Mathematics 23B; Fall 2010; V. Ginzburg
Practice Final

1. For each of the ten questions below, state whether the assertion is *true* or *false*. (You do not need to justify your answer.)

   (a) The area of the portion of the graph of $z = f(x, y)$ over a region D in the (xy)-plane is equal to
   $$\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA.$$

   (b) Let Ω be the region in $\mathbb{R}^3$ bounded by a surface S and $\mathbf{F}$ a C^1 vector field in $\mathbb{R}^3$. Gauss' theorem asserts that
   $$\iiint_\Omega \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S},$$
   where S is oriented inward.

   (c) Let S be the unit sphere. Then $\iint_S x^2 z\, dS = 0$.

   (d) Let z, r, and θ be cylindrical coordinates. Then
   $$\left|\frac{\partial(x, y, z)}{\partial(r, \theta, z)}\right| = r^2 \sin\theta.$$

   (e) Every onto map is necessarily one-to-one.

   (f) The integral $\iint_S f\, dS$, where f is a function, changes sign when the orientation of S is changed.

   (g) Let D be the region to which Green's theorem applies. Then the area of D is equal to $\frac{1}{2}\int_{\partial D}(xdy - ydx)$.

   (h) Let a surface S be oriented by the unit normal $\mathbf{n}$. Then $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n})\, dS$, for any vector field $\mathbf{F}$.

   (i) The vector field $\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$ is conservative.

   (j) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ for any C^2 vector field $\mathbf{F}$.

2. Evaluate the integral $\iint_D (\cos x - y)\, dx\, dy$, where the region D is bounded by $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$.

3. Evaluate the integral $\iiint_W z\, dx\, dy\, dz$, where W is the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

4. Find the area of the portion of the paraboloid $z = 9 - x^2 - y^2$ that lies over the $z = 0$ plane.

5. Find the volume of the solid inside the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

6. Evaluate the path integral $\int_C f\, ds$, where $f(x, y) = y \cos(2\pi x)$ and C is the triangle with sides $x = 0$, $y = 0$, and $x + y = 1$.

7. Evaluate the following line integrals:

   (a) $\int_C \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y) = -2(y + x)\mathbf{i} + (3x + 2y)\mathbf{j}$ and C is the ellipse $x^2/9 + y^2/3 = 1$ oriented counter-clockwise.

   (b) $\int_C (x\, dx + y\, dy + z^2\, dz)$, where C is the curve parametrized as $C(t) = (\cos t, \sin t, t)$ with $0 \le t \le 1$.

8. Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

   where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$ and S is the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$.

9. Let $F(z, y, z) = yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}$.

   (a) Evaluate $\nabla \times \mathbf{F}$. Is $\mathbf{F}$ conservative?

   (b) Find a function f such that $\mathbf{F} = \nabla f$.

10. Use Stokes' theorem to evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = -xy\mathbf{i} - xz\mathbf{j} - yz\mathbf{k}$ and C is the triangle with vertices $(0, 1, 0)$, $(0, 1, 5)$, and $(3, 1, 0)$ oriented by the ordering of the points.

11. Use Gauss' theorem to evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and S is the cylinder bounded by $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 1$ (the top and the bottom are included), oriented outward.