Math 23B Practice Midterm Solutions - 2015

1(a). True. Integrating the function 1 over a 2D region indeed gives its area. $\square$

1(b). False. There is a missing $||$ sign. The formula should read:

$$\int\int_D f(x,y)\, dA_{xy} = \int\int_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv}.$$

The notation $\frac{\partial(x,y)}{\partial(u,v)}$ means "the Jacobian determinant of the uv-to-xy transformation", but we require the *absolute value* of this determinant. The absolute value is critical.

The technical reason for the absolute value of the determinant is that dA_{xy} can be thought of as a "positive-valued area form" – it gives positive areas. Thus we must also require the transformed version $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv}$ to be a "positive-valued area form", which necessitates the absolute value. $\square$

1(c). False. Area is a positive value, but $x^2 - 1$ is negative between -1 and 1 (in other words, $y = x^2 - 1$ is the lower bound). The integral given will give -1 times the correct area. $\square$

1(d). True. Spherical coordinates (at least using the convention in the textbook) are given by

$$x = \rho\cos\theta\sin\phi,\ y = \rho\sin\theta\sin\phi,\ z = \rho\cos\phi.$$

Computing the absolute value of the Jacobian matrix:

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \left|\det\begin{bmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi}\end{bmatrix}\right| = \left|\det\begin{bmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi\end{bmatrix}\right|$$

(using the bottom row to generate subdeterminants)

$$= \left|\cos\phi\ \det\begin{bmatrix} -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi\end{bmatrix} - 0 + -\rho\sin\phi\ \det\begin{bmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi\end{bmatrix}\right|$$

$$= \left|\cos\phi\left(-\rho^2\sin^2\theta\sin\phi\cos\phi - \rho^2\cos^2\theta\sin\phi\cos\phi\right) - \rho\sin\phi\left(\rho\cos^2\theta\sin^2\phi - -\rho\sin^2\theta\sin^2\phi\right)\right|$$

$$= \left|\cos\phi\left(-\rho^2\sin\phi\cos\phi\right) - \rho\sin\phi\left(\rho\sin^2\phi\right)\right| = \left|-\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi\right|$$

$$= \left|-\rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right)\right| = \left|-\rho^2\sin\phi\right|$$

Since $\phi \in [0,\pi]$, $\sin\phi \geq 0$. The square of a number is always at least zero; $\rho^2 \geq 0$. Thus $-\rho^2\sin\phi$ is negative, so the above value, our answer, is $\rho^2\sin\phi$.

This, like the Jacobian for polar coordinates, is commonly-enough-used that it would be helpful to memorize. $\square$

1(e). False. A counterexample (a map that is one-to-one but is not onto) is $\arctan(x)$. It passes the horizontal line test and is therefore one-to-one, but its range is only $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, so, for example, the number 42 is not equal to $\arctan(x)$ for any x, and therefore $\arctan(x)$ is not onto. $\square$

1(f). False. Let $T(u,v) = (x(u,v), y(u,v))$ (we're just giving names to the coordinate functions). Then the Jacobian (aka the Jacobian determinant) is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix} = \det\begin{bmatrix} a & b \\ c & d\end{bmatrix} = ad - bc.$$

The answer differs by a negative sign. $\square$

1(g). False. The $dy\,dx$ is backwards (we must integrate the thing that has constant bounds-of-integration last). It should read:

$$\int\int_D f(x,y)\ dx\,dy = \int_a^b \int_{\gamma_1(y)}^{\gamma_2(y)} f(x,y)\ dx\,dy.$$

With $dy\,dx$, we wouldn't get a real number as an answer. $\square$

1(h). True. This is analogous to summing up n numbers and dividing by n to get their average value.

One could think of $f(x,y,z)$ having units of "function value" and dV as having units of volume. The upper integral gives an answer in units of "function value" times volume. The lower integral gives units of volume. Dividing gives units of "function value", and the specific number is the average value of the function over the region W. $\square$

1(i). False. Boundedness is not strong enough to guarantee integrability. We need to know more about the function's discontinuities. Theorem 2 (page 330) states that if $f\colon D \to \mathbb{R}$ is bounded and if the set of discontinuities of f lies on a finite union of graphs of continuous functions, then f is integrable.

For a counterexample to the problem, all we need to do is come up with a function that is "too discontinuous". One example is $g\colon [0,1]\times[0,1] \to \mathbb{R}$ defined by $g(x,y) = 0$ if x is rational and $g(x,y) = 1$ if x is irrational. Then g is discontinuous at every (x,y). But the square $[0,1]\times[0,1]$ isn't a *finite* union of graphs of continuous functions. $\square$

1(j). True. The integrand is an odd function in x, and the region D being integrated over is symmetric in x. Thus the integral, if split up using the $x=0$ axis and written as two integrals, the left and right integrals would cancel out. $\square$

2(a). The integrand is odd in y and the region is symmetric in the y coordinate. Thus the integral is zero. From a computational perspective,

$$\int\int_D y^3\,dx\,dy = \int_0^1 \int_{-e^x}^{e^x} y^3\,dy\,dx = \int_0^1 \left[\frac{1}{4}y^4\right]_{-e^x}^{e^x} dx = \frac{1}{4}\int_0^1 (e^x)^4 - (-e^x)^4\ dx = \frac{1}{4}\int_0^1 e^{4x} - e^{4x}\,dx.$$

The integrand cancels itself out and we get zero. $\square$

2(b). In order to figure out bounds of integration, we need to calculate the intersection of the two boundary curves.

$$2\sqrt{2}x^2 = y = \sqrt{x} \implies 4\cdot 2\cdot x^4 = x \implies 8x^4 - x = 0 \implies x\left(8x^3-1\right) = 0$$
$$\implies x = 0 \text{ or } 8x^3 - 1 = 0.$$

In the case where $8x^3-1=0$, we get $x^3 = \frac{1}{8}$ and therefore $x = \frac{1}{2}$. So the solutions are $x=0$ or $x=\frac{1}{2}$. We could plug these x values into the curves to see what the actual points of intersection are, but it turns out not to matter – all we need are x-bounds on the integral. Finally, we need to figure out which is the lower bound and which is the upper bound for y. Going half-way in the interval $x \in \left[0, \frac{1}{2}\right]$, and plugging in $\frac{1}{4}$ into both y-bounds, we get

$$y = 2\sqrt{2}\left(\frac{1}{4}\right)^2 = \frac{2\sqrt{2}}{16} = \frac{\sqrt{2}}{8} \text{ and } y = \sqrt{\frac{1}{4}} = \frac{1}{2} = \frac{4}{8}.$$

But $\sqrt{2} < 4$, so $\frac{\sqrt{2}}{8} < \frac{4}{8}$. Therefore $y = 2\sqrt{2}x^2$ is the lower bound and $y = \sqrt{x}$ is the upper bound. (This could be seen graphically by noticing that $y = 2\sqrt{2}x^2$ is a parabola pointing up and $y=\sqrt{x}$ is half a parabola pointing right, so $y = \sqrt{x}$ will go higher first).

$$\int\int_D xy\,dA = \int_0^{\frac{1}{2}} \int_{2\sqrt{2}x^2}^{\sqrt{x}} xy\,dy\,dx = \int_0^{\frac{1}{2}} x\left[\frac{1}{2}y^2\right]_{2\sqrt{2}x^2}^{\sqrt{x}} dx = \frac{1}{2}\int_0^{\frac{1}{2}} x\left(\left(\sqrt{x}\right)^2 - \left(2\sqrt{2}x^2\right)^2\right) dx$$

$$= \frac{1}{2}\int_0^{\frac{1}{2}} x^2 - 8x^5\,dx = \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{8}{6}x^6\right]_0^{\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{3}\left[x^3 - 4x^6\right]_0^{\frac{1}{2}} = \frac{1}{6}\left(\left(\frac{1}{2}\right)^3 - 4\left(\frac{1}{2}\right)^6\right)$$

$$= \frac{1}{6}\left(\frac{1}{8} - 4\cdot\frac{1}{64}\right) = \frac{1}{6}\left(\frac{1}{8} - \frac{1}{16}\right) = \frac{1}{6}\cdot\frac{1}{16} = \frac{1}{96}.$$

Sanity check: xy is positive in the region we've integrated over, so we should have a positive answer. $\frac{1}{96}$ is indeed positive. □

3(a). The region is a simple box, so there's no special setup. But in fact the integral is so regular that we can split it up and save some computation.

$$\iiint_W \frac{1}{xyz}\,dx\,dy\,dz = \int_1^e\int_1^e\int_1^e \frac{1}{xyz}\,dx\,dy\,dz = \int_1^e \frac{1}{z}\int_1^e \frac{1}{y}\int_1^e \frac{1}{x}\,dx\,dy\,dz.$$

The innermost integral $\int_1^e \frac{1}{x}\,dx$ is constant with respect to y and z. So we can pull it out of the integral.

$$= \left(\int_1^e \frac{1}{x}\,dx\right)\left(\int_1^e \frac{1}{z}\int_1^e \frac{1}{y}\,dy\,dz\right).$$

Same thing with $\int_1^e \frac{1}{y}\,dy$.

$$= \left(\int_1^e \frac{1}{x}\,dx\right)\left(\int_1^e \frac{1}{y}\,dy\right)\left(\int_1^e \frac{1}{z}\,dz\right).$$

There's nothing special about using the specific letters xyz for our integrals, we could just as well use a happy face or a cloud. But just to be mundane, let's use u.

$$= \left(\int_1^e \frac{1}{u}\,du\right)\left(\int_1^e \frac{1}{u}\,du\right)\left(\int_1^e \frac{1}{u}\,du\right) = \left(\int_1^e \frac{1}{u}\,du\right)^3.$$

The only computation we need to do is a single variable integral.

$$\int_1^e \frac{1}{u}\,du = [\ln|u|]_1^e = \ln|e| - \ln|1| = \ln e - \ln 1 = 1 - 0 = 1.$$

Cubing this, we get our answer $\left(\int_1^e \frac{1}{u}\,du\right)^3 = 1$. □

3(b). The region W is a quarter cylinder with a roof that slopes down to meet the origin, looking sort of like a half-wheel of cheese.

$x^2 + y^2 = 9$ is an infinite cylinder (along the z-axis) with radius 3. Limiting this to the first octant (where $x \geq 0$, $y \geq 0$ and $z \geq 0$), we have a quarter cylinder, cut off at $z = 0$. Adding the $x = 0$ bound doesn't change this shape. Adding the $y = z$ bound creates a "roof" for the cylinder, sloping down to 0 at the origin.

The integrand is z. The integrand and region suggest cylindrical coordinates. The bounds are $0 \leq r \leq 3$, $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq z \leq y = r\sin\theta$. Remember to multiply the integrand by r (this is the Jacobian for cylindrical coordinates).

$$\iiint_W z\,dV = \int_0^3\int_0^{\frac{\pi}{2}}\int_0^{r\sin\theta} z\,r\,dz\,d\theta\,dr = \int_0^3 r\int_0^{\frac{\pi}{2}}\left[\frac{1}{2}z^2\right]_0^{r\sin\theta} d\theta\,dr = \int_0^3 r\int_0^{\frac{\pi}{2}} \frac{1}{2}r^2\sin^2\theta\,d\theta\,dr$$

$$= \frac{1}{2}\int_0^3 r^3\int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2}\,d\theta\,dr = \frac{1}{2}\int_0^3 r^3\frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} dr = \frac{1}{4}\int_0^3 r^3\left[\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right] dr$$

$$= \frac{1}{4}\int_0^3 r^3\frac{\pi}{2}\,dr = \frac{\pi}{8}\left[\frac{1}{4}r^4\right]_0^3 = \frac{\pi}{32}3^4 = \frac{81\pi}{32}.$$

Sanity check: The first octant is where x, y, and z are nonnegative, so the integrand is nonnegative in this octant, so our answer should be nonnegative, which it is. □

4(a). The inequality $0 \le y \le x$ describes a 45 degree triangular region, starting at the origin, going out forever. The inequality $x^2 + y^2 \le \pi^2$ is a circle of radius π. The shape of the region is therefore a 45 degree sector of a circle of radius π, starting at $x = 0$, going counterclockwise. This suggests polar coordinates. The integrand we're given is $\cos\left(x^2+y^2\right)$ – the $x^2 + y^2$ and the circular region should be a big hint to use polar coordinates. $x^2 + y^2 = r^2$, so setting up our integral in polar coordinates (remembering the Jacobian for polar is r),

$$\int\int_D \cos\left(x^2+y^2\right)\, dx\, dy = \int_0^{\frac{\pi}{4}} \int_0^{\pi} \cos\left(r^2\right) r\, dr\, d\theta.$$

The inner integral does not depend on θ, so it is constant with respect to the outer integral, so we can pull it out as a constant.

$$= \left(\int_0^{\pi} \cos\left(r^2\right) r\, dr\right)\left(\int_0^{\frac{\pi}{4}} 1\, d\theta\right).$$

Using a u-sub for the left integral, $u = r^2$, $du = 2r\, dr$, $\frac{1}{2} du = r\, dr$, $u(0) = 0$, $u(\pi) = \pi^2$, and evaluating the easy right integral,

$$= \left(\frac{1}{2}\int_0^{\pi^2} \cos u\, du\right) [\theta]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\sin u\right]_0^{\pi^2} \frac{\pi}{4} = \frac{\pi}{8}\sin \pi^2.$$

□

4(b). W is given as the ball of radius 2 centered at the origin, and the integrand has $x^2 + y^2 + z^2$, both of which suggest spherical coordinates. The bounds for the region in spherical coordinates are $0 \le \rho \le 2$, $0 \le \theta \le 2\pi$, and $0 \le \phi \le \pi$. The Jacobian for spherical coordinates is $\rho^2 \sin\phi$. Note that $x^2+y^2+z^2 = \rho^2$.

$$\int\int\int_W e^{\left(x^2+y^2+z^2\right)^{3/2}} dV = \int_0^{2\pi}\int_0^{\pi}\int_0^2 e^{\left(\rho^2\right)^{3/2}} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

The inner two integrals don't depend on θ at all, so we can factor them out. Also note that $\left(\rho^2\right)^{3/2} = \rho^3$.

$$= \left(\int_0^{\pi}\int_0^2 e^{\rho^3}\rho^2 \sin\phi\, d\rho\, d\phi\right)\left(\int_0^{2\pi} 1\, d\theta\right) = \left(\int_0^{\pi} \sin\phi \int_0^2 e^{\rho^3}\rho^2\, d\rho\, d\phi\right)\left(\int_0^{2\pi} 1\, d\theta\right)$$

$$= \left(\int_0^2 e^{\rho^3}\rho^2\, d\rho\right)\left(\int_0^{\pi}\sin\phi\, d\phi\right)\left(\int_0^{2\pi} 1\, d\theta\right).$$

We were able to separate the integrals nicely. This is a good trick to do if possible, because it makes the computation much cleaner and less prone to error. Each integrand has an easy antiderivative.

$$= \left[\frac{1}{3}e^{\rho^3}\right]_0^2 [-\cos\phi]_0^{\pi} [\theta]_0^{2\pi} = \frac{1}{3}\left(e^8 - e^0\right)(-\cos\pi - -\cos 0)(2\pi - 0) = \frac{1}{3}\left(e^8 - 1\right)(- -1 - -1)\, 2\pi,$$

and the answer is $\frac{4\pi}{3}\left(e^8 - 1\right)$. □

5. Call the 3D region W. The volume of W is given by

$$\int\int\int_W 1\, dV.$$

The first two bounds, $z = 9 - x^2 - y^2$ and the xy-plane (i.e. $z = 0$) gives an upside-down paraboloid shape, whose boundary is at $0 = z = 9 - x^2 - y^2 \implies x^2 + y^2 = 9$ (a circle of radius 3). Adding in the last bound, the cylinder $x^2 + y^2 = 4$, we see that our xy-space region is only a disc of radius 2, which we'll call D. This is a z-elementary region, so it is natural to integrate z first. The bounds are $z = 0$ and $z = 9 - x^2 - y^2$ ($9 - x^2 - y^2$ is greater than 0 in D).

$$\int\int\int_W 1\, dV = \int\int_D \int_0^{9-x^2-y^2} 1\, dz\, dA_{xy} = \int\int_D [z]_0^{9-x^2-y^2}\ dA_{xy} = \int\int_D 9 - x^2 - y^2\, dA_{xy}.$$

Now we need to set up double-integral. It's a circle, and the integrand has a $-\left(x^2 + y^2\right)$ in it, so polar coordinates are suggested. The bounds of integration are simply $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Note that $9 - x^2 - y^2 = 9 - \left(x^2 + y^2\right) = 9 - r^2$. Remember the Jacobian for polar coordinates.

$$= \int_0^{2\pi} \int_0^2 \left(9 - r^2\right) r\, dr\, d\theta = \left(\int_0^2 9r - r^3\, dr\right)\left(\int_0^{2\pi} 1\, d\theta\right) = \left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_0^2 2\pi$$

$$= \left(\frac{9}{2} \cdot 4 - \frac{1}{4} \cdot 16\right) 2\pi = (18 - 4)\, 2\pi$$

and the answer is 28π. $\square$