Math 23B Practice Final Answers - 2011 (Courtesy of Victor Dods)

1(a). True. The area of a surface is given by the expression $\int\int_S 1\, dS$, and since we have a parametrization $\phi(x,y) = (x, y, f(x,y))$ with $S = \phi(D)$, this expands as

$$\int\int_S 1\, dS = \int\int_D 1\, \|T_x \times T_y\|\, dA_{xy}$$

$$= \int\int_D \left\| \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right) \right\| dA_{xy} = \int\int_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA_{xy},$$

as desired. □

1(b). False. The theorem statement is correct except that the orientation of S should be outward (referring to its unit normal vector field). □

1(c). True. The sphere is symmetric in the z direction (meaning if you flip it upside-down, it's the same surface), and the integrand is odd with respect to z. Therefore the integral is 0.

More concretely, if one were to parameterize the surface and set it up as a double integral, perhaps using cylindrical coordinates (in which the equation of a sphere is $r = \sqrt{1 - z^2}$), one would have a symmetric region D in θz-space, and the integrand would still be odd with respect to z. □

1(d). False. This is the Jacobian determinant for spherical coordinates (though using the letter r instead of the commonly used ρ). The Jacobian determinant for cylindrical coordinates is simply r. □

1(e). False. A counterexample is $f(x) = (x+1)\,x\,(x-1)$, which is a cubic polynomial and is therefore onto (extending down forever as x decreases, extending up forever as x increases), but it has 3 x-axis intercepts, so it doesn't pass the horizontal line test. □

1(f). False.

$$\int\int_S f\, dS = \int\int_D f(\phi(u,v))\, \|T_u \times T_v\|\, dA_{uv},$$

which doesn't change sign if the direction of the normal vector $T_u \times T_v$ changes, because the magnitude of a vector is never negative. □

1(g). True, with the caveat that the boundary ∂D is positively oriented. □

1(h). True. Given a parameterization $S = \phi(D)$,

$$\int\int_S F \cdot dS = \int\int_D F \cdot (T_u \times T_v)\; dA_{uv} = \int\int_D F \cdot (T_u \times T_v)\, \frac{\|T_u \times T_v\|}{\|T_u \times T_v\|}\, dA_{uv}$$

$$= \int\int_D F \cdot \frac{(T_u \times T_v)}{\|T_u \times T_v\|}\, \|T_u \times T_v\|\; dA_{uv} = \int\int_D (F \cdot n)\, \|T_u \times T_v\|\; dA_{uv} = \int\int_S (F \cdot n)\; dS,$$

as desired. □

1(i). False. There are several ways to show this (take a look at page 551 in Marsden and Tromba). The most straightforward way is showing that the curl of a vector field is zero (extending it to a vector field on $\mathbb{R}^3$ if necessary), but this requires the domain on which the vector field is defined to be simply connected (meaning there are no "holes" in the domain). This vector field is not defined at $(x,y) = (0,0)$, and therefore there is a "hole" at the origin (which extends to the entire z axis when considering an $\mathbb{R}^3$ vector field). So the curl test can't be used here.

In this case, if we can find a closed curve whose line integral is nonzero, we can show that the vector field is not conservative. Typically we want to pick an easy curve that goes around the "hole".

Take the unit circle for example, oriented counterclockwise (the orientation is not important here, so long as the resulting integral is nonzero). The parametrization is $\gamma(t) = (\cos t, \sin t)$, for $t \in [0, 2\pi]$, giving $\gamma'(t) = (-\sin t, \cos t)$. Then

$$\int_\gamma F \cdot ds = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^{2\pi} \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t} \right) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt = \int_0^{2\pi} 1\, dt = 2\pi \neq 0.$$

The integral is nonzero, so F can't be conservative (i.e. there is no function $g\colon \mathbb{R}^3 \to \mathbb{R}$ so that $\nabla g = F$). $\square$

1(j). True. We also have $\nabla \times (\nabla f) = 0$ for any twice-differentiable function $f\colon \mathbb{R}^3 \to \mathbb{R}$. In the diagram,

$$\{\text{scalar functions}\} \xrightarrow{\nabla} \{\text{vector fields}\} \xrightarrow{\text{curl}} \{\text{vector fields}\} \xrightarrow{\text{div}} \{\text{scalar functions}\}$$

applying any two arrows consecutively (i.e. applying the ∇ and then the curl operator, or applying the curl and then the div operator) always gives zero. $\square$

2. The region D is both x and y simple (draw a picture), bounded above by $y = \sin x$ and below by $y = 0$ (integrating as a y-simple region will be easier). In this problem, even though it states the integral with dx first and dy second, don't think that you're forced to integrate x first. This notation really only means integrating over the region D using the variables x and y, using whatever setup you like.

$$\int\int_D (\cos x - y)\, dx\, dy = \int\int_D (\cos x - y)\, dA = \int_0^\pi \int_0^{\sin x} (\cos x - y)\, dy\, dx$$

$$= \int_0^\pi \left[y \cos x - \frac{1}{2} y^2 \right]_0^{\sin x} dx = \int_0^\pi \left(\sin x \cos x - \frac{1}{2} \sin^2 x \right) - \left(0 \cos x - \frac{1}{2} 0^2 \right) dx$$

$$= \int_0^\pi \frac{1}{2} \sin(2x) - \frac{1}{2} \left(\frac{1 - \cos(2x)}{2} \right) dx = \int_0^\pi \frac{1}{2} \sin(2x) - \frac{1}{4} + \frac{1}{4} \cos(2x)\, dx$$

Here we could do a u-sub, or guess the antiderivatives of $\sin(2x)$ and $\cos(2x)$ and check our guesses by taking their derivatives; we get $-\frac{1}{2}\cos(2x)$ and $\frac{1}{2}\sin(2x)$ respectively. But notice that the period of $\sin(2x)$ and $\cos(2x)$ is π, and we're integrating over exactly one period-length. Using their periodicity, *and the fact that their average value is 0 over a single period* (this part is critical), we conclude that the integrals over the $\sin(2x)$ and $\cos(2x)$ terms are zero (note that this is not true in general for periodic functions, but only periodic functions whose average is zero).

Thus the above integral equals $\int_0^\pi -\frac{1}{4}\, dx = \left[-\frac{1}{4}x\right]_0^\pi = -\frac{\pi}{4}$. $\square$

3. The paraboloid-bounded solid described in the problem is radially symmetric (meaning when rotated about the z-axis, it is the same shape). Thus cylindrical coordinates would be a natural choice. $z = x^2 + y^2 = r^2$. We need to figure out which one of $z = r^2$ or $z = 4$ is the lower bound and which is the upper bound. At $r = 0$, $z = r^2 = 0^2 = 0 < 4$, so $z = r^2$ is the lower bound.

Again, just because the integral in the problem is presented as using $dx\, dy\, dz$, don't be fooled into thinking you must integrate a particular variable first. The freedom is yours to set up the integral over W however you like. We'll integrate z first, as the problem is presented clearly as a z-elementary region. Let D be the projection of W into the xy-plane. This is simply a circle having radius 2 (this is determined by computing the intersection of $z = r^2$ and $z = 4$, which is $r = 2$). We must be sure to remember the Jacobian of cylindrical coordinates; r.

$$\int\int\int_W z\, dx\, dy\, dz = \int\int_D \int_{r^2}^4 z\, dz\, r\, dA_{r\theta} = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 z\, dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left[\frac{1}{2} z^2 \right]_{r^2}^4 r\, dr\, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^2 \left(16 - r^4\right) r\, dr\, d\theta = \frac{1}{2} \int_0^{2\pi} \left[8r^2 - \frac{1}{6} r^6 \right]_0^2 d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} 32 - \frac{64}{6}\, d\theta = \frac{1}{2} \left(32 - \frac{32}{3}\right) \int_0^{2\pi} 1\, d\theta = \frac{1}{2} \left(\frac{64}{3}\right) 2\pi,$$

finally equalling $\frac{64\pi}{3}$. □

4. Here we're asked to find the surface area of a paraboloid, so we'll use a surface integral. Specifically $\int\int_S 1\, dS$. Because the surface is given by a graph, the parametrization is easy and we get to use a shortcut which tells us what the surface integral element dS is.

$$\phi(x, y) = (x, y, z)\,, \text{ with } z = 9 - x^2 - y^2$$

is the parametrization, noting that the region D over which this graph resides is bounded by the intersection of $z = 9 - x^2 - y^2$ and $z = 0$, in other words, $9 - x^2 - y^2 = 0$, implying $x^2 + y^2 = 3^2$, a circle of radius 3. With $f(x, y) = 9 - x^2 - y^2$. Then

$$\int\int_S 1\, dS = \int\int_D 1 \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA_{xy} = \int\int_D \sqrt{1 + (-2x)^2 + (-2y)^2}\, dA_{xy}$$

$$\int\int_D \sqrt{1 + 4\left(x^2 + y^2\right)}\, dA_{xy}.$$

The $x^2 + y^2$ and the circular shape of D suggest polar coordinates; the above integral equals

$$\int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2} r\, dr\, d\theta.$$

The inner integral is independent of θ, so the outer integral may be evaluated immediately, giving a 2π factor;

$$2\pi \int_0^3 \sqrt{1 + 4r^2} r\, dr.$$

A u-sub of $u = 1 + 4r^2$ renders

$$2\pi \left[\frac{1}{12} \left(1 + 4r^2\right)^{3/2} \right]_0^3 = \frac{\pi}{6} \left[\left(1 + 4 \cdot 3^2\right)^{3/2} - \left(1 + 4 \cdot 0^2\right)^{3/2} \right] = \frac{\pi}{6} \left(37^{3/2} - 1\right).$$

The answer is positive, which is good because we were calculating surface area. □

5. For a picture of this thing, look at http://mathworld.wolfram.com/SteinmetzSolid.html

The cylinders are along the z and y axes, and therefore the shape is symmetric when the y and z coordinates are changed. For each fixed value of x, the cross section of the solid is a square. What are the bounds of this square?

$$x^2 + y^2 \leq 1 \implies y^2 \leq 1 - x^2 \implies -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}$$

and

$$x^2 + z^2 \leq 1 \implies z^2 \leq 1 - x^2 \implies -\sqrt{1 - x^2} \leq z \leq \sqrt{1 - x^2}.$$

And of course, x only ranges between -1 and 1, as there is no cross section at other values of x. The area of the square cross section is then $\left(2\sqrt{1 - x^2}\right)^2$. Integrating area along length gives volume. The volume of the solid is

$$\int_{-1}^1 \left(2\sqrt{1 - x^2}\right)^2 dx = \int_{-1}^1 4\left(1 - x^2\right) dx = \int_{-1}^1 4 - 4x^2\, dx.$$

Notice that the function is even in x, and that the interval $[-1,1]$ over which we're integrating is symmetric. Thus the integral equals

$$2\int_0^1 4-4x^2\,dx = 2\left[4x-\frac{4}{3}x^3\right]_0^1 = 2\left(4-\frac{4}{3}\right) = 2\left(\frac{12-4}{3}\right) = 2\cdot\frac{8}{3} = \frac{16}{3}.$$

Sanity check: we were computing volume, so the answer should be positive, which it is. $\square$

6. To evaluate a path integral over the sides of a triangle, we must come up with a curve parametrization for each side. The orientation of the curve parametrization for a path integral doesn't matter. The sides given intersect at the points $(0,0)$, $(0,1)$ and $(1,0)$. Let the parametrizations be the following.

$$\begin{aligned} \text{Along } x=0: &\quad \gamma_1(t) = (0,t) \text{ with } t\in[0,1] \\ \text{Along } y=0: &\quad \gamma_2(t) = (t,0) \text{ with } t\in[0,1] \\ \text{Along } x+y=1: &\quad \gamma_3(t) = (t,1-t) \text{ with } t\in[0,1]. \end{aligned}$$

Then the path integral is

$$\int_C f\,ds = \int_{\gamma_1+\gamma_2+\gamma_3} f\,ds = \int_{\gamma_1} f\,ds + \int_{\gamma_2} f\,ds + \int_{\gamma_3} f\,ds$$

$$= \int_0^1 f(\gamma_1(t))\,\|\gamma_1'(t)\|\,dt + \int_0^1 f(\gamma_2(t))\,\|\gamma_2'(t)\|\,dt + \int_0^1 f(\gamma_3(t))\,\|\gamma_3'(t)\|\,dt.$$

We're also going to need the magnitude of the tangent vectors for each curve. $\gamma_1'(t) = (0,1)$ giving $\|\gamma_1'(t)\| = 1$. $\gamma_2'(t) = (1,0)$ giving $\|\gamma_2'(t)\| = 1$. $\gamma_3'(t) = (1,-1)$ giving $\|\gamma_3'(t)\| = \sqrt{2}$. The above integral equals

$$\int_0^1 t\cos(2\pi\cdot 0)\cdot 1\,dt + \int_0^1 0\cos(2\pi t)\cdot 1\,dt + \int_0^1 (1-t)\cos(2\pi t)\cdot 2\,dt$$

$$\int_0^1 t\,dt + \int_0^1 0\,dt + 2\int_0^1 \cos(2\pi t)\,dt - 2\int_0^1 t\cos(2\pi t)\,dt$$

(the last integral can be done by parts with $u=t$ and $dv = \cos(2\pi t)\,dt$)

$$= \left[\frac{1}{2}t\right]_0^1 + 0 + 2\left[\frac{1}{2\pi}\sin(2\pi t)\right]_0^1 - 2\left[\frac{t}{2\pi}\sin(2\pi t) + \frac{1}{4\pi^2}\cos(2\pi t)\right]_0^1$$

$$= \frac{1}{2} + \frac{2}{2\pi}(\sin(2\pi)-\sin(0)) - 2\left[\left(\frac{1}{2\pi}\sin(2\pi) + \frac{1}{4\pi^2}\cos(2\pi)\right) - \left(\frac{0}{2\pi}\sin(0) + \frac{1}{4\pi^2}\cos(0)\right)\right]$$

$$= \frac{1}{2} + \frac{2}{2\pi}(0-0) - 2\left[\left(\frac{1}{2\pi}\cdot 0 + \frac{1}{4\pi^2}\cdot 1\right) - \left(0 + \frac{1}{4\pi^2}\cdot 1\right)\right].$$

Everything cancels out except the $\frac{1}{2}$. $\square$

7(a). The main difficulty here is coming up with a parametrization for the ellipse. Note that an ellipse is just a scaled circle. The ellipse in question is $\frac{x^2}{3^2} + \frac{y^2}{\sqrt{3}^2} = 1$, and in this form, the major and minor axis lengths can be read clearly. In the x direction, it is 3. In the y direction, it is $\sqrt{3}$.

If we start with a circle parametrization $p(t) = (\cos t, \sin t)$ with $t\in[0,2\pi]$, and scale it in the x direction by 3 and in the y direction by $\sqrt{3}$, we get our parametrization.

$$C(t) = \left(3\cos t, \sqrt{3}\sin t\right); \qquad C'(t) = \left(-3\sin t, \sqrt{3}\cos t\right).$$

$$\int_C F \cdot ds = \int_0^{2\pi} F\left(C\left(t\right)\right) \cdot C'\left(t\right)\, dt = \int_0^{2\pi} \left(-2\sqrt{3}\sin t - 6\cos t, 9\cos t + 2\sqrt{3}\sin t\right) \cdot \left(-3\sin t, \sqrt{3}\cos t\right)\, dt$$

$$= \int_0^{2\pi} 6\sqrt{3}\sin^2 t + 18\sin t\cos t + 9\sqrt{3}\cos^2 t + 6\sin t\cos t\, dt$$

$$= \int_0^{2\pi} 6\sqrt{3}\cdot\frac{1}{2}\left(1-\cos 2t\right) + 9\sin 2t + 9\sqrt{3}\cdot\frac{1}{2}\left(1+\cos 2t\right) + 3\sin 2t\, dt$$

$$= \int_0^{2\pi} \frac{15\sqrt{3}}{2} + \frac{3}{2}\sqrt{3}\cos 2t + 12\sin 2t\, dt.$$

Here, since $\cos 2t$ and $\sin 2t$ are π-periodic and each of their average values is 0 over a period, and we're integrating over two whole periods, their contribution is zero. Thus the integral is

$$\left[\frac{15\sqrt{3}}{2}t\right]_0^{2\pi} = \frac{15\sqrt{3}}{2}2\pi.$$

The twos cancel out and the answer is $15\sqrt{3}\pi$. □

7(b). Plug and chug. First, evaluate $C'(t)$, which equals $(-\sin t, \cos t, 1)$.

$$\int_C x\, dx + y\, dy + z^2\, dz = \int_0^1 (\cos t)(-\sin t) + (\sin t)(\cos t) + \left(t^2\right)(1)\, dt$$

$$= \int_0^1 -\sin t\cos t + \sin t\cos t + t^2\, dt = \int_0^1 t^2\, dt = \left[\frac{1}{3}t^3\right]_0^1$$

and the answer is $\frac{1}{3}$. □

8. The surface given is a graph. We can see this by using $x^2+y^2+z^2 = 1$ to solve for z; $z = \pm\sqrt{1-x^2+y^2}$. But using the condition $z \geq 0$, we eliminate the $\pm$ and choose only the positive root. Given a graph, there is a really easy parametrization $Q(x,y) = (x, y, g(x,y))$, where $z = g(x,y)$. In this case, $g(x,y) = \sqrt{1-x^2-y^2}$. The domain for the parametrization is found by projecting the graph onto the xy plane. The projection of the upper hemisphere is a disc, in this case having radius 1. Let this be called D. Then

$$\int\int_S F \cdot dS = \int\int_D F\left(Q\left(x,y\right)\right) \cdot \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)\, dA_{xy}$$

(computing $\frac{\partial g}{\partial x} = \frac{1}{2}(-2x)\left(1-x^2-y^2\right)^{-1/2} = -\frac{x}{\sqrt{1-x^2-y^2}}$ and $\frac{\partial g}{\partial y} = \frac{1}{2}(-2y)\left(1-x^2-y^2\right)^{-1/2} = -\frac{y}{\sqrt{1-x^2-y^2}}$, and plugging in $F(Q(x,y))$)

$$= \int\int_D (x,y,1)\left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\right)\, dA_{xy} = \int\int_D \frac{x^2+y^2}{\sqrt{1-x^2-y^2}} + 1\, dA_{xy}$$

$$= \int\int_D \frac{x^2+y^2}{\sqrt{1-x^2-y^2}}\, dA_{xy} + \int\int_D 1\, dA_{xy}.$$

The second integral is simply the area of D, which is $\pi \cdot 1^2 = \pi$. The first integral suggests polar coordinates, because of the occurrence of x^2+y^2 and the radially-symmetric shape of the region D. Don't forget the Jacobian! The above equals

$$\int_0^{2\pi}\int_0^1 \frac{r^2}{\sqrt{1-r^2}} r\, dr\, d\theta + \pi = \left(\int_0^1 \frac{r^2}{\sqrt{1-r^2}} r\, dr\right)\left(\int_0^{2\pi} 1\, d\theta\right) + \pi$$

(the first integral can be done with the u-sub $u = 1 - r^2$, noting that in order to get rid of all instances of r, we can solve for $r^2 = 1 - u$; $u(0) = 1$, $u(1) = 0$, $du = -2r\,dr$, so $-\frac{1}{2}\,du = r\,dr$)

$$= \left(-\frac{1}{2}\int_1^0 \frac{1-u}{\sqrt{u}}\,du\right)(2\pi) + \pi = \left(-\int_1^0 u^{-1/2} - u^{1/2}\,du\right)\pi + \pi$$

$$= \left(-\left[2u^{1/2} - \frac{2}{3}u^{3/2}\right]_1^0\right)\pi + \pi = \left(-\left[(0-0) - \left(2 - \frac{2}{3}\right)\right]\right)\pi + \pi$$

giving an answer of $\frac{7\pi}{3}$. □

9(a). We could compute $\nabla \times F$ directly (in which case we'd find that it equals $(0,0,0)$), or apply a little theory: Notice that F is a gradient field – the component functions look similar – the i component is missing x, the j component is missing y, the k component clearly involved a derivative of z^2. Our guess is that $F = \nabla\left(xyz^2\right)$. Computing $\nabla\left(xyz^2\right)$, we see that our guess is correct. With $f(x,y,z) = xyz^2$, we have $F = \nabla f$. Then $\nabla \times F = \nabla \times (\nabla f) = 0$ because taking the curl of a gradient field is always zero.

F is conservative because it is a gradient field. Another argument is that it is conservative because it is defined on and differentiable on all of $\mathbb{R}^3$ and its curl is zero. □

9(b). This was found to be $f(x,y,z) = xyz^2$ in the previous part.

Without the guess-and-check method though, in case it wasn't obvious what the function f should be, one could construct it as follows. Using the gradient theorem (which states that $\int_C \nabla f \cdot ds = f\left(C_{\text{endpoint}}\right) - f\left(C_{\text{startpoint}}\right)$), we have

$$f(x,y,z) = \int_C \nabla f \cdot ds + f\left(C_{\text{startpoint}}\right),$$

where C is any path connecting $C_{\text{startpoint}}$ to (x,y,z). Because $C_{\text{startpoint}}$ is constant, $f\left(C_{\text{startpoint}}\right)$ is constant, and therefore doesn't really matter in terms of finding f (all antiderivatives differ by a constant term anyway). We can also take $C_{\text{startpoint}}$ to be arbitrary, say $(0,0,0)$. The simplest curve between $(0,0,0)$ and (x,y,z) is probably a straight line; $C(t) = t(x,y,z)$ with $t \in [0,1]$. $C'(t) = (x,y,z)$. Then

$$f(x,y,z) = \int_C \nabla f \cdot ds = \int_0^1 F(C(t)) \cdot C'(t)\,dt = \int_0^1 \left(ty\,(tz)^2, tx\,(tz)^2, 2\,(tx)\,(ty)\,(tz)\right) \cdot (x,y,z)\,dt$$

$$\int_0^1 t^3xyz^2 + t^3xyz^2 + 2t^3xyz^2\,dt = \int_0^1 4xyz^2t^3\,dt = 4xyz^2\int_0^1 t^3\,dt = 4xyz^2\left[\frac{1}{4}t^4\right]_0^1,$$

which equals xyz^2. Taking the gradient of this verifies that this is indeed an antiderivative of F (with respect to the gradient). □

10. Let S be the triangle bounded by C. The problem states that it is oriented by the order of the points. This gives a normal vector which points along the positive y axis (draw the figure and use the right-hand rule), in other words, $n(x,y,z) = (0,1,0)$. First we must make sure that F is differentiable, otherwise curl$F$ is not defined. Each component is a polynomial in x, y, z, so it is differentiable. Then by Stokes' theorem,

$$\int_C F \cdot ds = \int\int_S (\text{curl}F) \cdot dS = \int\int_S ((\text{curl}F) \cdot n)\,dS.$$

Now to compute curl$F$.

$$\text{curl}F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -xy & -xz & -yz \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(-yz) - \frac{\partial}{\partial z}(-xz)\right) - j\left(\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial z}(-xy)\right) + k\left(\frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial y}(-xy)\right)$$

$$= i(-z+x) - j(0-0) + k(-z+x).$$

The integral is equal to

$$\int\int_S (x-z, 0, x-z)\cdot(0,1,0)\; dS = \int\int_S 0\, dS$$

which evaluates to 0. □

11. In order to use Gauss' theorem, we must check that F is differentiable. It is, since its component functions are polynomials. Let V denote the solid bounded by S. By Gauss' theorem,

$$\int\int_S F\cdot dS = \int\int\int_V \text{div}F\, dV = \int\int\int_V 2x + 2y + 2z\, dV.$$

The solid V is a cylinder having radius 2 and height 1, so cylindrical coordinates would be most natural for this integral.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

and the Jacobian is r. The bounds of integration in these coordinates are $r \in [0,2]$, $\theta \in [0, 2\pi]$ and $z \in [0,1]$. The integral equals

$$\int_0^{2\pi}\int_0^2\int_0^1 (2r\cos\theta + 2r\sin\theta + 2z)\, r\, dz\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^2\int_0^1 r^2\cos\theta + r^2\sin\theta + rz\, dz\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^2\int_0^1 r^2\cos\theta\, dz\, dr\, d\theta + 2\int_0^{2\pi}\int_0^2\int_0^1 r^2\sin\theta\, dz\, dr\, d\theta + 2\int_0^{2\pi}\int_0^2\int_0^1 rz\, dz\, dr\, d\theta$$

$$\begin{aligned} = \quad & 2\left(\int_0^{2\pi}\cos\theta\, d\theta\right)\left(\int_0^2 r^2\, dr\right)\left(\int_0^1 1\, dz\right) \\ & +2\left(\int_0^{2\pi}\sin\theta\, d\theta\right)\left(\int_0^2 r^2\, dr\right)\left(\int_0^1 1\, dz\right) \\ & +2\left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_0^2 r\, dr\right)\left(\int_0^1 z\, dz\right). \end{aligned}$$

But the integrals $\int_0^{2\pi}\cos\theta\, d\theta$ and $\int_0^{2\pi}\sin\theta\, d\theta$ are zero, so the first two terms go away. The above equals

$$= 2\left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_0^2 r\, dr\right)\left(\int_0^1 z\, dz\right) = 2(2\pi)\left[\frac{1}{2}r^2\right]_0^2\left[\frac{1}{2}z^2\right]_0^1 = 4\pi\left(\frac{4}{2}\right)\left(\frac{1}{2}\right),$$

which equals 4π. □