$$

$$\Rightarrow \underbrace{DAd_g}_{Ad_g} : T_eG = \mathfrak{g} \circlearrowleft$$

Explicitly $\quad Ad_g(\eta) = \frac{d}{dt} g \exp(\eta t) g^{-1} \Big|_{t=0}$

Now write $\quad ad_\xi = \frac{d}{dt} Ad_{\exp(\xi t)} \Big|_{t=0}$

$$ad_\xi \eta = \frac{d}{dt} Ad_{\exp(\xi t)} \eta \Big|_{t=0}$$

<u>Fact</u> $\quad ad_\xi \eta = [\xi, \eta]$

$\Rightarrow$ G acts on $\mathfrak{g}^*$ via the dual action
Coadjoint actions Ad_g^* & ad_ξ^*

- $(Ad_g^* \lambda)(\eta) = \lambda(Ad_g \eta)$

- $(ad_\xi^* \underset{\mathfrak{g}^*}{\underset{\cap}{\lambda}})(\underset{\mathfrak{g}}{\underset{\in}{\eta}}) = \lambda(ad_\xi \eta)$

Ex $G = U(n)$ or $SO(n)$ ← compact
$\mathfrak{g} = u(n)$ or $so(n)$

$Ad_g h = g h g^{-1}$ ← matrices
$Ad_g \xi = g \xi g^{-1}$ ← matrices

compact $\Rightarrow$ biinvariant inner product on $\mathfrak{g}$ ← Ad-invariant
Explicitly: $\langle \xi, \eta \rangle = -\operatorname{tr} \xi \bar{\eta}$
$\Rightarrow$ $\mathfrak{g} = \mathfrak{g}^*$ equivariantly
With this identification

$$Ad_g \lambda = g \lambda g^{-1}, \quad \lambda \in \mathfrak{g} = \mathfrak{g}^*$$
$$ad_\xi \lambda = [\xi, \lambda]$$

Thm $\mathcal{O} \subset \mathfrak{g}^*$ is a coadjoint orbit
$\Rightarrow$ $\mathcal{O}$ has a canonical sympl str
(Kirillov - Kostant s.str).

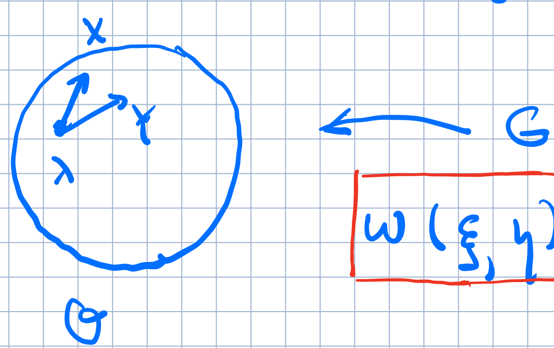
Several pfs: most natural via sympl reduction of the G-action on T^*G

<u>Direct pf:</u> Think of $S^2 \subset \mathbb{R}^3 = so(3)$
$G = SO(3)$

Construction of ω

$\lambda \in \mathcal{O} = G\cdot\lambda$, $X, Y \in T_\lambda \mathcal{O}$

$X = ad^*_\xi \lambda$, $Y = ad^*_\eta \lambda$



$$\boxed{\omega(\xi, \eta) := \lambda([\xi, \eta])}$$

similar

<u>Well defined:</u> need $ad^*_\xi \lambda = 0 \Rightarrow \lambda([\xi, \eta]) = 0$

But $\lambda([\xi, \eta]) =: (ad^*_\xi \lambda)(\eta) = 0$

<u>Non-deg:</u> ξ such that $\lambda([\xi, \eta]) = 0 \quad \forall \eta$

$\Leftrightarrow$ $(ad^*_\xi \lambda)(\eta) = 0 \ \forall \eta \Rightarrow ad^*_\xi \lambda = 0 \Rightarrow X = 0$ 91

Closed: outline Recall:

$d\omega(X,Y,Z)$

$= L_X\,\omega(Y,Z) - L_Y\,\omega(X,Z) + L_Z\,\omega(X,Y)$

$+\ \omega([X,Y],Z) - \omega([Y,Z],X) + \omega([X,Z],Y)$

Apply to our ω

$d\omega \Leftarrow$ Jacobi identity in $\mathfrak{g}$

$$[[\xi,\eta],\zeta] + [[\eta,\zeta],\xi] + [[\zeta,\xi],\eta] = 0$$

Rmk The construction for $\lhd$

$$S^2 \subset \mathbb{R}^3 = so(3) = so(3)^*$$

$$X,Y \in T_\lambda S^2$$

$$\omega_\lambda(X,Y) = \langle \lambda, \underbrace{[X,Y]}_{\text{cross product}} \rangle \leftarrow \text{inner product}$$

ω_λ ← area form on S^2 of radius $\|\lambda\|$

By the way $JX = [\nu, X]$

ν ← unit normal $= \dfrac{\lambda}{\|\lambda\|}$

$\lhd$

More general picture: Poisson str on $\mathfrak{g}^*$

Ex. $G = U(n)$

$\mathfrak{g} = \mathfrak{g}^* = \mathfrak{u}(n)$: $\bar{\xi}^T = -\xi$

action is conjugation : $g\xi g^{-1}$

Take
$$\xi = \begin{pmatrix} i\xi_1 & & 0 \\ & \ddots & \\ 0 & & i\xi_n \end{pmatrix} \qquad \xi_j \neq \xi_k$$

$\Rightarrow$ $\mathcal{O}$ = Full Flag manifold in $\mathbb{C}^n$ ← Recall the definition

$= U(n)/T^n$ ← $\mathrm{Stab}(\xi)$

← diagonal matrices $g\xi = \xi g$

$\Rightarrow$ symplectic structure on a Flag manifold (depends on ξ)

E.g. $U(2)$

$\mathcal{O} = \mathbb{F}l(1,2) = \mathbb{C}P^1 = S^2$ ← s.f. depends on ξ

Similarly partial flags, e.g. Grassmannians:

$$\xi = \begin{pmatrix} i\xi & & & & 0 \\ & \ddots & & & \\ & & i\xi & & \\ & & & i\xi' & \\ 0 & & & & \ddots \\ & & & & i\xi' \end{pmatrix} \Big\} k \qquad Gr(k,n) = Gr(n-k,n)$$

E.g. $k = 1$ or $n-1$ $\Rightarrow$ $\mathbb{C}P^n$

Rmk. Take $G = SO(n)$

$\mathfrak{g} = \mathfrak{g}^* = \mathfrak{so}(n) : \xi^T = -\xi$

Don't get all real (oriented or not) Grassmannians

(stabilizers are more complicated)

E.g. obviously cannot get $\mathbb{RP}^n$ or $S^{n>2}$

but can get some. E.g. $Gr^+(2,4)$, ...

or S^2

Homology of complex Flag manifolds
– sample Application of Morse Theory

M = Full Flag manifold in $\mathbb{C}^n = \mathcal{O}$
$= U(n)/\mathbb{T}^n \leftarrow$ diagonal matrices

$$\boxed{\textbf{Thm} \quad \sum_k \dim H_k(M) = n!}$$

Rmk One can also calculate individual dimension but this more involved.

Pf – Outline

$$M \subset u(n) = u(n)^*$$

- Then $\mathbb{T}^n \subset U(n)$ acts on M by conjugation. In the identification $M = U(n)/Stab(\xi) = \mathbb{T}^n$ this the left action

- By construction this action is Hamiltonian

Moment map

$$M \hookrightarrow u(n)^* \longrightarrow (t^n)^* = \mathbb{R}^n$$
$$u(n) \hookleftarrow t^n = i\mathbb{R}^n$$

- Fixed pts: obtained from ξ by permuting the components:
(Needs a pf...)

$$\xi = \begin{pmatrix} i\xi_1 & & \\ & \ddots & \\ & & i\xi_n \end{pmatrix} \Rightarrow n! \text{ fixed pts}$$

- $\eta \in \mathfrak{t}^n$ gives rise to a Hamiltonian v.f. on M with Hamiltonian
  $H: M \hookrightarrow \mathfrak{u}(n)^* \longrightarrow (\mathfrak{t}^n)^* \xrightarrow[\langle \cdot, \eta \rangle]{} \mathbb{R}$
- Assume that η generates $S^1 \subset \mathbb{T}^n$
  $\Rightarrow$ H generates an Ham S^1-action on M
- For "almost all" $S^1 \subset \mathbb{T}^n$
  $|M^{S^1} = M^{\mathbb{T}^n}| = n!$ ⟵ Discuss

<u>Reason:</u> only finitely many different stabilizers. Take $S^1 \pitchfork$ to all stabilizers.

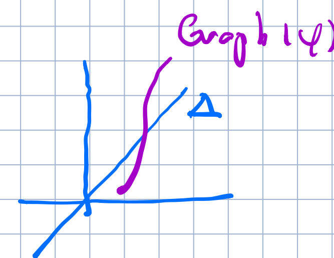
◁

- All points have even indices
  $\Rightarrow \partial_M = 0 \Rightarrow CM(H) = H_*(M)$
  $\Rightarrow \sum \dim H_k(M) = |M^{S^1}| = n!$

§9 Arnold's conjecture and all that

Some definitions - non-degeneracy

$\varphi : M \circlearrowleft$



Def • $x \in \mathrm{Fix}(\varphi)$ is non-deg if $\det(\underbrace{D\varphi_x}_{T_xM \circlearrowleft} - I) \neq 0$

No eigenvectors with eigenvalue 1.

$\iff \mathrm{Graph}(\varphi) \pitchfork_{\text{at } x} \Delta \subset M \times M$

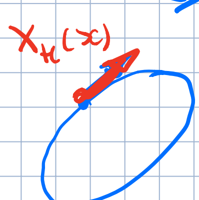
• φ is non-deg if all $x \in \mathrm{Fix}(\varphi)$ are non-deg

Rmk: In any reasonable class (all smooth, symplectic, vol. pres, Hamiltonian) non-deg is a generic condition

Rmk: Warning $H : \underset{\text{symplectic}}{M} \to \mathbb{R}$

$\varphi^t(x) \subset \{H = c\}$ • T-periodic
• non-constant

$\Rightarrow$ x is deg fixed pt for φ^T



$X_H(x)$ is an eigenvector with eigenvalue 1.

Arnold's conjecture

<u>Conj</u> (AC, I) $\varphi_H : (M, \omega) \circlearrowleft$ Ham

closed

$$\Rightarrow \quad \# \operatorname{Fix}(\varphi) \geq \underbrace{\begin{cases} \sum \dim H_i(M), & \varphi_H \text{ non-deg} \\ CL(M)+1, & \text{in general} \end{cases}}_{\text{same as for critical pts of smooth functions}}$$

<u>Rmk</u>
- One of the most important questions in sympl. geometry $\Rightarrow$ Floer theory
- <u>Motivation</u>: Poincaré's work in celestial mechanics (non-trivial path)
- Arnold stated it somewhat differently
- - Non-deg case is pretty much proved by now: Floer $- \dots -$ Fukaya et al
  - Deg case is still open for many manifolds
    $S^2 \times S^2 \times S^2 \leftarrow$ with areas $(1, \sqrt{2}, \sqrt{3}) \dots$

Some other variants: Lagrangian intersections

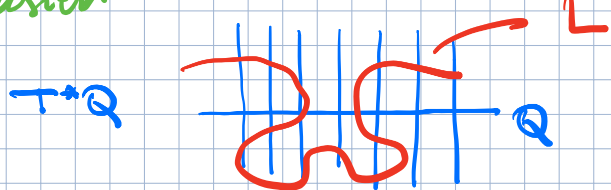
(Thm of Laudenbach - Sikorav) – Also conj by A.

$\varphi : T^*Q \circlearrowleft$ Hamiltonian diffeo

$L = \varphi(Q)$ zero section

$$\Rightarrow \quad \# L \cap Q \geq \begin{cases} \sum \dim H_i(L) : & \overbrace{L \pitchfork Q}^{\text{non-deg}} \\ cL(L)+1, & \text{general} \end{cases}$$

much easier



Ex. • Prove for $Q = S^1$ — explain

• $\alpha = df$, $L = \text{graph}(df) \subset T^*Q$

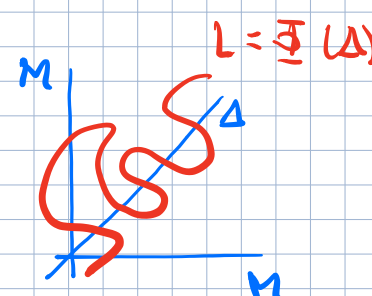
$L = \varphi(Q)$: $\varphi(q,p) = (q, p + df)$

$H(q,p) = f(q)$

AC, II $\Phi : \underset{(\omega, -\omega)}{M \times M} \circlearrowleft$ Ham. diffeo

$L = \Phi(\Delta) \leftarrow$ Lagr

$\# L \cap \Delta \geq \ldots\ldots$



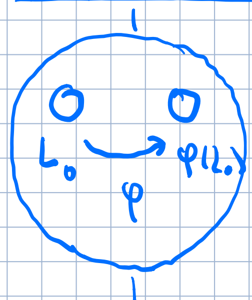
AC, II $\Rightarrow$ AC, I :

$\Phi(x,y) = (x, \varphi(y))$

Bottom line: in many instances one can expect L_0 & $\varphi(L_0)$ have many intersections

But not always:

Ex



$L_0 \subset S^2$ small circle (Lagrangian)

φ = rotation

$\varphi(L_0) \cap L_0 = \emptyset$

Back to AC, I

Ex $\quad H : M \to \mathbb{R}$ autonomous

$\mathrm{Crit}(H) \subset \mathrm{Fix}(\varphi)$

Note: $x \in \mathrm{Crit}(H)$ can be non-deg as a crit pt but deg as a fixed pt

E.g. $\quad H(p,q) = \pi(p^2 + q^2)$

But $\quad \varphi_H = id$

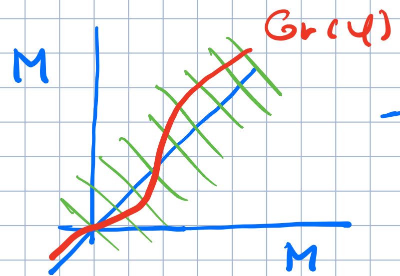
Further evidence

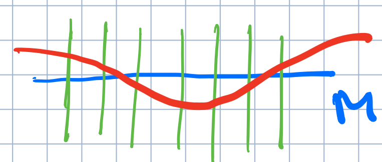
Thm (Weinstein, 70s)
$\varphi: M \circlearrowleft$ C^1-close to id, Ham
$\Rightarrow$ Fix$(\varphi) \geq \dots$

two versions
just $\varphi = \varphi^1_H$
or all φ^t_H, $0 \leq t \leq 1$

Pf

- $Gr(\varphi) \subset M \times M$ is C^1-close to Δ
  Weinstein tub nbd thm



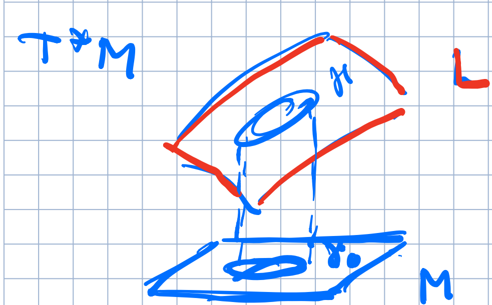


image$(Gr(\varphi)) \underset{C^1}{\approx}$ zero section
$\Rightarrow$ a section

$M \times M$ $\qquad$ T^*M

Lagrangian $\Leftrightarrow$ α is closed

- φ Hamiltonian $\Rightarrow$ $\underbrace{\alpha \text{ is exact}}$



$\Leftrightarrow \int_{\gamma_0} \alpha = 0$, $\gamma_0 \leftarrow$ any loop in M

$\Leftrightarrow \int_{\gamma} \lambda = p\,dq$, $\gamma \leftarrow$ any loop in L

- But $\int_\gamma \lambda = \mathrm{Flux}(\varphi)(\gamma_0) = 0$

Nuance: ◦ easy when $\varphi_H^t \overset{C^1}{\approx} id$

• Need work when only $\varphi \overset{C^1}{\approx} id$

($\Leftarrow$ Flux conjecture)

$\Rightarrow \quad \alpha = df$

$L \cap M \longleftrightarrow \mathrm{Crit}(f)$

Apply Morse Theory or LS Theory

$\triangleleft$

On the Pf of AC:

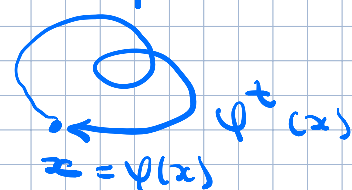
Floer Theory = Morse Theory for Action Functional

* M exact $\omega = d\lambda$

  E.g. $\mathbb{R}^{2n}$ or T^*M

$H : M \times S^1 \longrightarrow \mathbb{R}$, $\varphi = \varphi^1_H$

$\mathrm{Fix}(\varphi) \longleftrightarrow$ 1-periodic orbits of φ^t_H



$\Lambda = C^\infty(S^1, M)$

$A_H : \Lambda \longrightarrow \mathbb{R}$ action functional

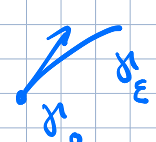
$$A_H(\gamma) = -\int_\gamma \lambda + \int_0^1 H_t(\gamma_t)\,dt$$

Thm (Least Action Principle)

$$\boxed{\underbrace{\mathrm{Crit}(A_H)} = \text{1-periodic orbits of } \varphi^t_H}$$

$\gamma_0 \in \mathrm{Crit}(A_H)$: $\forall\ \gamma_\varepsilon \leftarrow$ var. of γ_0

$$\frac{d}{d\varepsilon} A_H(\gamma_\varepsilon)\Big|_{\varepsilon=0} = 0$$



Pf: ex

* M closed, $\omega|_{\pi_2} = 0$: $\displaystyle\int_{S^2 \to M} \omega = 0$: E.g. T^{2n}, $\Sigma_{g\geq 1}$

But not $\mathbb{C}P^n$

$\Lambda_0 =$ contractible loops

$$A_H(\gamma) = \int_{\substack{D^2 \to M}} \omega + \int_0^1 H_t(\gamma(t))\,dt$$

well-defined : $\displaystyle\int_{S^2} \omega = 0$

$$A_H : \Lambda_0 \to \mathbb{R}$$

contr. 1-periodic orbits of φ_H^t $\subset$ $\mathrm{Fix}(\varphi)$

$\boxed{\textbf{Thm (LAP)} \quad \mathrm{Crit}(A_H) = \text{Contr. 1-periodic orbits of } \varphi_H^t}$

Now: do Morse or LS Theory for A_H

Extremely Difficult!

Maybe come back to it later.

Weinstein Conjecture

Contact Analog of AC

(M^{2n+1}, α) ← contact form

↑

closed manifold $\Rightarrow$ Reeb v.f. $\quad \alpha(R) = 1$, $\ i_R d\alpha = 0$

$\Rightarrow$ Reeb flow

Conj (WC): Every Reeb flow on a closed contact manifold has a periodic orbit.

Extremely difficult: by-and-large open

Some results

- $M = S^{2n-1} \hookrightarrow \mathbb{R}^n$ star-shaped (Weinstein, late 70s)
- $M \hookrightarrow \mathbb{R}^{2n}$ contact type (Viterbo, 80s)
- $\dim M^3$ Taubs (rel. recent)

§ 10 Elements of Poisson Geometry

Def A Poisson algebra:

- a commutative algebra A over $\mathbb{R}$ or $\mathbb{C}$
- A is also a Lie algebra
- Product rule

$$[f, gh] = [f,g]h + g[f,h]$$

Def A Poisson manifold: a manifold P s.t. $C^\infty(P)$ is given a str of Poisson algebra

Ex.
- A symplectic manifold with Poisson bracket

$$\{f, g\} = \omega(X_f, X_g) = -df(X_g)$$
$$= +L_{X_f} g = dg(X_f)$$

Note: Jacoby identity $\iff d\omega = 0$

- Any manifold with $\{f, g\} = 0$
- Any Lie alg with $f \cdot g \equiv 0$ is a Poisson algebra

Poisson bi-vector

Consider sections of $\Lambda^k TP^n$ ← any manifold

Notation Γ_k

∃ a bracket (Schouten – Nijenhuis)

$$[\ ,\ ] : \Gamma_k \times \Gamma_\ell \longrightarrow \Gamma_{k+\ell-1} \quad \text{s.t.}$$

- $\underbrace{[a, a\wedge b] = [a,b]\wedge c + (-1)^{(|a|-1)|b|}\, b \wedge [a, c]}_{\text{product rule}}$
- $\underbrace{[[a,b], c] = [a,[b,c]] + (-1)^{(|a|-1)(|b|-1)} [b, [a,c]]}_{\text{Graded Jacobi}}$

$[\ ,\ ]$ extends the Lie bracket

<u>Ex.</u> $\pi \in \Gamma_2 \Rightarrow [\pi,\pi] \in \Gamma_3$

<u>Prop</u>: Alt definition

- $(P, \pi \in \Gamma_2)$ set

$$\{f,g\} = \langle df \wedge dg, \pi \rangle \qquad (*)$$

By constr bilinear & Product rule

Jacobi $\Leftrightarrow [\pi,\pi] = 0$

- Conversely $(P, \{\ ,\ \})$ Poisson

$\Rightarrow \exists\ \pi \in \Gamma_2$ s.t. $(*)$ holds

and $[\pi,\pi] = 0$

On the pf

- $[\pi,\pi]=0 \iff$ Jacobi : calculation
- $\{,\} \rightsquigarrow \pi$

  π is characterized by

  $\langle \alpha\wedge\beta, \pi\rangle$ at every pt

  $\overset{"}{d}f\wedge dg \leftarrow$ at a pt

  $= \{f,g\}$

In coordinates: $\pi = \sum \pi_{ij} \frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$

Then $\pi_{ij} = \{x_i, x_j\} = \langle dx_i\wedge dx_j, \pi\rangle$

Hamiltonian v.f.

$$X_f := \text{"} df \lrcorner \pi \text{"} = [f,\pi]$$

$$L_{X_f} g = \{f,g\} = \langle df\wedge dg, \pi\rangle$$

$$\pi : T^*P \longrightarrow TP$$

$$\alpha \longmapsto \alpha\lrcorner\pi$$

$$df \longmapsto X_f$$

$$dx_i \longmapsto 2\sum_j \pi_{ij}\frac{\partial}{\partial x_j}$$

Ex. ω non-deg 2-form on M

$$\omega^\# : TM \longrightarrow T^*M \quad \text{non-deg}$$

$$X \longrightarrow i_X\omega$$

set $\pi = (\omega^\#)^{-1} : T^*M \longrightarrow TM$

$\pi \in \Gamma_2$: $\quad d\omega = 0 \iff [\pi,\pi] = 0$

- <u>Ham v.f.</u> $\subset$ <u>Poisson v.f.</u> $= \{X \mid L_X \pi = [X, \pi] = 0\}$

Note: even locally a Poisson v.f. need not be Ham: Big difference with sympl geom

E.g. $\pi = 0$: everything Poisson, nothing Hamiltonian

- $X_{\{f,g\}} = [X_f, X_g]$

  $[X_f, Y] = X_{-L_Y f}$: Ham. v.f. is an ideal in Poisson v.f.

Symplectic Foliation

- $\pi : T^*P \longrightarrow TP$
- $\mathrm{im}(\pi) = \mathcal{D}_x = \mathrm{span}(\text{Ham v.f. at } x)$
  - <u>singular</u> integrable distribution

    v, w tangent to $\mathcal{D} \Rightarrow [v, w]$ is tangent

<u>Thm</u> (Frobenius)

- $\forall x \quad \exists \quad F_x \hookrightarrow P$ (1-1 immersion)

  s.t. $\forall y \in F_x \quad T_y F_x = \mathcal{D}_y$
- π is tangent to F_x & symplectic there

  $\mathcal{F} = \{F_x\}$ = symplectic foliation

  $\Uparrow$ sympl. leaves

Examples

- (M, ω) symplectic $\quad \omega^{-1} = \pi$

  only one leaf

- $(P, \pi = 0)$ $\quad$ leaves = pts

- $P = (\mathbb{R}^2 \; x, y)$ $\quad \pi = f(x,y)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$

automatically $[\pi, \pi] = 0$

$$\{x, y\} = f(x,y)$$

Leaves: • pts on $\{f = 0\}$

• connected components of $\{f \neq 0\}$

- Dual of a Lie algebra: $\mathfrak{g}^*$

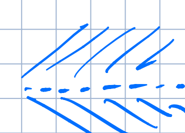
$$\{f, g\}(\underset{\mathfrak{g}^*}{\underset{\cap}{\lambda}}) = \lambda([\underset{\mathfrak{g}}{\underset{\cap}{df}}, \underset{\mathfrak{g}}{\underset{\cap}{dg}}])$$

Jacobi for $\mathfrak{g}$ $\Longleftrightarrow$ Jacobi for $\{ , \}$

Symplectic Leaves = coadjoint orbits

- Subexamples $\quad$ • $su(2) = so(3) = \mathfrak{g} = \mathbb{R}^3$

• $sl(2) = \mathbb{R}^3$

• $[X, Y] = Y$: $\mathbb{R}^2$: 

$$\{x, y\} = x$$

- $\mathbb{R}^3$ (x,y,z)

  $$\pi = a\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + b\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} + c\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}$$

  - $[\pi,\pi]=0$, $\operatorname{rk}\pi = 2$
  - leaves can be dense

  Even locally: Poisson $\neq$ Ham

  E.g. $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are all Poisson

<u>Poisson maps</u>

$$F : (P_1, \pi_1) \longrightarrow (P_2, \pi_2)$$

is Poisson if $\forall\, x \in P_1$

$$DF_x(\pi_1(x)) = \pi_2(F(x))$$

$\Leftrightarrow$ $F^* : C^\infty(P_2) \longrightarrow C^\infty(P_1)$ morphism of Poisson algebras: $F^*\{f,g\} = \{F^*f, F^*g\}$

$\dim P_1 = \dim P_2$ & π_1, π_2 symplectic

Poisson = symplectic

Moment maps and reduction revisited

(M, ω) symplectic

G = gp acting on M by symplectomor.

Def.

$\Phi : M \longrightarrow \mathfrak{g}^*$ is a moment map if

$\forall\ \xi \in \mathfrak{g} = \mathfrak{g}^{**}$

$X_{h_\xi} = \xi_M :=$ v.f. on M generated by ξ

$h_\xi : M \longrightarrow \mathfrak{g}^* \xrightarrow{\langle \xi, \cdot \rangle} \mathbb{R}$

Might or not exist.

Ex. $G = \mathbb{T}^k$, $\mathfrak{g}^* = \mathbb{R}^k$ with $\pi = 0$

Just as before

Note! Φ is Poisson $\Leftrightarrow$ Φ is equivariant

Can do reduction:

- $\Phi^{-1}(0)/G$ = symplectic
  Assuming smooth etc ...
- $\Phi^{-1}(\lambda)/G_\lambda$ = symplectic
  $G_\lambda = \mathrm{stab}(\lambda)$

E.g. Can get coadjoint orbits from T^*G ...

Poisson cohomology

Recall $\Gamma_k = \Gamma(\Lambda^k TP)$

$$0 \to \Gamma_0 = C^\infty \to \Gamma_1 = \mathfrak{X} \to \Gamma_2 \longrightarrow \dots$$
$$f \longmapsto X_f$$
$$X \longmapsto [X, \pi] \quad \dots$$

$\partial : \Gamma_k \to \Gamma_{k+1}$

$\quad w \mapsto [w, \pi]$

Jacobi & $[\pi, \pi] = 0 \Rightarrow \partial^2 = 0$

Def. Poisson cohomology
$$H^*_\pi(P) := H(\Gamma_*, \partial)$$

Usually very difficult to calculate

Ex. • $\pi = 0$: $H^k_\pi(P) = \Gamma_k$

• $\pi = \omega^{-1}$ symplectic : $TP \underset{\pi}{\overset{\omega}{\Longleftrightarrow}} T^*P$

$\quad (\Gamma_k, \partial) \longleftrightarrow (\Omega^k, d) \leftarrow$ de Rham

$\Rightarrow H^*_\pi(P) = H^*_{dR}(P)$

• $H^1_\pi(P) = \dfrac{\text{Poisson v.f.}}{\text{Ham v.f.}}$ cf. symplectic

• H^2_π = "deformation of π"

• $H^*_\pi(\mathfrak{g}^*) = H(\mathfrak{g}; C^\infty(\mathfrak{g}^*))$