Math 235, Dynamical systems
Winter 2022

Lecture 1
01/04-2022

→ Go through basic info

* No exams, no hw
* Problems stated in lectures }
  Up to them how much they take home
* OH: by appointment

Plan:

- basic concepts and examples
- elements of ergodic theory
- maps of S^1, the Denjoy example
- local normal forms, Hartman-Grobman and local analysis of DS
- hyperbolicity, horseshoes
- topological entropy

varying degree of detail

- Not a comprehensive course
- Examples are often non-trivial and very important

Math 235, Dynamical Systems, Winter 2022

- **Lectures:** TTh 1:30 - 3:05 PM, McHenry Clrm 1279 (the first two weeks remotely)

- **Instructor:** Viktor Ginzburg; office: McHenry 4124
  email: ginzburg(at)ucsc.edu

- **Office Hours:** TBA or by appointment

- **Text:** There will be no "official" textbook in this course. Some suggested reading and references:

  - *Introduction to the Modern Theory of Dynamical Systems* by A. Katok and B. Hasselblatt;
  - *Geometrical Methods in the Theory of Ordinary Differential Equations* by V.I. Arnold;
  - *Lectures on Dynamical Systems* by E. Zehnder;
  - *Measure and Category* by J.C. Oxtoby; more analysis & top dynamics
  - *Ergodic Theory* by I.P. Cornfeld, S.V. Fomin and Y.G. Sinai;
  - *Lecture Notes on Ergodic Theory* by C. Walkden;
  - *Dynamical Systems* by C. Robinson.

- **Tentative Syllabus:** This course will be a potpourri of dynamical systems, focusing on examples and main concepts and notions rather than technical proofs of general theorems. I plan to discuss or at least briefly touch upon some of the following topics and concepts:
  - elements of ergodic theory,
  - topological entropy,
  - structural stability,
  - maps of the circle and the Denjoy example,
  - local analysis and local normal forms,
  - hyperbolic dynamical systems.

  This will not be a comprehensive course in dynamical systems, but rather a non-technical overview of central notions and ideas. Examples are particularly important in dynamics and I will devote a lot of attention to them.

  COVID-19 Information: Please take care to comply with all university guidelines about masking in indoor settings, performing daily symptom and badge checks, testing as required by the campus vaccine policy, self-isolating in the event of exposure, and respecting others' comfort with distancing. Please do not come to class if your badge is not green. If you are ill or suspect you may have been exposed to someone who is ill, or if you have symptoms that are in any way similar to those of COVID-19, please err on the side of caution and stay home until you are well or have tested negative after an exposure.

- **Lecture notes (pdf files)** **The entire set (nearly 100MB).** **Weekly:**

  - **Week 1: Basic concepts; Examples: gradient flows, rotations of the circle, translations and linear flows on tori, the Kronecker theorem, geodesic flows** p. 3

  - **Week 2: Examples continued: geodesic flows on surfaces of negative curvature, the shift transformation. Elements of Ergodic Theory: invariant measures, some examples, the Poincare recurrence theorem.** p. 27

  - **Week 3: Elements of Ergodic Theory continued: the Birkhoff Ergodic Theorem; ergodicity and unique ergodicity; Examples: rotations of the circle (equidistribution), translations and linear flows on tori; toral automorphismns.** p. 51

  - **Week 4: Elements of Ergodic Theory continued: mixing; Bernoulli shifts; existence of invariant measures (the Krylov-Bogolubov theorem); the Oxtoby-Ulam theorem.**

p. 80

- Week 5: Homeomorphisms of the circle: general discussion (equivalence, structurale stability, etc); the rotation number. p 109
- Week 6: Homeomorphisms of the circle continued: properties of the rotation number; structurally stable diffeomorphisms of the circle; the Denjoy theorem; Digression: continious vs differentiable functions. p. 126
- Weeks 7-8: Homeomorphisms of the circle continued: the Denjoy example; Diophantine vs. Liouville numbers; Herman's Theorem and small denominators (examples). Local analysis: setting; Lyapunov and asymptotic stability; Lypunov functions. p. 139
- Week 9: Local analysis continued: asymptotic stability via linearization; non-degenerate and hyperbolic fixed points and equilibria; the Hartman-Grobman theorem (without proof); the linearization problem; resonances and the Poincare theorem on formal linearization). p. 163
- Week 10: Introduction to hyperbolic systems: horseshoes; hyperbolic maps and sets; structural stability (Anosov theorem's on structural stability of hyperbolic toral endomorphisms). p. 178

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§1. Introduction

What is a dynamical system?

- M a topological space (reasonably good), usually compact, e.g. a manifold

- $\to \varphi : M \to M$ a map, continuous or smooth, often but not always **invertible**. Interested in φ^k, $k \in \mathbb{N}$ or $\mathbb{Z}$

  $\mathbb{N}$ or $\mathbb{Z} \to C^0(M,M)$ or $C^\infty(M,M)$

  (semi) group homomorphism

time → discrete

time → continuous

$\to \varphi^t$ a flow on M

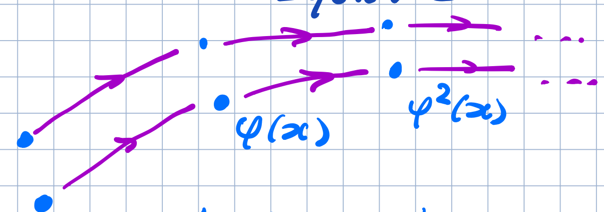
  $\mathbb{R} \to \mathrm{Homeo}(M)$ or $\mathrm{Diffeo}(M)$

  group homomorphism

  $t \mapsto \varphi^t$, $\quad \varphi^{t_1+t_2} = \varphi^{t_1}\varphi^{t_2}$

This set up models:

- M = the set of states of a (deterministic) system
- t, $t=k$ = time
- φ^t, φ^k = the evolution of the system



x = intial condition

Rmk can set $\varphi = \varphi^T$

- **Basic source: ODE = v.f.**

  - M a manifold (e.g. a domain in $\mathbb{R}^n$)
  - given a v.f. = ODE, complete
  - φ^t the flow of it:

    $\varphi^t(x)$ = sol with the initial condition x

    or $\varphi = \varphi^T$

- Dynamical systems $\supset$ ODE's
  But focus is different.

  In DS we are interested in
  - "global", geometrical features of φ, qualitative properties of φ
  - <u>not in</u> determining $\varphi^t(x)$ explicitly

  <u>E.g.</u> • The behavior of $\varphi^t(x)$ as $t \to \infty$ for x in a certain subset

  • Does $\varphi^t(x)$ comes back to x? How close? For "how many" x?

<u>A variant</u>: • M is a probability measure space
• φ measure preserving
...

• Basic Definitions and Terminology

Need some language to talk about qualitative properties

• $\{\varphi^k(x) \mid k \in \mathbb{N} \text{ or } \mathbb{Z}\}$ — positive semi-orbit
$\{\varphi^t(x) \mid t \in \mathbb{R}\}$

the orbit of x

Notation: $\mathcal{O}(x)$

• x is a fixed pt if

$\underbrace{\varphi(x) = x}_{\Rightarrow\ \varphi^k(x) = x\ \forall k}$ or $\varphi^t(x) = x \ \forall t$

• x is a periodic point if $\exists\ \underbrace{n, T}_{>0}$

$\varphi^{n}(x) = x$ or $\varphi^{T}(x) = x$

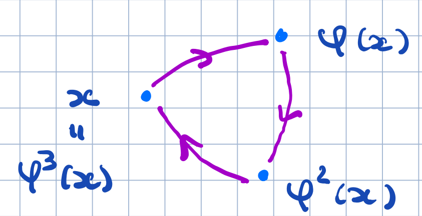
a period

Rmk • n is a period $\Rightarrow$ $2n$, $3n$, etc are also periods

• $\Rightarrow$ minimal period

Ex a fixed pt is also a periodic pt with <u>minimal</u> period 1

• The orbit through a periodic pt is a periodic orbit:



a periodic orbit with min period 3

- $X \subset M$ is an *invariant set* if $\underbrace{\varphi(X) \subset X}_{(\Rightarrow\ \varphi^k(X) \subset X,\ k \in \mathbb{N})}$ or $\varphi^t(X) \subset X$ $\forall t$
  usually closed

Ex An orbit Θ is an invariant subset
Θ is periodic $\Leftrightarrow$ Θ is closed
M is compact

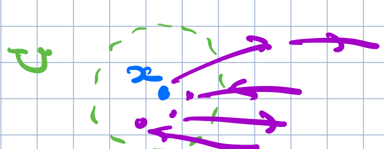
- φ (or φ^t) is *minimal* if M has no closed invariant subsets

  $\Leftrightarrow$ every orbit is dense

Ex. φ minimal $\Rightarrow$ no periodic orbits
(M compact)

- φ (or φ^t) is *topologically transitive* if $\exists$ a dense orbit
  $\Leftrightarrow$ every inv. subset is nowhere dense

- x is *recurrent* if x comes back to its arbitrarily small nbd infinitely many times:
  $\forall\, U \ni x \ \exists\, k_i \to \infty$ s.t. $\varphi^{k_i}(x) \in U$



Ex.
- x is periodic
- the orbit through x is dense

$\Rightarrow$ x is recurrent

- ω-limit set of x:

$\omega(x) = \{$all limits of $\varphi^{k_i}(x)$, $k_i \to \infty\}$

$$= \bigcap_{n=1}^{\infty} \overline{\{\varphi^k(x) \mid k \geq n\}}$$ closure

α-limit set: similarly but $-\infty$

similarly for flows

Ex.
- $\varphi^k(x) \xrightarrow[k\to\infty]{} y \Leftrightarrow y = \omega(x)$

- x is periodic $\Leftrightarrow$ $\omega(x) = \alpha(x) = \Theta(x)$
  $=$ the orbit through x

- the orbit through x is dense
  $\Leftrightarrow$ $\omega(x) = M = \alpha(x)$ invertible

- x is recurrent $\Leftrightarrow$ $x \in \omega(x) = \alpha(x)$

- For flows
  $$\omega(x) = \bigcap_{T \geq 0} \overline{\{\varphi^t(x) \mid t \geq T\}}$$
  is connected. Ex

- Many more to follow

Some examples of DS

- **Ex1** Gradient-like flows — boring DS!

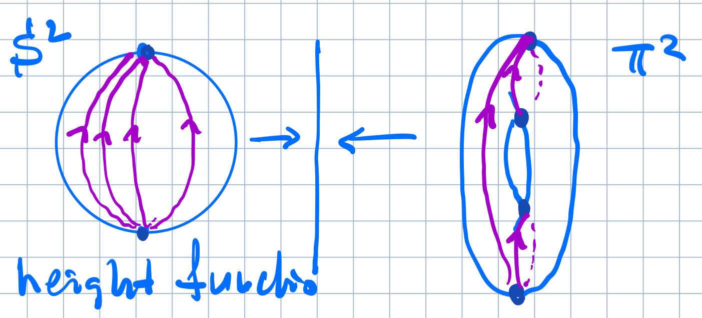
  - M is a closed manifold
  - $f: M \xrightarrow{C^\infty} \mathbb{R}$
  - X = gradient-like v.f.:

$$L_X f \geq 0$$

$$\underbrace{= \text{ only at } Crit(f)}$$

moreover: $\{X=0\} = Crit(f)$

Ex • $X = \nabla f$ for some R. m.



$$\frac{d}{dt} f(\varphi^t(x)) = L_X f(\varphi^t(x)) \geq 0$$

$$x \in Crit(f) \iff \longrightarrow = 0$$

$\Rightarrow$ • recurrent pts = periodic pts
= fixed pts
= $Crit(f)$

- No dense orbits
- $\forall x \quad \omega(x) \subset \mathrm{Crit}(x)$

  $\qquad\quad \alpha(x) \subset \mathrm{Crit}(x)$

If f is Morse:

$\omega(x)$ or $\alpha(x)$ is just one critical pt

Ex – hard

construct f such that
$\exists\, x$ s.t. $\omega(x) \subset \mathrm{Crit}(f)$
is a circle.

Ex2 Rotations of S^1

already much more interesting

$$S^1 = \mathbb{R}/\mathbb{Z} = \{|z|=1\} \subset \mathbb{C}$$

$$\varphi : S^1 \to S^1$$
$$\theta \mapsto \theta + \alpha \mod 1 \quad , \ \alpha \in S^1 \text{ fixed}$$
$$e^{2\pi i\theta} \mapsto e^{2\pi i \theta} e^{2\pi i \alpha}$$

Prop

two alternatives

- φ is periodic ($\varphi^q = id$)
  $\Leftrightarrow$ $\underline{\alpha \in \mathbb{Q}}$ $\quad : \alpha = \frac{p}{q}$
  ($\Leftrightarrow$ every pt is q periodic)
- φ is minimal (every orbit is dense)
  $\Leftrightarrow$ $\underline{\alpha \notin \mathbb{Q}}$

Pf $\quad \varphi^k(\theta) = \theta + k\alpha \mod 1$

- $\alpha \in \mathbb{Q} \quad : \alpha = \frac{p}{q}$

$\Leftarrow)$

$$\Rightarrow \varphi^q(\theta) = \theta + q\frac{p}{q} = \theta + p = \theta$$

$\Rightarrow)$ $\varphi^q = id$: $\varphi^q(\theta) = \theta + q\alpha = \theta \mod 1$

$$\Rightarrow q\alpha = p \in \mathbb{Z} \Rightarrow \alpha = \frac{p}{q}$$

• $\alpha \notin \mathbb{Q}$ Look at the orbit of $0 = 1$

$\Rightarrow \varphi^k(0) = k\alpha \neq 0$ in S^1

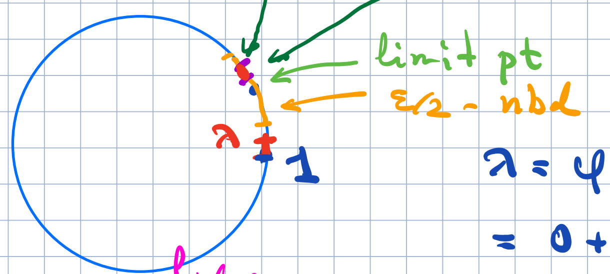
$\Rightarrow \varphi^{\ell}_{\ell\alpha}(0) \neq \varphi^{m}_{m\alpha}(0) \ \forall \ell, m$ $\qquad \underbrace{\varphi^{\ell - m}(0) = 0}_{(\ell - m)\alpha = 0 \bmod \mathbb{Z}}$

$\Rightarrow \varphi^k(0)$ has a limit pt

$\Rightarrow \forall \varepsilon > 0 \ \exists \ \ell_0, \ell_1$ s.t.

$0 \neq d(\underbrace{\varphi^{\ell_0}(0)}, \underbrace{\varphi^{\ell_1}(0)}) < \varepsilon$

distance in S^1, rotation invariant

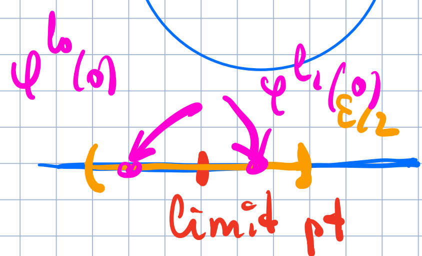


$\lambda = \varphi^{\ell_1 - \ell_0}(0)$

$= 0 + \ell_1\alpha - (0 + \ell_0\alpha)$

$= (\ell_1 - \ell_0)\alpha = \alpha'$

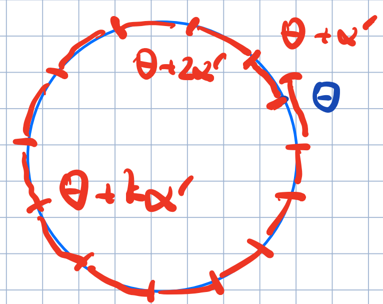
is ε-close to $0 \in S^1$



$\Rightarrow \forall \ \theta \in S^1$

$\varphi^{k(\ell_1 - \ell_0)}(\theta) = \theta + k(\ell_1 - \ell_0)\alpha$

is ε-dense



since ε is arbitrary

$\Rightarrow$ the orbit is dense

Rmk We will come back to this example many times and refine it.

Ex Prove that the decimal expansion of 2^k may begin with any finite sequence of digits:

Given $\lambda_1 \dots \lambda_s = \lambda$ $\exists$ k s.t.

$$2^k = \lambda_1 \dots \lambda_s \dots$$

Ex3 Translations on compact groups

G compact (metrizable) top gp
e.g. a Lie gp

$\varphi(x) = x \cdot \alpha$ $\qquad \alpha \in G$ fixed
"right" translation

$\Theta(1) = \{\alpha^k \mid k \in \mathbb{Z}\}$ subgroup abelian
use multiplicative notation

$\Rightarrow H = \overline{\Theta(1)}$ is a closed abelian subgroup

$\Rightarrow \quad \Theta(x) = \overbrace{x\Theta(1)}^{x\alpha^k} =$ translation of $\Theta(1)$

$\overline{\Theta(x)} = xH =$ —— · —— H

$\Rightarrow \varphi$ can be minimal only when G is abelian : $G = H = \overline{\Theta(1)}$

$\Rightarrow$ when G is a Lie gp then H is an extension of $\mathbb{Z}_r$ by $\mathbb{T}^n$

need to know a bit of Lie gps

$$1 \to \underbrace{\mathbb{T}^n}_{\text{connected component of id}} \to H \to \mathbb{Z}_r \to 1$$

Ex. $G = S^1$

The only closed subgroups are

- cyclic (roots of unity)
- S^1

$\Rightarrow$ H can only one of the two types:

$\rightarrow$ a cyclic subgroup $\Leftrightarrow \alpha \in \mathbb{Q}$

$\rightarrow$ $H = S^1 \Leftrightarrow \alpha \notin \mathbb{Q}$

To summarize $\varphi : G \to G$, $x \mapsto x \cdot \alpha$

compact (abelian)

- either all orbits $\mathcal{O}(x) = x\,\mathcal{O}(1)$ are dense
- or none of the orbits are dense:

$\mathcal{O}(x) \cap U \neq \emptyset \Leftrightarrow \mathcal{O}(1) \cap x^{-1}U \neq \emptyset$

$x\,\mathcal{O}(1)$

For group translations

minimal $\Leftrightarrow$ top transitive

Lecture 2
01/06-2022

Ex4 Translations of $\mathbb{T}^n$

- $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = \underbrace{S^1 \times \dots \times S^1}_{n}$

  $\theta = (\theta_1, \dots, \theta_n)$ mod 1

  $\varphi : \mathbb{T}^n \longrightarrow \mathbb{T}^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$

  $\theta \longmapsto \theta + \alpha$ use additive notation

Now $\exists$ more possibilities: orbits need not be either dense or periodic

But $\quad \mathcal{O}(x) = \{\varphi^k(x) \mid k \in \mathbb{Z}\} = \{x + k\alpha \mid k \in \mathbb{Z}\}$

$\qquad = x + \mathcal{O}(0) = x + \{k\alpha\}$

$\Rightarrow \quad \overline{\mathcal{O}(x)} = x + \underbrace{\overline{\mathcal{O}(0)}}$

closed abelian subgroup: an extension of $\mathbb{Z}_r$ by $\mathbb{T}^{m \leq n}$

$\Rightarrow$ $\begin{cases} \bullet \text{ All orbits are periodic} \\ \quad \Leftrightarrow \mathcal{O} \text{ is periodic} \end{cases}$ $\Leftrightarrow$ φ is periodic

$\quad\;\begin{cases} \bullet \text{ All orbits are dense} \\ \quad \Leftrightarrow \mathcal{O}(0) \text{ is dense} \end{cases}$ $\Leftrightarrow$ φ is minimal

characterize these two situations

translation (homeo) of a dense set is dense

- 0 is periodic $\Leftrightarrow \alpha \in \mathbb{Q}^n$

$q\alpha = 0 \mod \mathbb{Z}$ for some q

$\Leftrightarrow \underbrace{(q\alpha_1, \dots, q\alpha_n)}_{\varphi^q(0)=0} = 0 \mod \mathbb{Z}$

$\alpha_i = \frac{p_i}{q_i}$ $\quad i=1,\dots,n$ $\quad q = \operatorname{lcm}(q_1,\dots,q_n)$

$\alpha \in \mathbb{Q}^n$

- **Prop** φ is minimal

$\Leftrightarrow 1, \alpha_1, \dots, \alpha_n$ is linearly ind over $\mathbb{Q}$:

$r_0 \cdot 1 + \sum_{i=1}^{n} r_j \alpha_j = 0$, $r_j \in \mathbb{Q}$ $\Rightarrow$ all $r_j = 0$ $\quad j=0,\dots,n$

or $r_j \in \mathbb{Z}$

Rmk • $\mathbb{R}$ is a v.s. over $\mathbb{Q}$

$\dim_{\mathbb{Q}} \mathbb{R} = \infty$ (continuum)

$\Leftrightarrow$ $\dim_{\mathbb{Q}} \underbrace{\operatorname{span}(1, \alpha_1, \dots, \alpha_n)}_{\subset \mathbb{R}} = n+1$

• can replace $\mathbb{Q}$ by $\mathbb{Z}$

Rmk A lot can be in between these two cases:

can have

$1 \le \dim_{\mathbb{Q}}(1, \alpha_1, \dots, \alpha_n) \le n+1$

periodic all $\alpha_i \in \mathbb{Q}$ $\qquad$ minimal

Ex. $n=1$, $\alpha \in \mathbb{R}/\mathbb{Z}$, $\alpha \in \mathbb{R}$

$1, \alpha$ linearly ind over $\mathbb{Q}$

$\iff$ $\alpha \notin \mathbb{Q}$

$\iff$ φ minimal $\iff$ $\Theta(0)$ is dense

$\uparrow$ Last lecture

Pf Recall: for translations of compact (abelian) groups:

$$\left\{\begin{array}{l}\text{top transitive:}\\ \text{one dense orbit}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}\text{minimal:}\\ \text{all orbits are dense}\end{array}\right\}$$

i) lin dependent $\Rightarrow$ not minimal

$$\sum \underbrace{r_j}_{\in \mathbb{Z}} \alpha_j = -r_0 \in \mathbb{Z} \quad : \text{ a resonance relation}$$

not all $r_j \neq 0$

$$f : \mathbb{T}^n \longrightarrow S^1 \subset \mathbb{C}$$

$$\underbrace{f(\theta) = \exp\Big(2\pi i \sum_{j=1}^{n} r_j \theta_j\Big)}_{\text{trig. polynomial} \;\Rightarrow\; C^0}$$

$\rightarrow$ $f \not\equiv \text{const}$

$\rightarrow$ f is invariant: $f(\theta + \alpha) = f(\theta)$:

$$f(\theta+\alpha) = \exp\Big(2\pi i \sum_{j=1}^{n} r_j(\theta_j + \alpha_j)\Big)$$

$$= \exp\Big(2\pi i \sum r_j \theta_j\Big) \exp\Big(2\pi i \sum r_j \alpha_j\Big)$$

$$= f(\theta) \underbrace{\exp(-2\pi i\, r_0)}_{1} \qquad \in \mathbb{Z}$$

$$= f(\theta) \cdot 1$$

$\to$ $f(0)=1$, f is C^0 & $f \not\equiv 1$

$\Rightarrow$ • $\underbrace{X = \{\theta \mid f(\theta)=1\}}_{\supset\, \Theta(0) = \{k\alpha\}}$ proper invariant closed subset

• $\mathbb{T}^4 \setminus X \neq \emptyset$, open

$\Rightarrow$ $\Theta(0)$ is <u>not</u> dense

φ is <u>not</u> top transitive

$\Leftrightarrow$ not minimal

Pf $\underbrace{\text{top trans}}$ $\Rightarrow$ $\exists\, k$: $\varphi^k(U) \cap V \neq \emptyset$

$\exists$ a dense orbit : $\{\varphi^j(x)\}$ dense

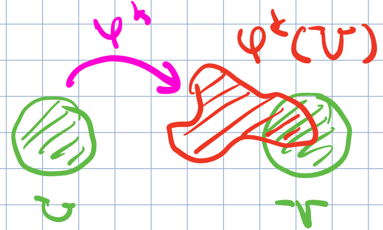
$\Rightarrow$ $\varphi^{j_0}(x) \in U$ & $\varphi^{j_1}(x) \in V$

$\varphi^{j_1 - j_0} \underbrace{\varphi^{j_0}(x)}_{\in U} \in V$

$\Rightarrow$ $\varphi^k(U) \cap V$, $k = j_1 - j_0$

2) lin independent $\Rightarrow$ minimal

Lemma M compact, separable metric space
$\varphi: M \to M$ is top transitive
$\Leftrightarrow$ $\forall$ U & $V \subset M$ open $\exists$ k s.t
$\varphi^k(U) \cap V \neq \emptyset$



Pf: Ex [KH]
(not entirely obvious for $\Leftarrow$)

Cor
φ is ~~top~~ transitive
$\Leftrightarrow$ $\forall$ U, V open invariant
$U \cap V \neq \emptyset$

$\Rightarrow$ every C^0 invariant function is const

could have used in 1), clear anyway

To the pf: by contradiction
- U, V open invariant
- Assume $U \cap V = \emptyset$

$f = \chi_U$: $f(x) = \begin{cases} 1 & x \in U \\ 0 & x \notin U \supset V \end{cases}$

$\Rightarrow$ $f \in L^2(\mathbb{T}^n)$, invariant
$f \not\equiv$ const $\Leftarrow f|_V = 0$
$f \not\equiv 1$ a.e.

$$f(\theta) = \sum_k f_k \cdot \exp(2\pi i \underbrace{\sum_j k_j \theta_j}_{\langle k, \theta\rangle})$$

f_k Fourier coeff

$\underbrace{k = (k_1, \dots, k_n)}_{\in \mathbb{Z}^n}$

$$f(\theta + \alpha) = f(\theta)$$

$$f(\theta+\alpha) = \sum_k f_k \underbrace{\exp(2\pi i \sum k_j(\theta_j + \alpha_j))}_{\exp(2\pi i \sum k_j \theta_j)\exp(2\pi i \sum k_j \alpha_j)}$$

$$= \underbrace{\sum_k \underbrace{f_k \exp(2\pi i \sum k_j \alpha_j)}_{\text{Fourier coeff}} \exp(2\pi i \sum k_j \theta_j)}_{f(\theta+\alpha)}$$

$$\Rightarrow \quad f_k = f_k \exp(2\pi i \sum k_j \alpha_j)$$

at least one $\neq 0 \Leftarrow f \not\equiv$ const
$k \neq 0$

$$\Rightarrow \quad \exp(2\pi i \sum k_j \alpha_j) = 1$$

$$\Rightarrow \quad \sum k_j \alpha_j \in \mathbb{Z}$$

$\Leftrightarrow \quad 1, \alpha_1, \dots, \alpha_n$ lin dependent over $\mathbb{Q}$

$\triangleleft$

• Linear flows on $\mathbb{T}^n$

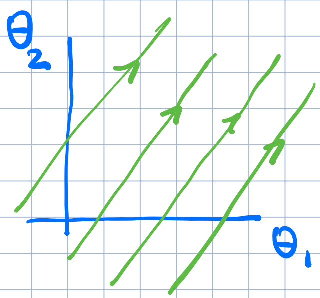
$\varphi^t : \mathbb{T}^n \to \mathbb{T}^n$

$\theta = (\theta_1, \dots, \theta_n)$

$\alpha = (\alpha_1, \dots, \alpha_n)$

$\varphi^t(\theta) = \theta + t\alpha$

$t \in \mathbb{R}$



Very similar to translations

Rmk only α_i/α_j matter

Prop • all ratios $\alpha_i/\alpha_j \in \mathbb{Q}$

$\iff$ all orbits are closed

• $\alpha_1, \dots, \alpha_n$ lin ind over $\mathbb{Q}$

$\iff$ all orbits are dense (minimal)

$\iff$ one orbit is dense (top. transitive)

no 1 here: it's easier for the orbits of φ^t to be dense than for $\varphi = \varphi^1$

Pf - Ex

- Digression to number theory: Kronecker thm

1D case

$\alpha \notin \mathbb{Q}$ $\quad \forall\ \lambda \in \mathbb{R}$ $\quad \forall\ \varepsilon > 0$

$\exists\ k, m \in \mathbb{Z}$ s.t.

$$|k\alpha + m - \lambda| < \varepsilon$$

This is $\Leftrightarrow$ $\{k\alpha\}$ dense in $S^1 = \mathbb{R}/\mathbb{Z}$

nD case

Thm (Kronecker)

$1, \alpha_1, \dots, \alpha_n$ lin ind over $\mathbb{Q}$

$\Leftrightarrow$ $\forall\ (\lambda_1, \dots, \lambda_n) = \lambda \in \mathbb{R}^n$, $\forall\ \varepsilon > 0$

$\exists$ $m = (m_1, \dots, m_n)$ and $k \in \mathbb{Z}$ s.t.

$$\underbrace{\| k\alpha + m - \lambda \| < \varepsilon}_{|k\alpha_i + m_i - \lambda_i| < \varepsilon}$$

Pf $\Leftrightarrow$ $\{k\alpha\}$ dense in $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$

Rmk Rich connections

DS $\longleftrightarrow$ number theory

Rmk Translations on compact Lie gps, $\mathbb{T}^n$, are <u>isometries</u>.

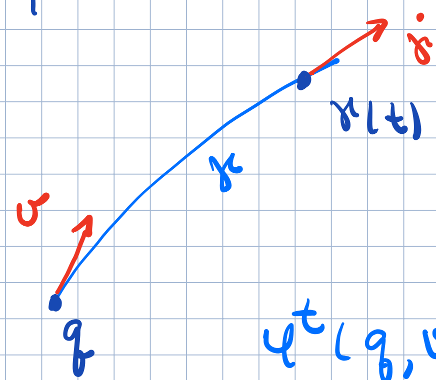
These examples exhaust all the dynamics complexity isometries can have.
Essentially nothing more complicated can happen

Ex 5 Geodesic flows

skipping details for now

- Q a Riemannian manifold, closed
- $M = STQ$ = unit tangent bundle

$\varphi^t : M \longrightarrow M$ the geodesic flow



γ the unit geodesic with

$\gamma(0) = q, \quad \dot{\gamma}(0) = v$

$$\varphi^t(q, v) = (\gamma(t), \dot{\gamma}(t))$$

Extremely important.

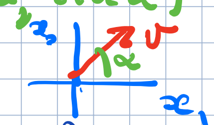
Ex. $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ as a Riemannian manifold

a flat torus

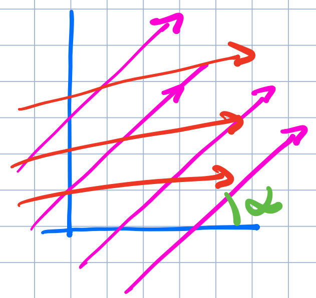
$$ST\mathbb{T}^2 = S^1 \times \mathbb{T}^2$$

$v = \alpha \qquad (x_1, x_2) = q$

$v = (\cos\alpha, \sin\alpha)$



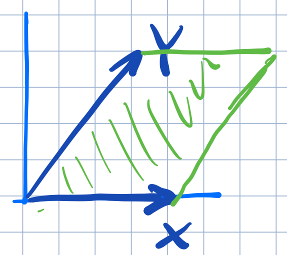
$$\underbrace{\varphi^t(x, \alpha) = (x_1 + t\cos\alpha, \; x_2 + t\sin\alpha, \; \alpha)}_{\text{parallel transport on } \mathbb{R}^2 \text{ in the direction of } v}$$



Depending on α
the orbits can be all
closed or all dense
in $\alpha = \text{const} = \alpha \times \mathbb{T}^2$

Rmk $\exists$ other flat metrics on $\mathbb{T}^2$:

$$\mathbb{T}^2 = \mathbb{R}^2/\Gamma\,, \qquad \Gamma = X\cdot\mathbb{Z} + Y\cdot\mathbb{Z}$$
$$X, Y \in \mathbb{R}^2$$



some of them are isometric
and some are not

But their geodesic flows are very similar

More interesting examples:
surfaces of const neg. curvature
Later?

Lecture 3
01/11-2022

Ex 6 Geodesic flows: surfaces of neg curvature

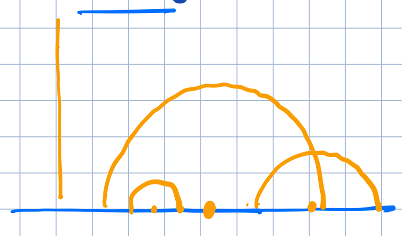
Hyperbolic plane

- $\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$ upper half plane, $z = x+iy$

  Riemannian metric of const curv $= -1$ $\leftarrow$ hyperbolic metric

  $$\frac{dx^2+dy^2}{y^2}$$

- Geodesics: circles with centers on the x-axis including vertical lines:



Return to this a bit later

Ex After knowing that $PSL(2,\mathbb{R})$ are isometries:

- check that a vertical line is a geodesic
- check that for any circle $\exists\, g$ sending a vertical line to that circle

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- **Isometries:**

$$SL(2,\mathbb{R}) \longrightarrow \underbrace{PSL(2,\mathbb{R})}_{SL(2,\mathbb{R})/\{\pm I\}} \longrightarrow Iso(\mathbb{H})$$

orientation preserving isometries

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xmapsto{\text{"2} \to \text{1"}} \underbrace{z \mapsto \frac{az+b}{cz+d}}_{\text{fractional-linear transformation}}$$

$ad - bc = 1$

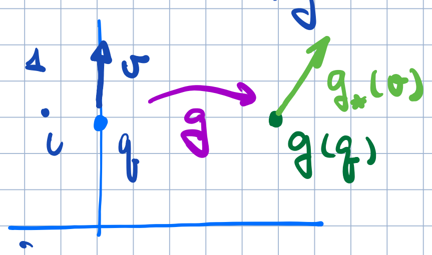
not hard

Ex • Check that this is an action by isometries

- Check that $PSL(2,\mathbb{R}) \longrightarrow Iso(\mathbb{H})$ is an isomorphism: every isometry has this form

Hint • $g \in Iso(\mathbb{H})$ is completely determined by $g(q,v)$ $\quad (q,v)$ - fixed

$q \in \mathbb{H}$, $v \in T_q\mathbb{H}$, $\|v\|$



$q = i$
$v = 1$

- For any $(p,w) \in ST\mathbb{H}$

$\exists!\, g$ s.t. $g(q,v) = (p,w)$

- $PSL(2,\mathbb{R}) \longrightarrow ST\mathbb{H}$

$g \longmapsto g(q,v)$

is an diffeo

• Compact surfaces with hyperbolic metrics

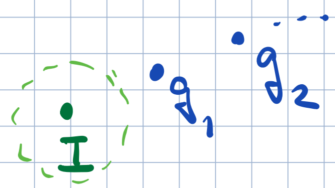
Fact (not obvious)

$\Sigma_{g\geq 2}$ $\quad \exists\ \Gamma \subset PSL(2,\mathbb{R})$ (or $SL(2,\mathbb{R})$)

$\uparrow$ a discrete subgroup

$\rightarrow$ $\exists$ a nbd U of I s.t. $U \cap \Gamma = \{I\}$

s.t. $\Sigma_g \underset{\text{diffeo}}{\cong} \mathbb{H}/\Gamma$



Cor Σ_g admits a metric of constant curvature -1 (a hyperbolic metric)

Cor • $\mathbb{H} \underset{\text{diffeo}}{\cong} \mathbb{R}^2$ is the universal covering of Σ_g

• $\Rightarrow$ $\pi_{n\geq 2}(\Sigma_g) = 0$

Cor $\quad ST\Sigma_g \cong \underbrace{\Gamma \backslash SL(2,\mathbb{R})}_{\text{an algebraic model for } ST\Sigma_g}$

Rmk Γ is not unique: different metrics on $\Sigma_{g\geq 2}$ with curv $= -1$.

- **Algebraic construction:**
  - $\Gamma \subset SL(2,\mathbb{R})$ a discrete subgroup s.t. $M = \Gamma \backslash SL(2,\mathbb{R})$ is compact and smooth
  - $g(t) : \mathbb{R} \longrightarrow SL(2,\mathbb{R})$ a one parameter subgroup

$\Rightarrow$ a flow on M

$$\varphi^t(x) = x \cdot g(t)$$

Ex. Taking Γ as before we get a flow on $ST\Sigma_{g\geq 2}$

Specific examples

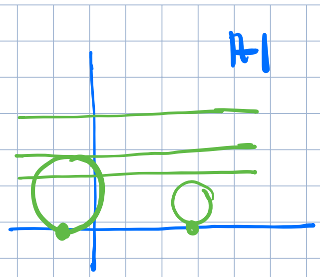
- $g(t) = \begin{pmatrix} \cos t, & \sin t \\ -\sin t, & \cos t \end{pmatrix} \in SO(2) \subset SL(2,\mathbb{R})$ elliptic

$\Rightarrow$ all orbits are periodic

- $g(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ Horocycle flow parabolic

the projection of any orbit to Σ_g (or $\mathbb{H}$) is a geodesic circle of curvature $k=1$

Ex: what are these?



e.g. horizontal lines

Ex: Prove that the horocycle flow in $ST\Sigma_g$ has no closed orbits

(Need to show that no orbit can close up as $\mathbb{H} \longrightarrow \Sigma_g$

- $g(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in SL(2, \mathbb{R})$ hyperbolic subgroup

the geodesic flow:
some orbits are closed (closed geodesics)
but some are dense!

Two non-obvious facts:
- the union of periodic orbits is dense
- $\exists$ dense orbits: top transitive

Rmk Generalizes to groups other than $SL(2, \mathbb{R})$; important!

Further reading

- [CFS]: § 4.4
- [KH]: § 5.4

Ex 7 • Shift transformations

"Symbolic Dynamics"

• Preliminaries – pt set topology

• A = compact metric space

• $A^{\mathbb{Z}} = \cdots \times \overset{-1}{A} \times \overset{0}{A} \times \overset{1}{A} \times \cdots$

Elements: (bi)infinite sequences

$x = \{x_i \in A \mid i \in \mathbb{Z}\}$

• With product topology:

open sets $\cdots \times U_{-1} \times U_0 \times U_1 \times \cdots = \prod U_i$

where all but a finite number $U_i = A$

Fact $A^{\mathbb{Z}}$ is compact

• metric

$$d(x,y) = \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}}\, d(x_i, y_i)$$

can put any conv. series

Rmk

$$\dim A^{\mathbb{Z}} = \Big(1 + 2\sum_{i=1}^{\infty} \frac{1}{2^i}\Big) \operatorname{diam} A$$

$$= 1 + \frac{\cancel{\tfrac{1}{2}}\; 2}{1-\frac{1}{2}}$$

$$= 3 \operatorname{diam} A$$

observation:

$x_i = y_i$ for $|i| \le N$

$$\Rightarrow \quad d(x,y) \le \underbrace{2 \sum_{i=N+1}^{\infty} \frac{1}{2^i} \cdot \operatorname{diam} A}_{2 \cdot \frac{1}{2^{N+1}} \frac{1}{1-\frac{1}{2}} \cdot \operatorname{diam} A}$$

$$\Rightarrow \quad d(x,y) \le \frac{\operatorname{diam} A}{2^{N-1}}$$

$$\begin{array}{ccccccccccc}
\cdots & x_{-N} & \cdots & x_{-1} & x_0 & x_1 & \cdots & x_N & \cdots \\
\neq & \parallel & & \parallel & \parallel & \parallel & & \parallel & \neq \\
\cdots & y_{-N} & \cdots & y_{-1} & y_0 & y_1 & \cdots & y_N & \cdots
\end{array}$$

$$\underbrace{y_{-N} \cdots y_{-1}\ y_0\ y_1 \cdots y_N}_{\text{agree}}$$

$\Rightarrow \quad d(x,y)$ is small

• Shift Transformation

set • $A = \{0, 1\}$ $\qquad d(0,1) = 1$

$M = A^{\mathbb{Z}}$

$= $ sequences of 0 & 1's

• $\varphi : M \to M$ shift to the left

$\varphi(x)_i = x_{i+1}$, a homeo

$$\cdots x_{-2}\, x_{-1}\, x_0\, x_1\, x_2 \cdots$$

$$\cdots \leftarrow \leftarrow \leftarrow \leftarrow \cdots$$

Rmk: variants

• Replace $A = \{0,1\}$ by the alphabet $A = \{1, \dots, n\}$. Similar properties

• Replace $A^{\mathbb{Z}}$ by

$M = A^{\mathbb{N}} = A \times A \times \dots$

$= $ one sided, infinite seq

$\varphi : M \to M$, left shift

$\varphi(x_0 x_1 x_2 \dots) = x_1 x_2 x_3 \dots$

C^0, but not invertible

Interpretation:

$A = $ collection of states

$x \in A^{\mathbb{Z}}$ a process

$x_0 = $ state at $t = 0$

$x_1 = $ — · — · $t = 1$

— · — · —

Properties

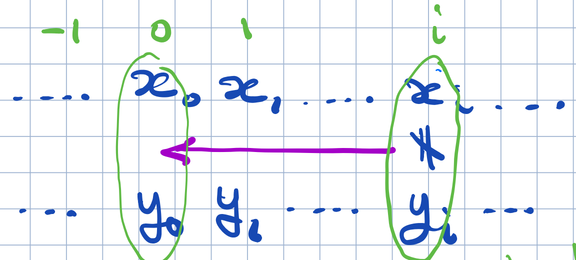
- φ is very far from an isometry:
  φ is expansive
  $\exists\ \varepsilon > 0$ s.t. $\forall\ x \neq y\ \exists\ k$ with
  $$d(\varphi^k(x), \varphi^k(y)) > \varepsilon$$

Pf $\quad \varepsilon = 1, \qquad x \neq y \Rightarrow \exists\ i : x_i \neq y_i$

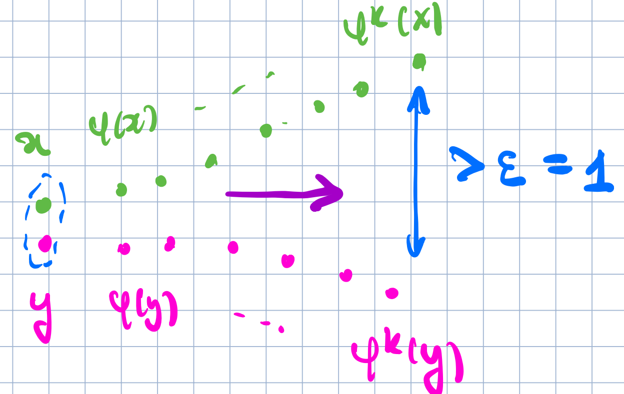
$k = -i$

$$\varphi^k(x)_0 = x_i$$
$$\varphi^k(y)_0 = y_i$$

$$d(\varphi^k(x), \varphi^k(y)) \geq d(\varphi^k(x)_0, \varphi^k(y)_0) = 1$$



contribute with weight 1 $\qquad$ contribute with weight $\frac{1}{2^{|i|}}$



$\triangleleft$

• Periodic pts = periodic sequences

$\Rightarrow p(k) = |\, k\text{-periodic pts}\,|$

$= 2^k$

If $|A| = n$, $p(k) = n^k$

similar to the geodesic flow of a hyperbolic surface

• Periodic pts are dense

<u>Pf</u> Given x and $\varepsilon > 0$ take N so that

$$\frac{1}{2^{N-1}} < \varepsilon$$

length $2N+1 = k$

$$\text{set} \quad \hat{x} = \overbrace{x_{-N}\, x_{-N+1} \dots x_0 \dots x_N}^{\text{length } 2N+1 = k}$$

$$y = \dots \hat{x}\,\hat{x}\,\hat{x} \dots$$

$$\Rightarrow \quad y_i = x_i \qquad |i| \le N$$

$$\Rightarrow \quad d(x,y) \le \frac{\operatorname{diam} A}{2^{N-1}} < \varepsilon \qquad \operatorname{diam} A = 1$$

$$\begin{array}{cccccccccc} {\scriptstyle -N-1} & {\scriptstyle -N} & & {\scriptstyle -1} & {\scriptstyle 0} & {\scriptstyle 1} & & {\scriptstyle N} & {\scriptstyle N+1} & \\ \dots x_{-N-1} \,| & x_{-N} & \dots & x_{-1} & x_0 & x_1 & \dots & x_N \,| & x_{N+1} & = x \end{array}$$

$$\underbrace{x_{-N} \dots x_N}_{\hat{x}}$$

$$\dots x_N \,|\, \overbrace{x_{-N} \dots\dots x_0\, x_1 \dots x_N}^{\hat{x}} \,|\, x_{-N} \dots = y \quad \triangleleft$$

• φ is top. transitive : similar to the geodesic flow of a hyperbolic metric

$\exists$ a dense orbit.

Pf: • $M = A^{\mathbb{Z}}$ is separable :

$\exists$ a countable dense set

(e.g. can take periodic pts)

Denote these set by

$\{x^0, x^1, x^2, \dots\}$ each of these is a bi-inf sequence

$\forall\ y \in M \quad \exists\ x^{i_s}$ s.t.

$d(y, x^{i_s}) \to 0, \quad i_s \to \infty$

• let $\hat{x}^i$ be the finite sequence

$x_{-i} \dots\dots x_0 \dots\dots x_i$

and

$$\begin{array}{l} \quad\quad\quad \dots\dots -1 \quad 0 \quad 123 \quad 4\dots 8 \quad \dots \\ z = \dots 0 \dots 0\ \hat{x}^0\ \hat{x}^1\ \hat{x}^2\ \hat{x}^3 \dots \end{array}$$

Claim $\{\varphi^k(z)\}$ is dense

Pf • Given y & $\varepsilon > 0$

Pick $i = i_s$ so large that

• $d(y, x^i) < \frac{\varepsilon}{2}$

• $\frac{1}{2^{i-1}} < \frac{\varepsilon}{2}$

• pick k so that
$\hat{x}^i$ is centered at 0 in $\varphi^k(z)$

$$\begin{array}{llllllllll}
\cdots & -1 & 0 & 1\,2\,3 & 4\ldots 8 & \ldots & & \\
z = \ldots 0 \ldots & 0 & \hat{x}^0 & \hat{x}^1 & \hat{x}^2 & \hat{x}^3 & \ldots & \hat{x}^i
\end{array}$$

$$\xleftarrow{\qquad\qquad \varphi^k \qquad\qquad}$$

$$\Rightarrow \quad d(x^i, \varphi^k(z)) \leq \frac{1}{2^{i-1}} < \varepsilon/2$$

• $$d(y, \varphi^k(z)) \leq \underbrace{d(y, x^i)}_{<\ \varepsilon/2} + \underbrace{d(x^i, \varphi^k(z))}_{<\ \varepsilon/2} < \varepsilon$$

$\triangleleft$

Ex. Show that $M = A^{\mathbb{Z}}$ is homeo to the Cantor set

Rmk $\exists$ C^∞ φ: surface $\circlearrowleft$ on
φ: disk $\circlearrowleft$ on manifold $\circlearrowleft$
s.t. $\exists$ K $\leftarrow$ invariant subset
with $\varphi|_K \cong (A^{\mathbb{Z}}, \text{shift})$

These are "horseshoes"
very common & important

Further Reading:
[KH] § 1.9

We will keep returning to shifts...

§2 Elements of Ergodic Theory

Lecture 4
01/13-2022

Setup

- Let now (M, μ) be a measure space:

  Usually assume:
  - μ is a probability measure: $\mu(M) = 1$
  - If M is a metric space, then μ is a Borel measure: μ is defined on all open sets ($\Rightarrow$ on all Borel sets)

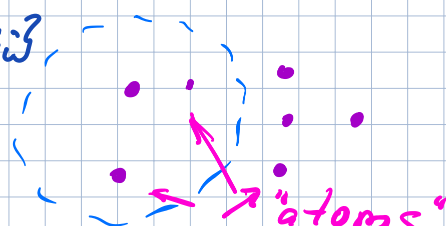
Ex: smooth measure
- M^n closed orientable manifold
- $\omega \in \Omega^n(M)$ $\quad \omega > 0$
- $\mu(U) = \int_U \omega$

Ex "measures supported on finite sets"

$X \subset M$ finite $\quad X = \{x_i\}$

$$\mu(U) = \frac{1}{|X|} \, |X \cap U|$$

$$= \frac{1}{|X|} \sum \delta_{x_i}$$



Ex. linear combination:

μ_0 & μ_1 as above $\Rightarrow$ so is $\lambda\mu_1 + (1-\lambda)\mu_0$

$\forall \quad 0 \le \lambda \le 1$

Rmk: supp μ

closure

$$\operatorname{supp}\mu = \overline{\{x \mid \forall\ U = \text{nbhd of } x,\ \mu(U) > 0\}}$$

$$\operatorname{supp}(\lambda\mu_1 + (1-\lambda)\mu_0) = \operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1$$

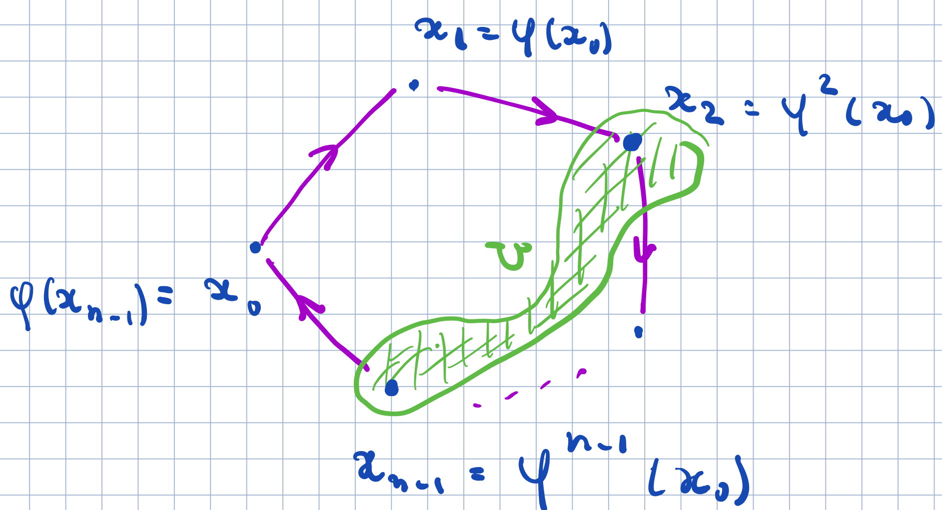
- $\varphi : M \to M$ is measure preserving and C^0 or homeo when M is also a metric space

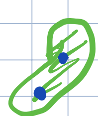
<u>Ex.</u> $\varphi : M \to M$

$X = \{x_0, \dots, x_{n-1}\}$ a periodic orbit

$\Rightarrow$ an invariant measure

$$\mu(U) = \frac{1}{n} |X \cap U| = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{x_i}$$



$\mu($  $) = \dfrac{\#\{x_i \text{ in } U\}}{n}$

= frequency of entering U

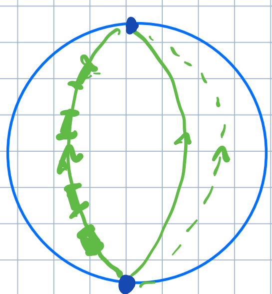
Revisiting our main examples from the measure theory perspective

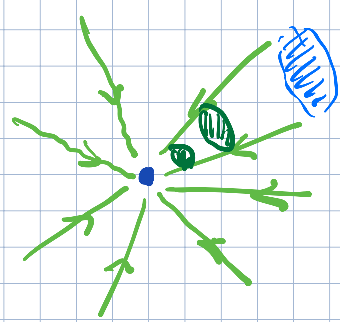
Ex: Gradient flows:

Ex • For any invariant Borel measure

$$\operatorname{supp}\mu \subset \operatorname{Crit}(f) = \operatorname{Fix}(\varphi)$$

In particular when $\operatorname{Crit}(f)$ are isolated, the only inv measures come from fixed pts





Ex. Rotations of S^1 or translations of $\mathbb{T}^n$

- The standard measure $d\theta$ or $d\theta_1 \wedge \dots \wedge d\theta_n$ is obviously invariant

Rmk An isometry of a Riemannian manifold always preserves the Riemannian vol

- Depending on α, there could be other invariant measures
  e.g. $\alpha \in \mathbb{Q}$ then $\theta \mapsto \theta + \alpha$ has periodic orbits, etc

We'll look into these maps some more later

- $\alpha = \frac{p}{q}$, $\varphi^q = id$
  $\Rightarrow$ $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ - action on S^1

$S^1 \xrightarrow{\pi} S^1/\mathbb{Z}_k \leftarrow$ circle

every invariant measure has the form
$= \pi^*($ a measure on $S^1/\mathbb{Z}_k)$

Ex. Geodesic flows have a natural invariant measure
Three ways to see:

1) Q^n R. manifold

$$TQ \overset{\cong}{\longleftrightarrow} T^*Q \longleftarrow \text{symplect}$$
$$v \longleftrightarrow \langle v, \cdot \rangle$$

$\Rightarrow$ TQ also gets a sympl. str ω
Geodesic flow is the Ham flow of $H(v) = \frac{1}{2}\langle v, v\rangle$

$M := STQ = \{H = 1/2\} \leftarrow$ regular level

Ex $\exists$ $\nu \in \Omega^{2n-1}(TQ)$ s.t.

- $\nu \wedge dH = \omega^n$ near $\{H = 1/2\}$
- $\nu|_M$ is unique & $\nu|_M \neq 0$

Could use the notation:

$$\nu|_M = \frac{\omega^n}{dH}$$

Invariant by construction?
(energy & ω conservation)

$$\left.\begin{array}{l}\text{A variant}\\ \text{\& Example}\end{array}\right\}: \left.\begin{array}{l}\text{a vol form}\\ \text{\& } \{H=c\}\end{array}\right\} \Rightarrow \begin{array}{l}\text{a vol form}\\ \text{on } H=c\end{array}$$

- $\mathbb{R}^3$ $\quad dx\wedge dy\wedge dz = \eta$
- $H(x,y,z) = x^2+y^2+z^2$

  $S^2 = \{H=1\}$ regular level
- $\exists\ \nu$ s.t.

$$\nu\wedge dH = \eta \quad \text{near } S^2$$

$$\nu = \frac{x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy}{6(x^2+y^2+z^2)}$$

$$dH = 2(x\,dx + y\,dy + z\,dz)$$

$$\nu\wedge dH = \frac{1}{6(x^2+y^2+z^2)}\big(2x^2\,dx\wedge dy\wedge dz$$

$$+\,2y^2\,dx\wedge dy\wedge dz + 2z^2\,dx\wedge dy\wedge dz\big)$$

$$= dx\wedge dy\wedge dz = \eta$$

- $\nu|_{S^2}$ is unique and

$$= \frac{1}{6}(x\,dy\wedge dz - \ldots) = \frac{1}{6}\text{ area form on } S^2$$

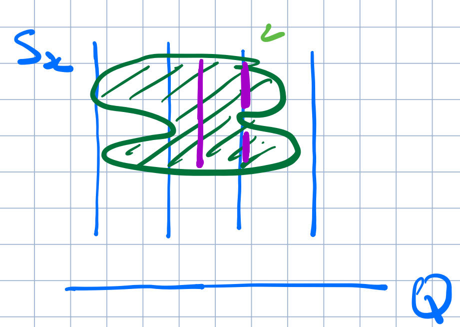
2) STQ has a natural measure

$$\begin{array}{c} STQ \\ \big\downarrow \pi \\ Q \end{array} \qquad \mu(U) = \int_{\pi(U)} \mu_{S_x}(S_x \cap U)\, dx$$

fibre $S_x \subset T_x Q$

natural measure on S_x

vol form on Q



Invariance not-obvious

3) For $ST\Sigma_g = \Gamma \backslash SL(2,\mathbb{R})$, i.e. $Q = \Sigma_g$ with a hyperbolic metric

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad - bc = 1 \right\}$$

$$\nu = \frac{da \wedge db \wedge dc}{|d|}$$

bi-invariant Haar measure

$\Rightarrow$ descends to an invariant measure on $\Gamma \backslash SL(2,\mathbb{R})$

Rmks • ν is preserved by all 1-parameter subgroups of $SL(2,\mathbb{R})$

• Other invariant measures (e.g. from periodic orbits)

Ex. **Shift transformations**

μ_A = a measure on A

$\Rightarrow$ • a Borel measure on $A^{\mathbb{Z}} = M$

$U = \quad \ldots \times U_{-1} \times U_0 \times U_1 \times \ldots$

all but a finite number $= A$

"a cylinder"

$\mu(U) = \prod \mu(U_i)$, then extend

• μ is shift-invariant

Sub-Ex $\quad A = \{1, \ldots, n\}$

$1 \geq p_i \geq 0$ s.t. $\sum p_i = 1$

probability of i

$\mu(\{i\}) = p_i$

E.g. $p_i = 1/n$

Rmk : • Thus we have many invariant measures on $A^{\mathbb{Z}}$

• $\exists$ other invariant measures : e.g. periodic orbits

• Poincaré Recurrence Thm

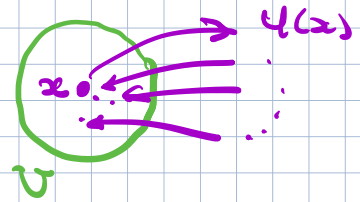
Simple and very important

Thm (PR) ← different versions

- $\varphi: M \circlearrowleft$, $\mu =$ invariant Borel measure
- $U \subset M$, measurable (e.g. open), $\mu(U) > 0$

$\Rightarrow$ for a.a. $x \in U$ the orbit $\{\varphi^k(x) \mid k \in \mathbb{N}\}$ visits U again (can set visit time $k \geq$ any n)

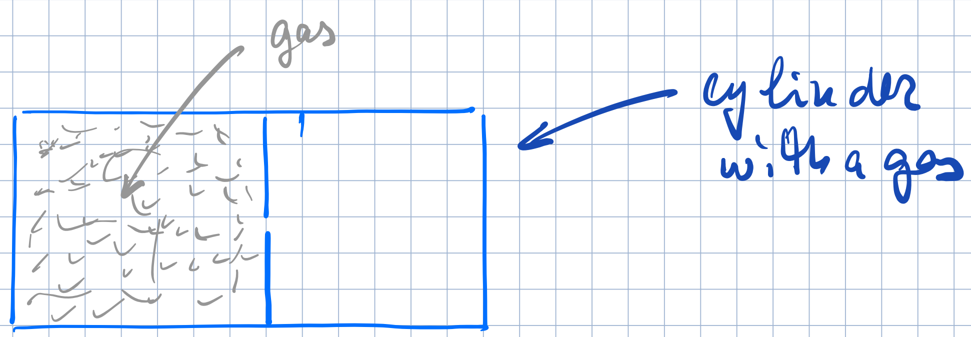
Cor Assume μ is such that $\mu(\text{open}) > 0$,
$\varphi: M \circlearrowleft$ μ-preserving
$\Rightarrow$ a.e. pts are recurrent:
$\varphi^k(x)$ comes back and close to x



Interpretation: $U =$ event, $\mu(U) =$ probability (no matter how small)

$x, \varphi(x), \varphi^2(x), \ldots$ a process
$\Rightarrow$ every possible event will eventually happen again if it happens once

Ex



- Initial condition: gas in one half of the cylinder
- this is a positive (but close to 0) probability event
- $\exists$ time $T > 0$ such that the gas on its own will again concentrate in one half of the cylinder
- Why don't we observe this? The reason is that T is huge! Longer than the existence of the universe!

Lecture 5
01/18-2022

Pf: Poincaré Recurrence

Observation: $\mu(X \cap \varphi^k(X)) = 0 \quad \forall k$

$$\Rightarrow \mu(X) = 0$$

Pf: $\mu(X \cap \varphi^k(X)) = 0 \Leftrightarrow \mu(\underbrace{\varphi^i(X) \cap \varphi^j(X)}_{\text{Apply } \varphi^{-i}}) = 0$

$k = j-i$

$$\mu(\varphi^i(X) \cap \varphi^j(X)) = \mu(X \cap \varphi^{j-i}(X)) = 0$$

all disjoint

$$\mu\left(\coprod_{k=0}^{\infty} \varphi^k(X)\right) = \sum \mu(\varphi^k(X)) = \infty \cdot \mu(X) \le 1$$

$$\Rightarrow \mu(X) = 0$$

Given $U \subset M$, need to show a.a. $x \in U$ come back to U:

$$X = \{x \in U \mid \varphi^k(x) \notin U \ \forall\ k = 1, 2, \ldots\}$$

Claim $X \cap \varphi^k(X) = \emptyset$

Indeed: $\varphi^k(X) \cap \underset{\underset{X}{\cup}}{U} = \emptyset$

Observation $\Rightarrow \mu(X) = 0$. $\triangleleft$

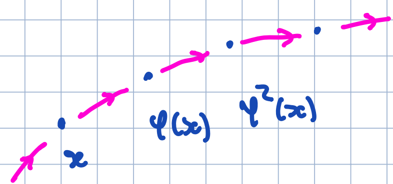
• Birkhoff Ergodic Theorem

Setting: as before

- $\varphi : M \to M$, μ φ-invariant, prob
- $f \in L^1(M)$ — $\int |f| \, d\mu < \infty$

might or not exist

- $\bar{f}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\varphi^i(x))$ ⟵ time average



Averaging f along the orbit of x

<u>Ex.</u> x is periodic with min period k:

$$x = \varphi^k(x)$$

$$\Rightarrow \quad \bar{f}(x) = \frac{1}{k}\left(f(x) + f(\varphi(x)) + \dots + f(\varphi^{k-1}(x)) \right)$$

<u>Thm</u> (Birkhoff Ergodic Thm)

- $\bar{f}(x)$ exists for a.a. x,
- $\bar{f}$ is φ-invariant : $\bar{f} \circ \varphi = \bar{f}$
- $\bar{f}$ is L^1 & $\int \bar{f} \, d\mu = \int f \, d\mu$

Non-trivial; see e.g. [KH] § 4.1 (c)
[FKS] § 1.2

<u>Rmk:</u> <u>Two perspectives:</u>

1. μ is natural and fixed: φ varies within the class of μ-preserving maps, or a smaller class.

<u>Ex.</u> Hamiltonian systems:
a symplectic form preserved by φ and more...
$\Rightarrow \omega^n$ a natural inv measure, etc

2. φ is given and we are intersted in all invariant measures...

Pf of implications:

- invariance: $\bar{f}(\varphi(x)) = \bar{f}(x)$ for a.a. x

$$\bar{f}(x) = \lim_{n\to\infty} \frac{1}{n}\left(f(x) + \dots + f(\varphi^{n-1}(x))\right)$$

both $\exists$ for a.a. x

$$\bar{f}(\varphi(x)) = \lim_{n\to\infty} \frac{1}{n}\left(f(\varphi(x)) + \dots + f(\varphi^{n}(x))\right)$$

$$\bar{f}(\varphi(x)) - \bar{f}(x) = \lim_{n\to\infty} \frac{1}{n}\left(f(\varphi^{n}(x)) - f(x)\right)$$

$f \in L^1$
φ pres μ

$$= 0 \quad \text{for a.a. } x$$

- integral:

$$\int \bar{f}\, d\mu = \int \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\varphi^i(x))\, d\mu(x)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \int f\circ\varphi^i\, d\mu$$

all equal to $\int f\, d\mu$

$$= \lim_{n\to\infty} \frac{1}{n}\cdot n \cdot \int f\, d\mu$$

$$= \int f\, d\mu$$

$\triangleleft$

Variants

- $$\overline{f}(x) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{i=-n}^{n} f(\varphi^i(x))$$

  a.e. equal to the previous one

- For flows:

$$\overline{f}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\varphi^t(x))\,dt$$

$$\overset{a.e.}{=} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(\varphi^t(x))\,dt$$

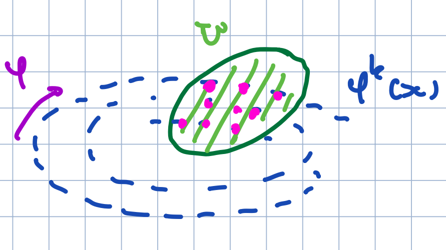
Similar statement: $\overline{f}$ exists a.e. and

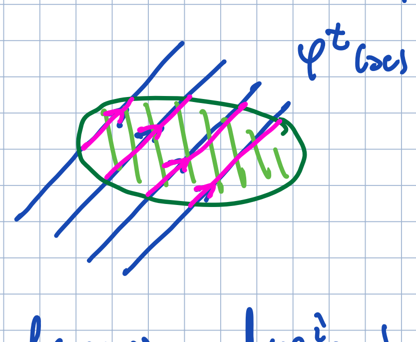
$$\int \overline{f}\,d\mu = \int f\,d\mu$$

Ex - Interpretation

$U \subset M$, $f = \chi_U$ characteristic function

$\Rightarrow \overline{f} =$ average time $\varphi^i(x)$ or $\varphi^t(x)$ spends in U





$$\overline{\varphi}(x) = \lim \frac{1}{n}\left(f(x) + f(\varphi(x)) + \dots + f(\varphi^i(x)) + \dots \right)$$

• Ergodicity and unique ergodicity

Def φ on μ is ergodic if for every measurable inv set A either $\mu(A) = 1 = \mu(M)$ or $\mu(A) = 0$

Rmk : no "proper" inv. subsets

measure theoretic

Prop : (φ, μ) is ergodic

$\Leftrightarrow$ $\forall f \in L^1$ $\bar{f} = \text{const} = \int f\, d\mu$:

$$\bar{f}(x) = \underbrace{\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\varphi^i(x))}_{\text{time average}} = \underbrace{\int_M f\, d\mu}_{\text{space average}}$$

for a.a. x

$\Leftrightarrow$ The only inv L^1-functions are constant functions

Equivalent, clearly.

$\Leftrightarrow$ $\forall$ a.a. x the average time $\underbrace{\varphi^k(x) \text{ spends}}_{\text{observed probability}}$ in U = $\mu(U)$

probability of U

characteristic function

Pf. • φ is ergodic:

every invariant χ_U is (a.e.) constant

or every inv U is a.e. M or $\emptyset$.

$\Big\{$ look at $f \geq$ const; use the fact that χ_U's are L^1-dense

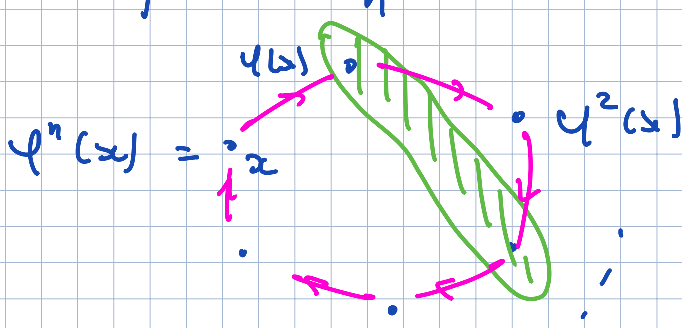
• every $f \in L^1$ is constant (a.e.) $\triangleleft$

Ex. • $x, \varphi(x), \dots, x = \varphi^n(x)$ a periodic orbit

• μ the associated measure

$$\mu = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\varphi^i(x)}$$

$$\Leftrightarrow \quad \mu(U) = \frac{1}{n} \left| \varphi^i(x)\text{'s in } U \right|$$



$\Rightarrow$ μ is ergodic

• **Def** φ is <u>uniquely</u> (or strongly) ergodic if $\exists$ exactly one invariant measure μ for which φ is ergodic.

natural measure

Rmk φ is ergodic with respect to μ, look at all inv. measures, none of them is ergodic ... More later when we study the space of inv. measures

Examples — wait!

<u>But an important pt:</u>

φ uniquely ergodic for μ $\leftarrow$ <u>"continuous"</u>
open sets: $\mu > 0$

$\Rightarrow$ no periodic orbits (e.g. fixed pts)

every per orbit gives rise to an ergodic measure

Thm M is compact, $\mu(M)=1$, $\varphi \in C^0$, μ is invariant

1) • φ ergodic $\Rightarrow \exists$ an orbit dense in $\operatorname{supp}\varphi$

2) • φ uniquely ergodic

$\Rightarrow$ every orbit is dense in $\operatorname{supp}\varphi$

Cor Assume that $\mu(\text{open}) > 0$ $\forall$ open:
($\Rightarrow$ $\operatorname{supp}\varphi = M$)

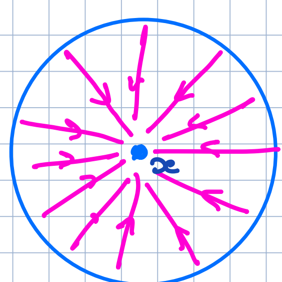
• φ ergodic $\Rightarrow \exists$ a dense orbit: top transitive

• φ uniquely ergodic

$\Rightarrow$ all orbits are dense: minimal

Rmk: Similar for flows

Rmk - Counterexample to Cor



$\exists\, \varphi$ uniquely ergodic without dense orbits

The only inv measure is δ_x but no dense orbits

The reason: δ_x is not "continuous"

Rmk Ic 1) we'll show that $O(x)$ is dense in $\operatorname{supp}\mu$ for μ-a.a. $x \in \operatorname{supp}\mu$

Parallel between measure and topology

Measure		**Topology**
inv measure μ	$\longrightarrow$ not 1-1 but 'onto' later	inv set supp μ
ergodic	$\Longrightarrow$ $\not\Longleftarrow$	top transitive: $\exists$ dense orbit
uniquely ergodic	$\Longrightarrow$ $\not\Longleftarrow$	minimal: every orbit is dense
......		
more info because a lot of measures are involved		less info more robust

__Rmk__ Similar for flows

On the pf of the Thm

Fact: M compact metric space
$\varphi : M \to M$ homeo (or just C^0)
$\Rightarrow$ φ has an (ergodic) inv. measure

Borel, probability

To be discussed later

Pf of Thm

1) μ ergodic $\Rightarrow$ $\exists$ an orbit dense in supp μ

- Can assume $M = \operatorname{supp}\mu$ ← compact metric space
- Let $\{U_j\}$ be a base of the induced top on $M = \operatorname{supp}\mu$ $\Rightarrow$ $\mu(U_j) > 0$

for any open V $\exists$ $U_j \subset V$

$\mathcal{O}(x)$ is dense $\Leftrightarrow$ $\forall j$: $\mathcal{O}(x) \cap U_j \neq \emptyset$

- ergodicity $\Rightarrow$

characteristic function of U_j

$\forall j$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{U_j}(\varphi^i(x)) = \mu(U_j) > 0$$

for a.e. x : $x \in X_j$: $\mu(X_j) = 1$

$\Rightarrow \quad \chi_{U_i}(\varphi^i(x)) > 0$ for some i

$\Rightarrow \quad \varphi^i(x) \in U_j$: $\mathcal{O}(x)$ enters U_j

$X = \bigcap X_j$ full measure

$\forall x \in X \quad \mathcal{O}(x)$ intersects $U_j \ \forall j$

$\triangleleft$

Rmk We have proved that
$\mathcal{O}(x)$ is dense in $\operatorname{supp}\mu$
for μ-a.a. $x \in \operatorname{supp}\mu$

2) μ uniquely ergodic $\Rightarrow \varphi$ is minimal in $\operatorname{supp}\mu$

Assume $\mathcal{O}(x)$ is not dense in $M = \operatorname{supp}\mu$

$\overline{\mathcal{O}(x)} \subsetneq M = \operatorname{supp}\mu$ compact metric space

$\Rightarrow \exists$ an ergodic inv measure ν
with $\operatorname{supp}\nu \subset \overline{\mathcal{O}(x)} \subsetneq M$

$\Rightarrow \nu \neq \mu \quad \rightarrow\leftarrow$ with unique ergodicity

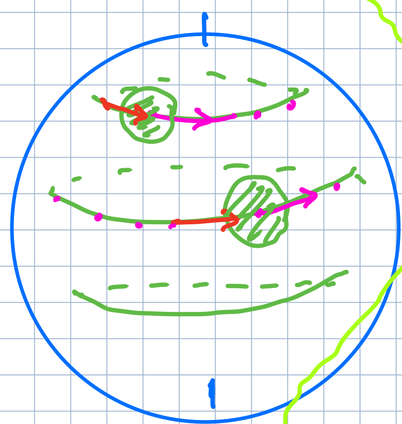
Fact

$\triangleleft$

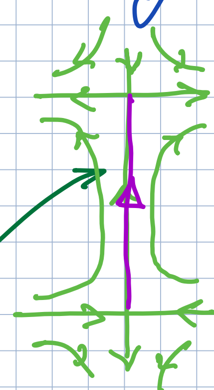
Discussion:
Recurrence vs Ergodicity

(PR): a.a. pts are recurrent: come back arbit. close to themselves

holds unconditionally (μ is inv)



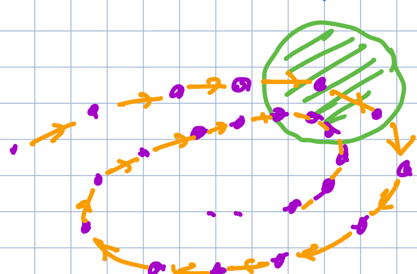
- Rot of S^2 in α
- Regardless of α, $\varphi^k(x)$ gets arbitrarily close to x



Rmk a.a. is essential: pts here do not come back

Ergodicity: ← does not hold unconditionally

a.a. pts enter every set U with frequency $\mu(U)$



U

Rmk:

ergodic $\Downarrow$ transitivity

uniq erg $\Downarrow$ minimal

- Examples

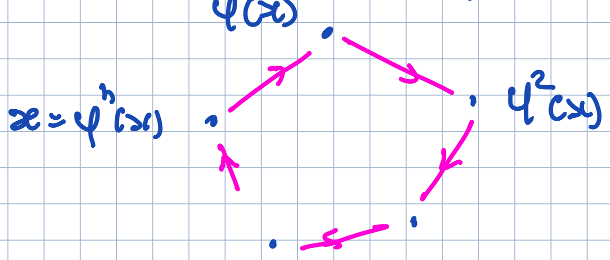
- Gradient flows:
  nothing interesting

  For any μ : $\operatorname{supp} \mu \subset \operatorname{Crit}(f) = \operatorname{Fix}(\varphi)$

  and μ is ergodic $\Leftrightarrow$ $\mu = \delta_x$, $x \in \operatorname{Crit}(f)$

Rmk - Ex: a discr inv measure μ

ergodic $\Leftrightarrow$ $\mu = \delta_{O(x)}$ $\leftarrow$ periodic

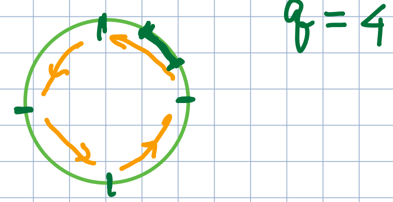


$$\mu = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\varphi^i(x)}$$

- Rotations of $S^1 = \mathbb{R}/\mathbb{Z}$

$\varphi : \theta \mapsto \theta + \alpha \qquad \alpha \in \mathbb{R}/\mathbb{Z}$

preserves $\mu = d\theta$



- $\alpha \in \mathbb{Q} \Rightarrow$ periodic : $\varphi^q = id, \quad \alpha = \frac{p}{q}$

$\Rightarrow$ every orbit is periodic

$\Rightarrow$ not ergodic

- $\alpha \notin \mathbb{Q}$

same condition as minimality

minimality

Thm $\alpha \notin \mathbb{Q} \Rightarrow \underbrace{\varphi \text{ is uniquely ergodic}}$

Pf. : Focus on ergodicity (not unique)

Need to show: $\forall g \in L^1(S^1)$

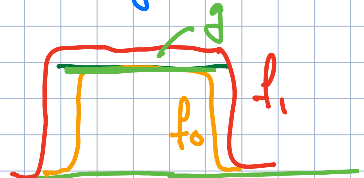
$$* \quad \frac{1}{n}\sum_{k=0}^{n-1} g(\theta + k\alpha) \longrightarrow \int g \, d\theta$$

$\Updownarrow$

For a.a. θ : in fact for all $\theta \in S^1$

$\Updownarrow$

Enough to do this for $g = \chi_U$ open



$f_0 \le g \le f_1, \; 0 \le f_1 - f_0 \le \varepsilon$

$\Updownarrow$

Enough to do this for $C^0(S^1)$

Lemma (Weyl)

$\forall f \in C^0(S^1)$

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\theta + k\alpha) \xrightarrow[\text{uniformly}]{} \int_{S^1} f\, d\theta$$

Pf. Trig polynomials are C^0-dense in $C^0(S^1)$

$\Rightarrow$ Enough to prove this for trig polynomials

$$\sum_{l=n}^{m} a_l e^{2\pi i l \theta}$$

Compare with the pf of minimality for translations of $\mathbb{T}^n$ ∇_0

$\Rightarrow$ Enough to prove this for $f = e^{2\pi i l \theta}$:

$$\boxed{\frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi i(\theta + k\alpha)\cdot l} \xrightarrow[\text{unif}]{} 0}$$

$l \neq 0$

$$\frac{1}{n}\left|\sum_{k=0}^{n-1} e^{2\pi i l(\theta+k\alpha)}\right|$$

$$= \frac{1}{n}\underbrace{\left|e^{2\pi i l\theta}\right|}_{1}\cdot\left|\sum_{k=0}^{n-1} e^{2\pi i l k\alpha}\right|$$

$$= \frac{1}{n}\frac{\left|1-e^{2\pi i l n\alpha}\right|}{\underbrace{\left|1-e^{2\pi i l\alpha}\right|}}$$

bounded by $2 = 1+1$

$\neq 0 \;\; \forall l : \alpha \notin \mathbb{Q}$

discuss later

$$\leq \frac{1}{n}\frac{2}{\left|1-e^{2\pi i l\alpha}\right|} \longrightarrow 0$$

$\triangleleft$

Rmk For unique hyperbolicity

Criterion: – exactly what we proved!

Assume that $\frac{1}{n}\sum_{k=0}^{n-1} f(\varphi^k(x)) \xrightarrow{\text{unif}} \int f\,d\mu$

for every $f \in C^0$

$\Rightarrow$ φ is uniquely ergodic $\triangleleft$

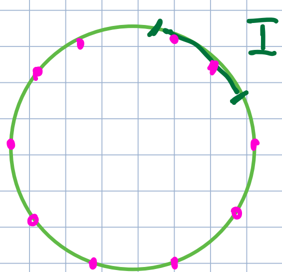
Digression to number theory:

uniform distribution

Def. A seq $x_k \in \mathbb{R}$ or $\mathbb{R}/\mathbb{Z} = S^1$
is uniformly distributed (mod 1)
if $\forall$ $I \subset S^1$ (an interval)

$$\underbrace{\lim_{n\to\infty} \frac{1}{n} \left|\{0 \le k \le n-1 \mid x_k \in I\}\right|}_{\text{frequency with which } x_k \text{ enters } I} = \underbrace{\mu(I)}_{\text{length of } I}$$

$$\frac{\mu(I)}{\mu(S^1)=1}$$



Rmk x_k is uniformly distributed
$\iff$ $x_k + \text{const}$ is

Pf: replace I by $I + \text{const}$

Q Which sequences are uniformly distributed?

Important in number theory:
books and books ...

By def:

$\varphi : S^1 \circlearrowleft$ is ergodic

$\iff \forall$ a.a. $x \in S^1$, $\varphi^k(x)$ is uniformly distributed

Cor $\alpha \notin \mathbb{Q} \iff k\alpha$ is uniformly distr. mod 1

More involved dynamical systems arguments (see e.g. [Walkden])

Thm (Weyl)

$x_k = \alpha_n k^n + .. + \alpha_1 k + \alpha_0$ is uniformly distr if at least one of $\alpha_1, \dots, \alpha_n \notin \mathbb{Q}$

etc.

Ex. $n=1$ $\quad x_k = \alpha_1 k + \alpha_0$

unif distr $\iff \alpha_1 \notin \mathbb{Q}$

- Translations of $\mathbb{T}^n$ and flows on $\mathbb{T}^n$

Very similar

$$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = \overbrace{S^1 \times \dots \times S^1}^{n}$$

$$\alpha = (\alpha_1, \dots, \alpha_n)$$

$$\varphi : \Theta = (\theta_1, \dots, \theta_n) \longmapsto \Theta + \alpha, \quad \underbrace{\alpha = (\alpha_1, \dots, \alpha_n)}_{\text{fixed}}$$

$$\mathbb{T}^n \longrightarrow \mathbb{T}^n$$

$\mu = d\theta_1 \dots d\theta_n$ is φ-invariant

Thm - Ex

same condition as minimality

φ is (uniquely) ergodic

$\iff$ $1, \alpha_1, \dots, \alpha_n$ are lin. ind over $\mathbb{Q}$

Pf - Ex : reduce to $e^{2\pi i \langle k, \theta \rangle} = f$

$k = (k_1, \dots, k_n)$

Likewise — uniform distributions ...

For flows on $\mathbb{T}^n$

$$\varphi^t(\theta) = \theta + t \cdot \alpha$$
$$= (\theta_1 + t\alpha_1, \dots, \theta_n + t\alpha_n)$$

<u>Thm - Ex</u>

φ^t is (uniquely) ergodic

$\iff$ $\underbrace{\alpha_1, \dots, \alpha_n \text{ are lin ind over } \mathbb{Q}}_{\text{same condition as minimality}}$

Toral Endomorphisms

a new class of examples

- $A \in \underbrace{SL(n, \mathbb{Z})}_{\substack{\text{integer entries} \\ \det A = 1}}$ $\iff A^{-1} \in SL(n, \mathbb{Z})$

- $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ $\qquad A : \mathbb{Z}^n \circlearrowleft$

$\Rightarrow \quad A : \mathbb{T}^n \longrightarrow \mathbb{T}^n$

$\qquad\qquad x \longmapsto Ax$

- $\mu = dx_1 \dots dx_n$ is invariant : $\det A = 1$
- what are fixed/periodic pts

$\rightarrow \quad A0 = 0 \Rightarrow 0 \in Fix(A)$

$\Rightarrow \delta_0$ is invariant

$\Rightarrow A$ is not uniquely ergodic, not minimal

$\rightarrow$ Other fixed or periodic pts

x is k-periodic

$A^k x = x$ in $\mathbb{T}^n$

Lifting to $\mathbb{R}^n$: $A^k x = x +$ integer vector

Def A is hyperbolic if

A has no eigenvalues

with abs value 1

Ex $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \leftarrow$ Arnold's cat map is hyperbolic

Ex A is hyperbolic $\Rightarrow$ per. pts are dense

[KH]?

e.g. A is hyperbolic

Thm A has no eigenvalue which is a root of unity $\Leftrightarrow$ A is ergodic

$\Downarrow$

Cor A has no eigenvalue which is a root of unity $\Rightarrow$ A has a dense orbit

In fact $\mathcal{O}(x)$ is dense for a.a. $x \in \mathbb{T}^n$

Rmk:
- when $n=2$: no root of unity $\Leftrightarrow$ hyperbolic
- $n>2$: Probably not?

Pf:

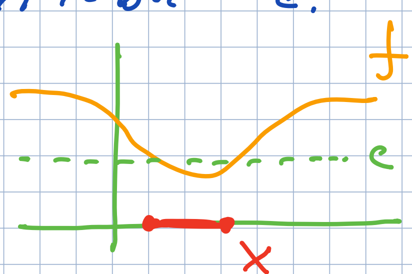
Recall:

- A ergodic: every invariant set has $\mu = 0$ or $\mu = 1$

- A not ergodic: $\exists$ inv. set X with $0 < \mu(X) < 1$

  $\iff$ $\exists$ an inv $f \in L^\infty$ s.t. $f \neq$ const (a.e.)   can have L^1 or L^2

  - Take $f = \chi_X$
  - In the other direction take $X = \{ x \mid f(x) \le c \}$ for a suitable c.



$\Rightarrow$ Need to show

A hyperbolic $\iff$ every invariant $f \in L^\infty$ is constant

$\Leftarrow$) Assume that A has an eigenvalue which is a root of unity.

Goal: construct an invariant function $f \in L^\infty$

- The same is true for $B := A^T$:

$\exists\ q \geq 1$ s.t. $B^q v = v$, $v \in \mathbb{R}^n$, $\neq 0$

$\Leftrightarrow (\underbrace{B^q - I}_{\text{in } M(n,\mathbb{Z})}) v = 0$

<u>Ex.</u> Prove that $v \in \mathbb{Q}^n$ and hence can assume $v \in \mathbb{Z}^n$

- Take the smallest q with this property

$$\Rightarrow\quad e^{2\pi i \langle v, A^q x\rangle} = e^{2\pi i \langle B^q v, x\rangle} = e^{2\pi i \langle v, x\rangle}$$

Set $f(x) = \sum_{j=0}^{q-1} e^{2\pi i \langle v, A^j x\rangle} \in L^2(\mathbb{T}^n)$, $C^\infty(\mathbb{T}^n)$

$$= \sum_{j=0}^{q-1} e^{2\pi i \langle B^j v, x\rangle}$$

Ex. $f \neq \text{const} \Leftarrow v \neq 0$

Use the fact that q is minimal deg

Claim: $f(Ax) = f(x)$ invariant

Pf
$$
\begin{aligned}
f(Ax) &= \sum_{j=0}^{q-1} e^{2\pi i \langle B^{j+1} v,\, x\rangle} \\
&= \sum_{j=1}^{q-1} e^{2\pi i \langle B^{j} v,\, x\rangle} + e^{2\pi i \langle B^{q} v,\, x\rangle} \\
&= f(x)
\end{aligned}
$$

$\triangleleft$

$\Rightarrow$) No roots of unity

$\Rightarrow$ No inv. functions

(L^2 or L^∞ or L^1 ...)

Assume f is invariant

$f \in L^2$ and $f(A^m x) = f(x)$ $\forall\, m \in \mathbb{Z}$

a.a. x

Fourier expansions

$$f(x) = \sum_{\ell \in \mathbb{Z}^n} f_\ell \exp(2\pi i \langle \ell, x \rangle)$$

$$\|$$

Need $f_\ell = 0$

$\ell \neq 0$

$$f(A^m x) = \sum_{k \in \mathbb{Z}^n} f_k \exp(2\pi i \langle k, A^m x \rangle)$$

$$= \sum_{k \in \mathbb{Z}^n} f_{\underset{B^{-m}\ell}{k}} \exp(2\pi i \langle \underbrace{B^m k}_{\ell \in \mathbb{Z}^n}, x \rangle)$$

$B = A^T$

$$f(x) = \sum_\ell f_\ell \exp(2\pi i \langle \ell, x \rangle)$$

$$\| \quad \Rightarrow \quad \|$$

$$f(A^m x) = \sum_\ell f_{B^{-m}\ell} \exp(2\pi i \langle \ell, x \rangle)$$

$\forall\, m \in \mathbb{Z}$

$\forall\, \ell \in \mathbb{Z}^n$

$$\dots = f_{B^{-1}\ell} = f_\ell = f_{B\ell} = f_{B^2\ell} = \dots$$

i.e. $\boxed{f_\ell = f_{B^m \ell}}$ $\quad \forall\, m \in \mathbb{Z}\ \ \forall\, \ell \in \mathbb{Z}^n$

Goal: $\ell \neq 0 \Rightarrow f_\ell = 0$

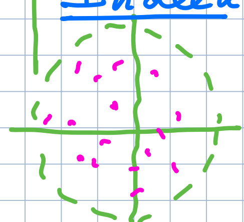
Hence $f = \text{const}$

Note: $\underbrace{|f_\ell| \to 0 \text{ as } |\ell| \to \infty}$ $\Leftarrow f \in L^2$

Assume $f_\ell \neq 0$, $\ell \neq 0$

$\Rightarrow$ $B^m \ell$, $m \in \mathbb{Z}$ can take only finitely many values

Indeed



Assume not: then $B^{m_s}\ell \to \infty$ for some $m_s \to \pm\infty$

only finitely many $\mathbb{Z}^n$ pts in a ball

$\Rightarrow f_\ell = f_{B^{m_s}\ell} \to 0$

$\Rightarrow f_\ell = 0$

$\Rightarrow \quad B^{m_1} l = B^{m_0} l \qquad m_1 > m_0$

$\Rightarrow \quad B^{\underbrace{m_1 - m_0}_{q}} l = l$

$\Rightarrow \quad B^q l = l$ ← eigenvector

$\Rightarrow$ an eigenvalue which is a root of unity!

$\rightarrow\leftarrow$

$\triangleleft$

Go through the Thm and pf again

Lecture 7

01/25-2022

- A stronger ergodicity property: Mixing (Digression)

Def $\varphi : (M, \mu) \circlearrowleft$ is mixing if

Roots in probability theory

$\forall A, B$ (measurable)

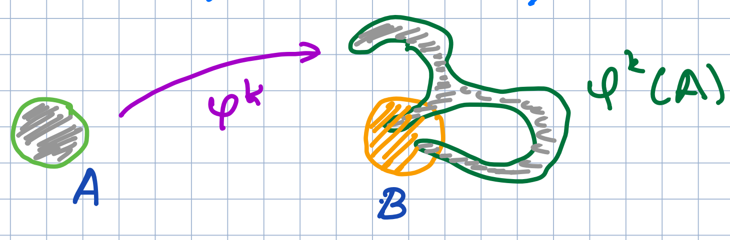
$$\mu(\varphi^{-k}(A) \cap B) \xrightarrow[k\to\infty]{} \mu(A)\mu(B)$$

Rmk & Observations

- $\varphi^{-k}(A) = \{x \mid \varphi^k(x) \in A\}$ is defined even when φ is not invertible
- Prefer to think as

$$\mu(\varphi^k(A) \cap B) \longrightarrow \mu(A)\mu(B)$$

- $\mu(A) \neq 0$, $\mu(B) \neq 0$
$\Rightarrow \varphi^k(A) \cap B \neq \emptyset \quad k \gg 0$



- Topological counterpart (top mixing)
$\forall$ open sets U, V

$$\varphi^{-k}(U) \cap V \neq \emptyset \quad \forall \text{ large } k$$

- Mixing $\Rightarrow$ Top Mixing
  when $\mu(\text{open}) > 0$

- Mixing $\Rightarrow$ Ergodic

Pf • Ergodic $\Leftrightarrow$ $\forall$ inv set A

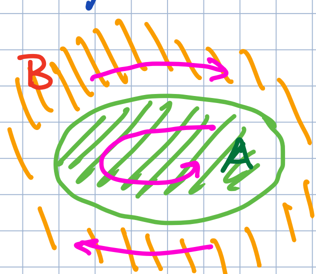
$$\underbrace{\mu(A) = 0 \text{ or } \mu(A) = 1}$$

$$\Leftrightarrow \mu(A)\,\mu(\underbrace{M \setminus A}_{B}) = 0$$

• φ mixing, $\underbrace{A \text{ invariant}}$: Need
$B = M \setminus A$

$$\underbrace{\varphi^{-k}(A)}_{A} \cap \underbrace{B}_{M\setminus A} = \emptyset$$

$$0 = \mu(\varphi^{-k}(A) \cap B) \longrightarrow \underbrace{\mu(A)\,\mu(M \setminus A) = 0}_{\Rightarrow \text{ ergodic}}$$



A & B never mix

$\triangleleft$

Rmk • Similarly for flows
• Other notions of mixing (weak, etc) ...

Ex

Examples

- isometries are not (top) mixing
  $\Rightarrow$ rotations of S^1, translations of $\mathbb{T}^k$, lin flows on $\mathbb{T}^k$ are not mixing

mixing $\Rightarrow$ ergodicity
$\;\;\;\;\;\;\;\;\;\;\not\Leftarrow$ unique ergodicity

- Hyperbolic $A: \mathbb{T}^2 \to \mathbb{T}^2$ are mixing [KH]

$$\begin{array}{ccc} & \text{ergodicity} & \\ \nearrow\!\!\!\!\nearrow & & \Uparrow \\ \text{mixing} & \begin{array}{c}\not\Leftarrow \\ \not\Rightarrow\end{array} & \text{unique ergodicity} \end{array}$$

- Examples continued: Shift Transformations

Setting:

- $\mathbb{Z}_n = \{0, \dots, n-1\}$ an alphabet
- $M = \mathbb{Z}_n^{\mathbb{Z}} = \{ x = \dots x_{-1} x_0 x_1 x_2 \dots \}$
  = bi-inf sequences
  compact metric space
- $\varphi : M \to M$ shift to the left

$$\varphi(x)_i = x_{i+1} \quad , \quad \text{homeo}$$

$$\dots \; x_2 \; x_{-1} \; \boxed{x_0} \; x_1 \; x_2 \; x_3 \; \dots$$

- Rmk $\mathbb{Z}_n$ a gp
  $\Rightarrow$ $M = \mathbb{Z}_n^{\mathbb{Z}}$ is a compact top. gp
  φ is a gp homomorphism
- Inv measures. (Bernoulli measures)
  - Fix $0 < p_i < 1$ $i = 0, \dots, n-1$
    $\Sigma p_i = 1$ probability of $a_i \in \mathbb{Z}_n$

- Cylinders

$I = (i_1, \dots, i_s)$ multiindex

$Y = (a_1, \dots, a_s) \in Z_m^s$

$$C_Y^I = \{x \mid x_{i_j} = a_{i_j} \mid \forall\ i_j \in I\}$$

$\dots\dots x_{-1} x_0 x_1 \dots x_{i_j} \dots$

$\qquad\qquad\qquad\quad \shortparallel$

$\qquad\qquad\qquad\quad a_{i_j}$

These also form a base of the top of $Z_m^{\mathbb{Z}}$

<u>Def</u>

$p_i = \mu(a_i)$

$$\mu(C_Y^I) = p_{a_1} \cdot p_{a_2} \cdot \dots \cdot p_{a_s}$$

μ is the product measure

$\Rightarrow$ Extend to a measure

$\Rightarrow$ a φ invariant measure μ on M (Probability, Borel)

<u>Ex</u> $p_i = \frac{1}{n}$: all a_i have the same prob

$\Rightarrow$ the Haar measure on $Z_n^{\mathbb{Z}}$ (invariant under right or left translations, $\mu(\text{open}) > 0$

Recall top properties of φ:

- dense periodic pts = periodic seq x_i
  $p(k) = n^k \leftarrow$ # of per pts of per k
- top transitive: $\exists$ a dense orbit

On the measure theory side:

Thm φ is mixing for μ

$\Rightarrow$

Cor φ is ergodic & top mixing

Rmks:

- φ is not uniquely ergodic (per. orbits or different $\{p_i\}$) and not minimal
- similarity with hyperbolic $A: \mathbb{T}^n \to \mathbb{T}^n$

Pf

- Observations: [KH] enough to check mixing when A & B are cylinders

$$\cdots\; x_{i_1} \;\cdots\; x_j \;\cdots\; x_{i_2} \;\cdots$$
$$\cdots\; \overset{\shortparallel}{a_{i_1}} \;\cdots\; \overset{\shortparallel}{a_{i_2}} \cdots = Y$$

Need $\mu(\varphi^{-k}(C_Y^I) \cap C_X^J) \to \mu(C_Y^I)\,\mu(C_X^J)$ for any two such cylinders different length

- Note . $\varphi^{-k}(C_Y^I) = C_Y^{I+k}$

$I+k = (i_1+k, \dots, i_s+k)$

$\Rightarrow \forall\, I \;\&\; J$

$$\varphi^{-k}(C_Y^I) = C_Y^{I+k}$$

disjoint from J when k is large

- Recall $\qquad Y = (a_1, \dots, a_s)$

$$\mu(C_Y^I) = p_{a_1} \cdots p_{a_s}$$

- $L \cap J = \emptyset \qquad Y = (a_1 \dots a_s)$

$$C_Y^L \cap C_X^J = C_{Y \cup X}^{L \cup J} \qquad X = (b_1, \dots, b_r)$$

$$\Rightarrow \boxed{\underbrace{\mu(C_Y^L \cap C_I^J)}_{a_1 \dots a_s b_1 \dots b_r} = \underbrace{\mu(C_Y^L)}_{a_1 \dots a_s} \cdot \underbrace{\mu(C_X^J)}_{b_1 \dots b_r}}$$

- $L = I + k \qquad k$ is large

disjoint from J

$$C_Y^L = C_Y^{I+k} = \varphi^{-k}(C_Y^I) \qquad \text{disjoint}$$

$$\Rightarrow \quad \varphi^{-k}(C_Y^I) \cap C_X^J = C_Y^L \cap C_X^J$$

$$\Rightarrow \quad \mu(\varphi^{-k}(C_Y^I) \cap C_X^J) = \mu(\varphi^{-k}(C_Y^I))\, \mu(C_X^J)$$

$\Rightarrow$ mixing

$\triangleleft$

Probabilistic Interpretation

- $\mathbb{Z}_2 = \{0, 1\}$; $\boxed{p_0 = p_1 = \frac{1}{2}}$ unbiased coin
- $M =$ bi-inf sequences of 0 & 1's
- each sequence
  = sequence of coin tosses
  $0 =$ heads
  $1 =$ tails
  an experiment or a trial

- $I = \{0, \dots, m\}$
  $Y = \{b_0, \dots, b_m\}$ $\qquad b_i = 0$ or 1

  C^I_Y = event : the first $m+1$ tosses give outcome Y : $\mu(C^I_Y) = \dfrac{1}{2^{m+1}}$

  $\varphi^{-k}(C^I_Y) = C^{k+I}_Y$
  = event : the tosses $k, \dots, k+m$ give outcome Y

$$\lim_{k \to \infty} \frac{1}{k} \{0 \le i \le k-1 \mid \varphi^k(x) \in C^I_Y\}$$

= frequency with which the sequence Y occurs in x

<u>Ergodicity</u> $\Rightarrow$ for almost all trials x

$$\underbrace{\text{the frequency}}_{\substack{\text{something we} \\ \text{can measure} \\ \text{by experiment,} \\ \text{"statistics"}}} = \underbrace{\text{probability of } Y}_{\underbrace{\mu(C^{I}_{Y})}_{\text{abstract notion}}}$$

<u>Ex.</u> Interpret mixing in terms of conditional probability.

Lecture 8
01/27 - 2022

- Existence of invariant measures

M compact metric space (separable)
$\varphi : M \longrightarrow M$ homeo or just C^0
We have used:

Fact: φ has an invariant (ergodic) measure.

Goal: justify this

Thm (Krylov - Bogolubov)
M compact, $\varphi : M \xrightarrow{C^0} M$
$\Rightarrow \exists$ an invariant Borel probability measure

Preliminaries: M as above

- $C^0(M)$ = Banach space with
  $$\|f\| = \sup_{x\in M} |f(x)|$$

- Dual space:
  $$C^0(M)^* = \{ \Phi : C^0(M) \to \mathbb{R} \mid \text{bounded} \}$$

Thm (Riesz Representation Thm)

$C^0(M)^*$ = the space of finite Borel measures μ (not necessarily pos)

$$\Phi(f) = \int_M f \, d\mu$$

Rmk
- $\mu = \mu_+ - \mu_-$ ← pos. measures
- Φ pos: $f \geq 0 \Rightarrow \Phi(f) \geq 0$
  $\Rightarrow \mu$ is pos
- $\Phi(1) = 1 \Rightarrow \mu$ is probability: $\int \mu = 1$
- $\Phi(f \circ \varphi) = \Phi(f) \quad \forall f$
  $\Leftrightarrow \mu$ is φ invariant

Pf

Idea: For $x \in M$ set

$$\mu_x(V) = \lim_{k\to\infty} \frac{1}{k}\{0 \le i \le k-1 \mid \varphi^i(x) \in V\}$$

as in Birkhoff ergodic theorem, or

$$\Phi_x(f) = \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} f(\varphi^i(x))$$

and define μ_x by

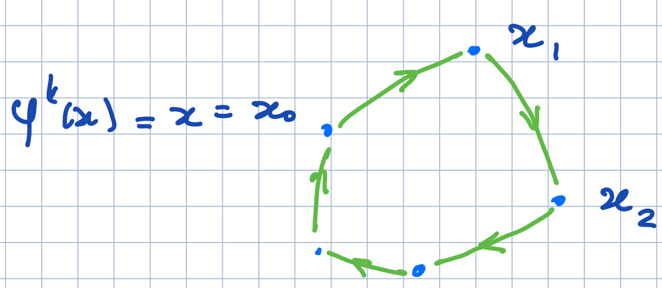
Riesz

$$\Phi_x(f) = \int f \, d\mu_x$$

Then μ_x is an invariant probability measure assuming that the limits exist

Ex: x is k-periodic

$x = x_0, \; x_1 = \varphi(x), \dots \; x_i = \varphi^i(x), \; x_k = \varphi^k(x) = x_0$



$\Rightarrow \quad \mu_x = \frac{1}{k} \sum \delta_{x_i}$ invariant Borel probability measure

Implementation

- Let $f_j \in C^0(M)$, $j = 1, 2, \ldots$ be a countable collection dense in C^0 (with respect to the sup-norm.)

- Pick x and consider

$$a_k^j = \frac{1}{k} \sum_{i=0}^{k-1} f_j(\varphi^i(x)) \longleftarrow \underline{\text{bounded}}\ \forall j.$$

diagonal process

$\Rightarrow$ $k_s(1) \xrightarrow[s \to \infty]{} \infty$ $\quad a^1_{k_s(1)}$ converges as $s \to \infty$

$\Rightarrow$ $k_s(1)$ contains a subsequence

$k_s(2) \xrightarrow[s \to \infty]{} \infty$ $\quad a^2_{k_s(2)}$ also converges

$\ldots$

Set $k_s = k_s(s)$ subsequence in all of them

$\Rightarrow$ $a^j_{k_s} \xrightarrow[k_s \to \infty]{} a^j$ $\quad \forall j$:

$$\exists \lim_{k_s \to \infty} \frac{1}{k_s} \sum_{i=0}^{k_s - 1} f_j(\varphi^i(x)) = a^j \quad \forall j$$

$\Rightarrow$

$$\exists \lim_{k_s\to\infty} \frac{1}{k_s}\sum_{i=0}^{k_s-1} f(\varphi^i(x)) =: \Phi_x(f)$$

exists

$\{f_j\}$ dense in $C^0(M)$

Riesz Representation theorem

$\Rightarrow \exists\ \mu_x$ s.t.

$$\Phi_x(f) = \int f\, d\mu_x$$

check (Ex):

- $f \geq 0 \Rightarrow \Phi_x(f) \geq 0$
- $\Phi_x(1) = 1$

$\Big\}$ clear

- $\Phi_x(f\circ\varphi) = \Phi_x(f)$ $\}$ calculation

$\Rightarrow$ μ_x is positive, probability and invariant

$\triangleleft$

Rmk $\quad \operatorname{supp}\mu_x \subset \overline{\mathcal{O}(x)}$

Rmk (Ex) A short cut with more functional analysis:

Set
$$\Phi_x^{(k)}(f) := \frac{1}{k}\sum_{i=0}^{k-1} f(\varphi^i(x)) = \left(\frac{1}{k}\sum_{i=0}^{k-1} \delta_{\varphi^i(x)}\right)(f)$$

$$|\Phi_x^{(k)}(f)| \underset{=}{\leq} \|f\|$$

$$\Rightarrow \quad \|\Phi_x^{(k)}\| \underset{=}{\leq} 1 :$$

$$\Phi_x^{(k)} \in \underbrace{\text{unit ball in } C^0(M)^*}_{\text{week* compact (sequentially)}}$$

Alaoglu's thm

Alaoglu thm $\Rightarrow$ $\Phi_x^{(k)}$ contains a pt-wise converging subsequence:

$$\Phi_x^{k_i}(f) \longrightarrow \Phi_x(f) \quad \forall f$$

This is essentially the def of Φ_x

Now finish the pf as above. $\triangleleft$

Rmk Can also take
$$\lim \frac{1}{2k+1}\sum_{i=-k}^{k} f(\varphi^i(x))$$
when φ is invertible

Rmk How often does the lim exist?

$$\lim_{k\to\infty} \frac{1}{k} \underbrace{\underbrace{\sum_{i=0}^{k-1} f(\varphi^i(x))}_{\sum_{i=0}^{k-1} \delta_{\varphi^i(x)} f}}_{\Phi_x^k(f)}$$

Ex or [KH]

Answer: for any φ-inv μ the limit exist for μ-a.e. x $\forall f$

Hint: combine the Birkhof ergodic theorem with the pf of Krylov-Bogolubov thm

• How do ergodic measures enter this picture?

Notation: $\mathcal{M}_\varphi = \{\varphi\text{-inv. prob. Borel measures}\}$

$$\mathcal{M} = \mathcal{M}_\varphi \hookrightarrow C^0(M)^*$$

$$\mu \longmapsto \Phi_\mu := \left(f \longmapsto \int_M f \, d\mu \right)$$

• The image is in the unit sphere $\|\Phi\| = 1$ and weak* compact & $\Phi_\mu(1) = 1$

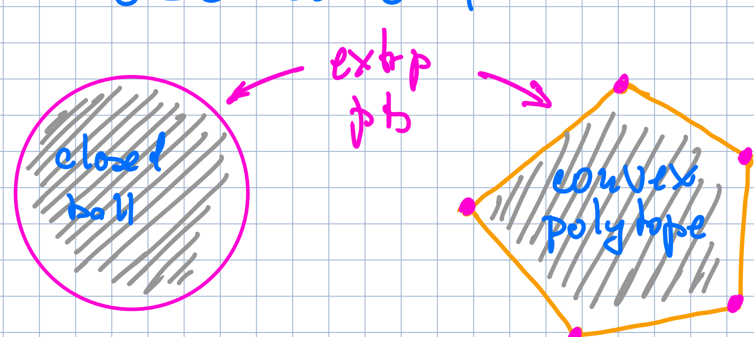
• $\mathcal{M}_\varphi$ is convex:

$$0 \le t \le 1, \quad \mu = (1-t)\mu_0 + t\,\mu_1$$

$$\mu \in \mathcal{M}_\varphi \Leftarrow \mu_0 \in \mathcal{M}_\varphi, \quad \mu_1 \in \mathcal{M}_\varphi$$

Def μ is an <u>extreme</u> pt of $\mathcal{M}$ if for any such decomposition $t=0$ or $t=1$

can be any convex set



Notation: $\mathrm{Ext}(\mathcal{M})$

<u>Cor</u> $\underbrace{\mathrm{Ext}}_{\text{ext pts}}(\mathcal{M}) \neq \emptyset$

<u>Rmk</u> In general: <u>closure</u>

$$\overline{\text{convex Hull}(\underbrace{\mathrm{Ext}(\mathcal{M})}_{\text{ext. pts}})} = \mathcal{M}$$

convex

In general, when $\dim = \infty$, even the fact that

$$\mathrm{Ext}(\mathcal{M}) \neq \emptyset$$

is not obvious

Thm $\left\{\begin{matrix}\text{Ergodic}\\ \text{measures}\end{matrix}\right\} = \left\{\begin{matrix}\text{Extreme}\\ \text{of pts } \mathcal{M}_\varphi\end{matrix}\right\}$

Pf

"$\supset$" $\quad \mu \in \mathcal{M}_\varphi$, not ergodic:

$\exists\, A$ with $0 < \mu(A) < 1$

Set $\mu_X(Y) = \dfrac{\mu(X \cap Y)}{\mu(X)}$ $\Big\}$ restriction to X measure

$\mu_0 = \mu_A, \quad \mu_1 = \mu_{M \setminus A}$

$\Rightarrow \quad \mu = \mu(A)\mu_A + (1 - \mu(A))\mu_{M \setminus A}$

$\Rightarrow \quad \mu$ is not an extreme pt

"$\subset$" **Idea** • μ_0, μ_1 = extreme pts, ($\Rightarrow$ ergodic)

enough to have μ_0 extr

• $\mu_0 \neq \mu_1$: $\exists\, A$ $\mu_0(A) \neq \mu_1(A)$

Form:

$\mu = (1-t)\mu_0 + t\mu_1$ not extreme : $t \neq 0, 1$

Want to show not ergodic

(A particular case)

Assume it is

Birkhoff.

$$\frac{1}{k}\sum_{i=0}^{k-1} \chi_A(\varphi^i(x)) \nearrow \mu(A) = (1-t)\mu_0(A) + t\mu_1(A)$$
$$\qquad\qquad \longrightarrow \mu_0(A)$$

$\Rightarrow \mu_0(A) = \mu_1(A) \quad \rightarrow\leftarrow$

A catch: - need x to "a.a." for μ_0 & μ_1

- might not exist
- But then $\operatorname{supp}\mu_0 \cap \operatorname{supp}\mu_1 = \emptyset$ and μ is again not ergodic

- A more serious problem: not every non-extreme pt can be decomposed as $\mu = (1-t)\mu_0 + t\mu_1$
- Not literally, but ...

Need some functional analysis: Choquet's thm

<u>Cor</u> Every $\varphi: M \to M$ has an ergodic measure

<u>Cor</u> The following def of unique ergodicity are equivalent:

- ergodic and an ergodic measure is unique
- ergodic and inv measure is unique.

How common is ergodicity?

Setting:

- M a compact manifold (perhaps with boundary or corners)
- μ = smooth measure (Lebesgue)

E.g. M = closed ball or I^n = cube

- $H = \{\varphi : M \to M \mid \mu\text{-pres homeo}\}$ with sup-topology:

$$d(\varphi, \psi) = \sup_{x \in M} (\varphi(x), \psi(x))$$

H has the Baire property:
a countable intersection of open & dense sets is dense

a residual set: dense G_δ or more generally containing a dense G_δ

Thm (Oxtoby - Ulam)

Ergodic φ form a residual subset of H, $\dim \geq 2$

Ex. Show that not true when $\dim = 1$

Cor Top transitive φ (i.e. with a dense orbit) form a residual subset of H

Rmk • Nothing like that is true for C^∞-diffeos pres μ!

KAM

At least when $\dim M = 2$ or in the Ham case or...

...

• C^1 or C^k – more subtle

Avila - Crovisier - Wilkinson

ArXiv 1408.4252

• Not easy to construct $\varphi : D^2 \to D^2$ with a dense orbit

Direct pf of Cor - Outline

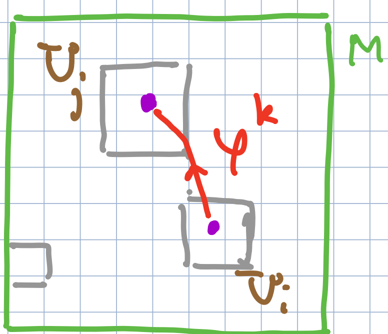
Following [Oxtoby]

- $M =$ square $[0,1]\times[0,1]$

  $\mu =$ Lebesgue measure

  $\{\varphi : M \to M \ \mu\text{-pres. homeo}\} = H$

- $\{U_i\} =$ collection of open squares in M with rational vertices (top base)



$$E_{ij} = \{\varphi \mid \exists k \geq 1 : \varphi^{-k}(U_j) \cap U_i \neq \emptyset\}$$

clear　　key pt

Claim $\forall\, i,j$ E_{ij} is open and dense

Thm $\Leftarrow$ Claim:

$$\bigcap_{i,j} E_{i,j} =: E \leftarrow \text{residual}$$
$$\Psi \ni \varphi$$

$G_j := \bigcup_{k=1}^{\infty} \varphi^{-k}(U_j)$ is open & dense:

$$U_i \cap \bigcup_{k=1}^{\infty} \varphi^{-k}(U_j) \neq \emptyset \Leftarrow \varphi \in E$$

Baire: $G = \bigcap G_j$ is residual in M

$$\Rightarrow G \neq \emptyset$$

$x \in G$ $\quad \forall j: \quad x \in \bigcup_{k=1}^{\infty} \varphi^{-k}(U_j)$

$$\Rightarrow \exists k: \quad \varphi^k(x) \in U_j$$

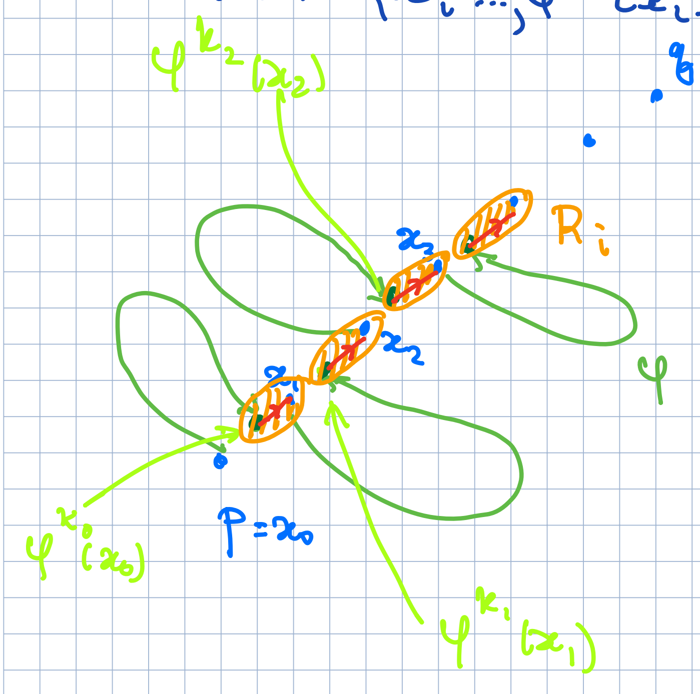
$$\Rightarrow \mathcal{O}(x) \text{ is dense.}$$

Rmk We have shown that for a residual set of φ's the set of x with $\mathcal{O}(x)$ dense is residual.

Idea of the pf of the Claim

- Given i, j and φ need to find an arbitrarily small ψ and $p \in U_i$ s.t. $(\psi\varphi)^k(p)$ U_j for some k
  $\Rightarrow \psi\varphi \in E_{ij}$ & $\psi\varphi \approx \varphi$
- Can assume that periodic pts of φ form a meager set
  – such φ's form a residual set

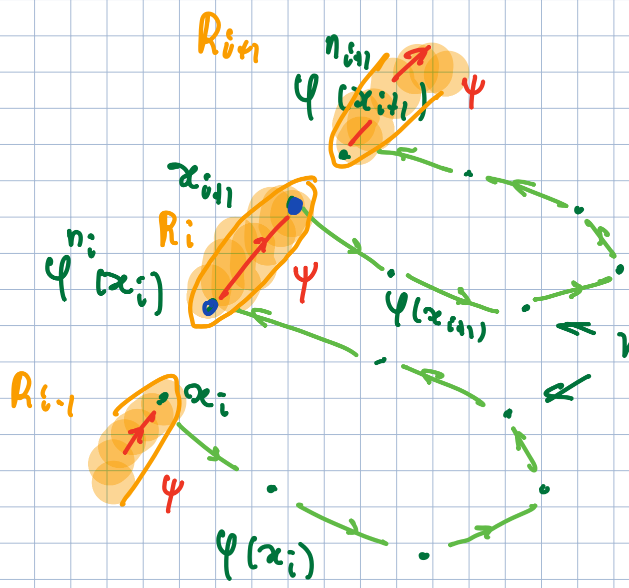
- Pick $p \in U_i$ & $q \in U_j$ and "connect" them by a seg. disj orbit $\{x_i, \dots, \varphi^{n_i}(x_i)\}$



- F = union of these orbits

- R_i's : disjoint small open sets
  $F \cap R_i = \{\varphi^{n_i}(x_i), x_{i+1}\}$

$R = \coprod R_i$



Local pic enlarged

none of these other pts are in R

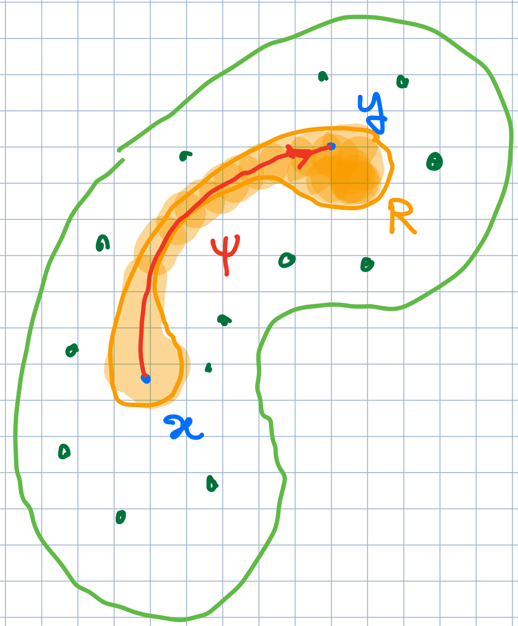
- ψ : · $\operatorname{supp}\psi \subset \bigsqcup R_i$
  · $\psi(\varphi^{n_i}(x_i)) = x_{i+1}$
  · $\|\psi\|_{C^0}$ is small

- $\Rightarrow \quad (\varphi\psi)^{n_0+\dots+n_k}(p) = q$

Rmk $\quad x_i, \varphi^{n_i}(x_i), x_{i+1}, \varphi^{n_{i+1}}(x_{i+1})$ are quite close

$\Rightarrow$ size $(R_i) \sim d(x_i, \varphi^{n_i}(x_i))$

$\Rightarrow$ cannot make $\|\psi\|_{C^1}$ small $\quad \triangleleft$

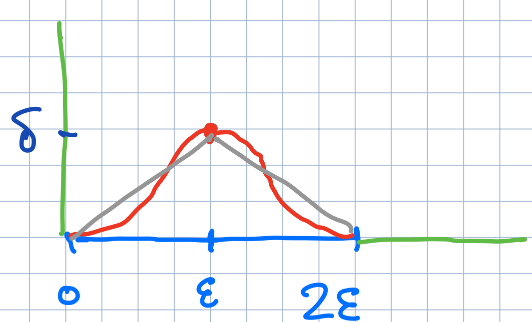


$p \approx q$, $\operatorname{supp} \psi \subset R$

$\Rightarrow \psi \approx id$

$\not\Rightarrow \psi \overset{C^1}{\approx} id$

$f(\varepsilon) = \delta$



- if δ is small $\Rightarrow f$ C^0 small

$\max |f| = \delta$

- f C^1-small?

$f' \sim \frac{\delta}{\varepsilon}$

$\Rightarrow$ If $\delta \ll \varepsilon$ can mak f C^1-small

If $\delta \approx \varepsilon$ cannot

§ 3 Homeomorphisms of S^1

Lecture 9
02/01-2022

Generalities: Equivalence of Dynamical systems

Setting

homeo or diffeo in some reasonable class

$\varphi, \psi : M \longrightarrow M$

a closed manifold

"Def" φ & ψ are equivalent if

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & M \\ h \uparrow & & \uparrow h \\ M & \xrightarrow{\psi} & M \end{array}$$

$\varphi = h \psi h^{-1}$

φ is "conj" to ψ

- usually h is roughly of the same type as φ & ψ (e.g. volume preserving)
- But usually h is only C^0, even when φ & ψ are C^k, $1 \leq k \leq \infty$

Want h to preserve the most essential features: periodic orbits, top transitivity, ergodicity ...

Q: Any hope of "classification"?

Rmk: In some limited number of cases, Yes. Overall, No

Related notion: structural stability

"Def" φ is str. stable if

$$\psi \underset{C^k}{\approx} \varphi \Rightarrow \psi \sim \varphi$$

top. conj

Rmk: rare but interesting

str stable property:

φ has it & $\psi \underset{C^k}{\approx} \varphi \Rightarrow \psi$ has it

Rmk: more reasonable

Rmk (Flows)

The true conj: $\psi^t = h\varphi^t h^{-1}$ (even when $h \in C^0$) is usually true restrictive (see below)

$\Rightarrow$ usually just want h to send orbits to orbit but not to preserve time-parametrizations

Some comments and Examples

- "Persistence" of fixed pts under small perturbations

- $\varphi: M \xrightarrow{C^1} M$ ← manifolds

- $\varphi(p) = p$

$\underbrace{D\varphi_p}_{\text{the "linearization"}} : T_pM \longrightarrow T_pM$ invertible

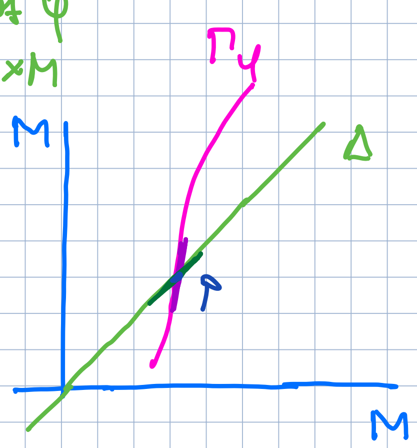
$\varphi(x) = D\varphi_p(x) + \dots$
$p = 0$

Def p is non-deg if
- $D\varphi_p$ does not have 1 as eigenvalue
  i.e. $I - D\varphi_p$ is invertible

Ex ⇕

- $\Gamma_\varphi \pitchfork \Delta$ at p

graph of φ in $M\times M$ — the diagonal



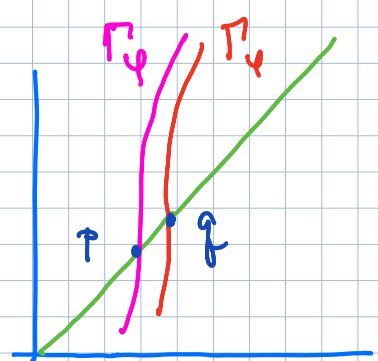
Prop - Ex

Assume that p is non-deg and $\psi \underset{C^1}{\approx} \varphi$

$\Rightarrow$ • Near p $\exists$ a fixed pt q of ψ

• $D\psi_q \approx D\varphi_p$ $\leftarrow$ use a chart containing p & q

inverse function thm



slope of Γ_φ at p

$\approx$

slope of Γ_ψ at q

Def - Rmk x = q-periodic pt is non-deg if $x \in \mathrm{Fix}(\varphi^q)$ is non-deg for φ^q

min period

Rmk $\psi = h\varphi h^{-1}$, $h \in C^1$

$p \in \mathrm{Fix}\,\varphi \Rightarrow q = h(p) \in \mathrm{Fix}(\psi)$

$$\begin{tikzcd}
T_pM \arrow[r, "D\varphi_p"] & T_pM \\
T_qM \arrow[u, "Dh"] \arrow[r, "D\psi_q"'] & T_qM \arrow[u, "Dh^{-1}"']
\end{tikzcd}$$

if $h \in C^1$

$\Rightarrow \underbrace{\text{eigenvalues of } D\varphi_p = \text{eigenvalues of } D\varphi_q}_{\text{easy to change by a } C^1\text{-small pert}}$

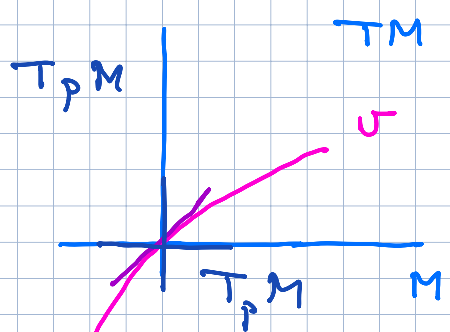
$\Rightarrow$ Not much hope for "classification" and str stability if $h \in C^1$

Rmk (Flows)

φ^t = flow of vect. field v

$v(p) = 0 \Rightarrow p \in \mathrm{Fix}(\varphi^t) \quad \forall t$

In a chart: $v(x) = Dv_p(x) + \dots$



$Dv_p : T_pM \to T_pM$

Def p is non-deg if

- Dv_p is "non-deg" does not have 0 as an eigenvalue
- $\mathrm{Graph}(v) \pitchfork \underbrace{M}_{\text{zero section}} \subset TM$

Ex. State and prove an analogue of Prop for flows

Rmk Assume that $\psi^t = h\varphi^t h^{-1}$ $\forall t$
"the conjugation preserves time"

$\Rightarrow$ γ = periodic orbit of φ^t with period T

$\Updownarrow$

$h(\gamma)$ = periodic orbit of ψ^t with period T

The period is very easy to change by a small perturbation:
E.g. $v \rightsquigarrow (1+\varepsilon)v$

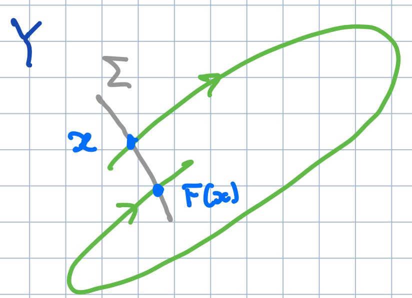
$\Rightarrow$ Not much hope for str. stability or classification when h preserves time.

- Conceptually:

maps in dim n $\leftrightsquigarrow$ flows in dim $n+1$

Cross-sections: • φ^t flow on Y^{n+1} generated by v

Y



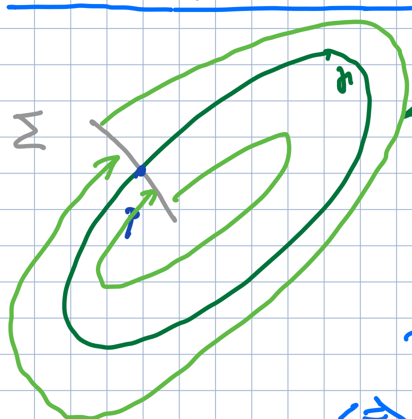
• $\Sigma^n \subset Y^{n+1}$, $v \pitchfork \Sigma$ and the return map F is defined

$\Rightarrow$ Dynamics of F captures a lot of dynamics of φ^t:

periodic orbit of F $\hookrightarrow$ Periodic orbits of the flow

Global cross-sections rarely exist

Poincaré return map



periodic orbit γ of φ^t

$F : (\text{nbd of } p \text{ in } \Sigma) \to \Sigma$

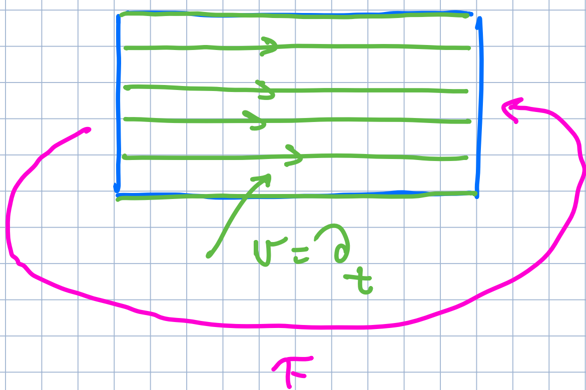
p is non-deg for F

$\overset{\text{def}}{\Leftrightarrow}$ γ is non-deg for v (or φ^t)

Mapping torus

$F : M \longrightarrow M$

$$Y = \underset{t}{M \times [0,1]} / \sim \qquad (x,0) \sim (F^{-1}(x), 1)$$



- v descends to Y $\Rightarrow$ flow φ^t
- $\Sigma = M \times 0$ is a closed section
- F = return map

$$\mathrm{Per}(F) = \mathrm{Per}(\varphi^t)$$

F is top transitive, ergodic, minimal etc $\iff$ φ^t is top transitive, ergodic, minimal etc

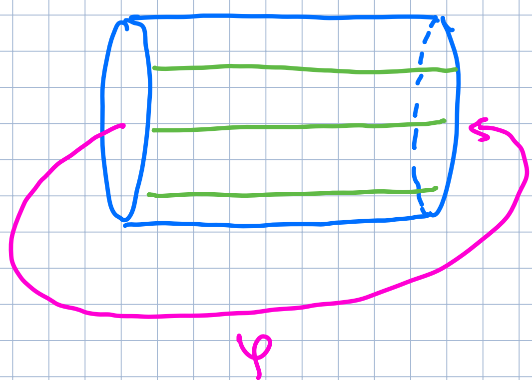
Dynamics of F $\longleftrightarrow$ Dynamics of φ^t

Ex - Ex

$M = S^1$, $\varphi : S^1 \xrightarrow{C^k} S^1$ $\quad k = 0, 1, \ldots$

orientation pres, homeo or diffeo

$$Y = S^1 \times [0,1] / (x,0) \sim (\varphi(x), 1)$$



Prove that Y is a C^k-manifold which is diffeo ($k \geq 1$) or homeo (C^0) to $\mathbb{T}^2$

Lecture 10
02/03-2022

Specializing

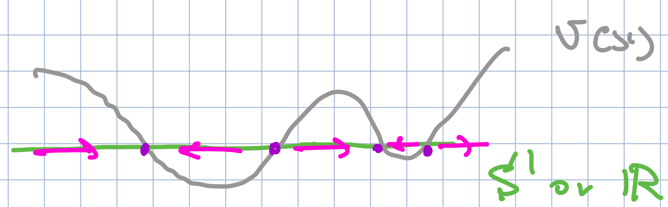
Dynamics on S^1: questions

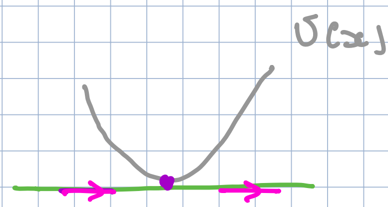
- Flows on S^1 or $\mathbb{R}$ are rather simple

→ $\dot{x} = v(x)$ can be integrated explicitly

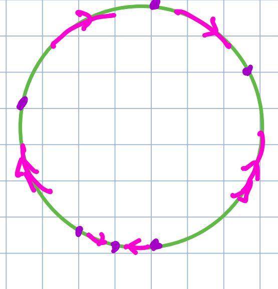
$$\frac{dx}{v(x)} = dt \quad \ldots$$

→ Easy to visualize:





- Fixed pts = zeros of v
- no periodic orbits or interesting dynamics



But homeo or diffeo's

$$\varphi : S^1 \to S^1, \quad S = \mathbb{R}/\mathbb{Z}$$

can already be very interesting

Some questions:

- What are str. stable maps?
- R_α rotation by α

  $\theta \to \theta + \alpha$

  Is R_α equiv R_β $\alpha \neq \beta$?

  Cases: $\alpha = \frac{p}{q}$, $\beta \notin \mathbb{Q}$ or $\alpha, \beta \notin \mathbb{Q}$ or

  $\alpha = \frac{p_1}{q}$, $\beta = \frac{p_2}{q}$

- Is $\varphi(\theta) = \theta + \alpha + \varepsilon \sin(2\pi\theta)$ equivalent to R_α

- Can we have φ without periodic orbits and dense orbits?

Trying to answer $\Rightarrow$ unexpected results

Rotation number

- classification questions $\leftrightarrow$ invariants
- Rotation number $\rho : \underbrace{\mathrm{Homeo}_+(S^1)}_{H} \longrightarrow S^1$

Construction: $\varphi \in H = \mathrm{Homeo}_+(S^1)$

$S^1 = \mathbb{R}/\mathbb{Z}$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "F"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\Uparrow \exists"] & \mathbb{R} \arrow[d, "\pi"] \\ S^1 \arrow[r, "\varphi"'] & S^1 \end{tikzcd}$$

a lift of φ to $\mathbb{R} \to \mathbb{R}$

$x = e^{2\pi i \theta}$

$\varphi(x) = e^{2\pi i F(\theta)}$

use x for S^1 & $\mathbb{R}$

Properties

- $F : \mathbb{R} \to \mathbb{R}$ is a homeo and str. monotone increasing
- $F(x+1) = F(x) + 1$
- for any two lifts F_0 & F_1 of φ

  $$F_1 - F_0 = \text{const} \in \mathbb{Z}$$
- $F^k = \underbrace{F \circ \dots \circ F}_{k}$ is a lift of φ^k

Set $\rho_x(F) := \lim_{k\to\infty} \frac{1}{k} F^k(x)$ can take $|k| \to \infty$

<u>Prop</u> • the limit exists

• $\rho_x(F)$ is ind. of x : $\rho(F)$

• $\rho(F_1) - \rho(F_0) \in \mathbb{Z}$
for any two lifts F_1 & F_0 of φ

well-defined

<u>Def</u> The rotation number of φ:
$$\rho(\varphi) = \left(\rho(F) \bmod 1\right) \in S^1$$

other ways to write $\rho_x(F)$:

set $a(x) = a_\varphi(x) :== F(x) - x$
$a : \mathbb{R} \to S^1 \to \mathbb{R}$ 1-periodic :

<u>Pf.</u> $a(x+1) = F(x+1) - (x+1) = F(x) - x$ ◁

$$F(x) = x + a(x)$$
$$F^k(x) =: x + a_k(x)$$?

$$F^2(x) = F(x + a(x)) = x + a(x) + a(x + a(x))$$
$$= x + \underbrace{a(x) + a(\varphi(x))}_{a_2(x)}$$

$$a_k(x) = a(x) + a(\varphi(x)) + \dots + a(\varphi^{k-1}(x))$$

$$\Rightarrow \rho_x(F) = \lim_{k\to\infty} \frac{F^k(x) - x}{k}$$ ← can always add

$$= \lim_{k\to\infty} \frac{a_k(x)}{k}$$

Ex. 1) $\varphi(x) = R_\alpha(x) = x+\alpha$ ← rotation in α

$$F(x) = x + \underbrace{\alpha}_{a(x)} + \text{an integer}$$

$$F^k(x) = x + \underbrace{k\alpha}_{a_k(x)}$$

$$\Rightarrow \rho(R_\alpha) = \alpha$$

2) $\varphi(x) = x + \alpha + \varepsilon \sin(2\pi x)$

Things get complicated

$\rho(\varphi)$ depends on α & ε !

Pf of the proposition

- **Independence of x**

F^k = a lift of φ^k : monotone $\nearrow$

$F^k(x+1) = F^k(x)+1$

$x \leq y \leq x+1$

$\Rightarrow \quad F^k(x) \leq F^k(y) \leq F^k(x)+1$

$$\frac{1}{k}\Big|\underbrace{F^k(x) - F^k(y)}_{\leq 1}\Big| \leq \frac{1}{k} \longrightarrow 0$$

when $x \leq y \leq x + n$, $n \in \mathbb{Z}$

$$\frac{1}{k}\left|F^k(x) - F^k(y)\right| \leq \frac{n}{k} \longrightarrow 0$$

- **Existence**

Lemma Assume a_k an arb seq s.t. $a_{n+m} \leq a_n + a_m + L$

$\Rightarrow \lim\limits_{k\to\infty} \frac{a_k}{k}$ exists in $\mathbb{R} \cup \{-\infty\}$

e.g. $L = 0$

$a_{n+m} \leq a_n + a_m$

subadditive

Pf of the Lemma set $a := \liminf_{k \to \infty} \frac{a_k}{k}$ Assume $a > -\infty$

$a = -\infty$: ex

Take n so large that

$$\frac{a_n}{n} \le a + \varepsilon/3 \quad \text{and} \quad \frac{L}{n} \le \frac{\varepsilon}{3}$$

Note
$$a_{2n} \le 2a_n + L$$
$$a_{3n} \le 3a_n + 2L$$
$$\dots$$
$$a_{\ell n} \le \ell a_n + (\ell - 1)L$$

write $k = n \cdot \ell + r$, $0 \le r \le n-1$

$$\frac{a_k}{k} = \frac{a_{n\ell + r}}{k} \le \frac{a_{n\ell} + a_r + L}{k}$$

$$\le \frac{a_{n\ell}}{k} + \frac{a_r + L}{k}$$

bounded

$\to 0 \quad k \to \infty$

$$\le \frac{\ell a_n + (\ell-1)L}{\ell n + r} + \frac{a_r + L}{k}$$

$$\le \frac{\ell a_n}{\ell n + r} + \frac{(\ell-1)L}{\ell n + r} + \frac{a_r + L}{k}$$

$$\le \underbrace{\frac{a_n}{n}}_{< a + \varepsilon/3} + \underbrace{\frac{L}{n}}_{< \varepsilon/3} + \underbrace{\frac{a_r + L}{k}}_{< \varepsilon/3} \le a + \varepsilon$$

$k \to \infty$

$$\Rightarrow a < \frac{a_k}{k} < a + \varepsilon \Rightarrow \lim \text{ exists}$$

Back to the pf.

Recall $a_k(x) = F^k(x) - x =: a_k$

<u>Claim</u> $a_{n+m} \le a_n + a_m + 1$

<u>Pf</u> $a_{n+m} = F^{n+m}(x) - x$

$$= \underbrace{F^n(F^m(x))}_{F^n(y)} - \underbrace{F^m(x)}_{y} + \underbrace{F^m(x) - x}_{a_m}$$

$$x + k \le y \le x + k + 1$$

$$\left.\begin{array}{l} F^n \nearrow \\ F^n(z+1) = F^n(z) + 1 \end{array}\right\} \Rightarrow$$

$$\underbrace{F^n(x+k)}_{\substack{\parallel \\ F^n(x)+k}} \le F^n(\ y\ ) \le \underbrace{F^n(x+k+1)}_{F^n(x)+k+1}$$

$$F^n(\ y\ ) - \ y \ \le \underbrace{F^n(x) + k + 1 - x + k}_{F^n(x) - x + 1 = a_n + 1}$$

$\lhd$

Lemma + Claim $\Rightarrow$ $\lim \in [-\infty, \infty)$ exists

<u>Ex</u>: $\frac{a_n}{n}$ bounded from below

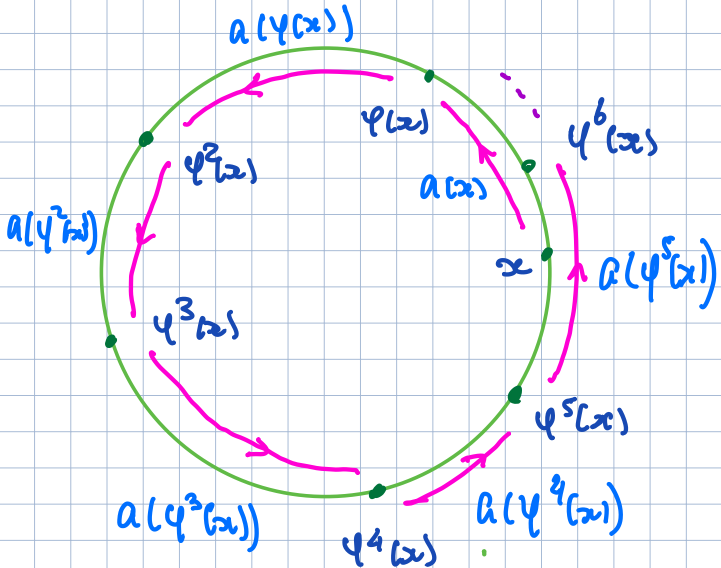
$\Rightarrow \lim \in \mathbb{R}$

$\lhd$

Lecture 11

02/08-2022

To summarize:



$\varphi: S^1 \to S^1$

$$\rho(\varphi) = \lim_{k\to\infty} \frac{1}{k}\left(a(x) + a(\varphi(x)) + \dots + a(\varphi^{k-1}(x))\right)$$

exists and ind of x

Prop - Properties

(a) ρ is conjugation invariant

$$\rho(h\varphi h^{-1}) = \rho(\varphi)$$

(b) $\rho(\varphi) = \frac{p}{q} \in \mathbb{Q} \iff \varphi$ has a q-periodic orbit

($\Rightarrow$ all periodic orbits of φ has the same minimal period q)

(c) $\rho: H \to S^1$ is continuous with respect to the sup-norm

Pf

a) F = a lift of φ

$H = \text{—} \cdot \text{—} \cdot\cdot - h \quad : \quad H(0) \in [0,1)$

$\Rightarrow \quad H^{-1} = \text{—} \cdot \text{—} \cdot - h^{-1} \quad H^{-1}(0) \in [0,1)$

$\Rightarrow \quad HFH^{-1} = \text{—} \cdot \text{—} \cdot\cdot h\varphi h^{-1}$

$0 \le H(1) = H(0) + 1 \le 2$

$$\Rightarrow \quad \underbrace{|H(x) - x|}_{\text{periodic}} \le 2 \quad \forall x \in [0,1) \quad (1)$$

$\Rightarrow \forall x \in \mathbb{R}$

$\Rightarrow \quad |H^{-1}(x) - x| \le 2 \quad$ (similar) $\quad (2)$

similarly

$|y - x| < 2 \Rightarrow |F^n(y) - F^n(x)| \le 3 \quad (3)$

$$|HF^nH^{-1}(x) - F^n(x)|$$
$$\le \underbrace{|H(F^nH^{-1}(x)) - F^nH^{-1}(x)|}_{\le 2 \text{ by } (1)} + \underbrace{|F^n\underbrace{H^{-1}(x)}_{|y-x|\le 2} - F^n(x)|}_{|F^n(y) - F^n(x)| \le 3 \text{ by } (3)}$$
$$\le 2 + 3 = 5$$

$$\Rightarrow \frac{1}{n}|HF^nH^{-1}(x) - F^n(x)| \le \frac{5}{n} \to 0$$

$$\Rightarrow \quad \rho_x(HFH^{-1}) = \rho_x(F) \Rightarrow \rho(h\varphi h^{-1}) = \rho(\varphi)$$

b)

$\Leftarrow$) $x_0, \varphi(x_0), \dots, \varphi^{q-1}(x_0), \varphi^q(x_0) = x_0$

$\Rightarrow$ $F^q(x_0) = x_0 + p$

$F^{kq}(x_0) = x_0 + \underbrace{kp}_{a_{kq}}$

$\Rightarrow \rho(\varphi) = \lim_{k\to\infty} \frac{a_{kq}}{kq} = \frac{p}{q}$

$\Rightarrow$ [KH] or [Arnold] or Ex

Hint: • Reduce to φ^q has a fixed pt where $\rho(\varphi) = \frac{r}{q}$: Then $\rho(\varphi) = 0$

• F = lift with $F(0) \in [0,1)$

$0 < a(x) = F(x) - x < 1$

$\Rightarrow \delta < a(x) < 1-\delta \qquad \delta > 0 \ \forall x$

• $\Rightarrow \delta \leq \frac{F^n(0)}{n} \leq 1-\delta \Rightarrow \rho(\varphi) \neq 0$

c) [KH]

$\triangleleft$

Cor (of a) : $\rho(\varphi) \neq \rho(\psi) \Rightarrow \varphi$ & ψ are not conj

$\Rightarrow \underbrace{R_\alpha}_{\text{rot by } \alpha}$ is top conj $R_\beta \Leftrightarrow \alpha = \beta$ & $R_\alpha = R_\beta$

Applications

1. str stable diffeo's of S^1

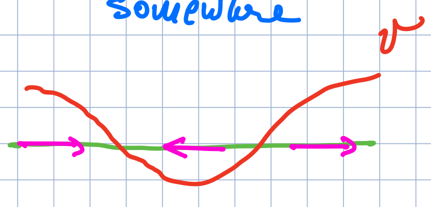
Q: Do we have str stable diffeos of S^1?

Ex 0. R_α is never str stable

$R_\beta \approx R_\alpha$ $\alpha \approx \beta$ but $R_\beta \not\sim R_\alpha$ $\alpha \neq \beta$

Ex 1. $v : \mathbb{R} \to \mathbb{R}$ 1-periodic

Ex $\underbrace{v(x) = 0}_{\text{somewhere}} \Rightarrow v'(x) \neq 0$



$x' = v(x)$ DE on $\mathbb{R}$ or S^1

$\forall t \neq 0$ $\varphi = \varphi^t : S^1 \to S^1$ is non-deg

all periodic orbits are non-deg

$Per = Fix = \{v = 0\}$

$\Rightarrow \rho(\varphi) = 0$

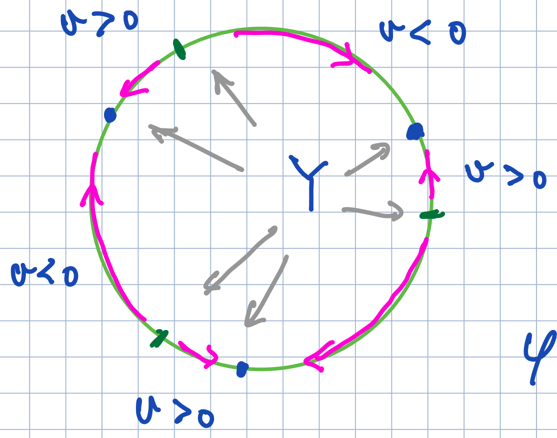
Prop

Pf of str stability is tedious but not difficult

Ex 2

Ex

- v = v.f. on S^1
- $Y = \{v = 0\}$ invariant under $R_{1/q}$
  $\neq \emptyset$



- $v' \neq 0$ at $Y \neq \emptyset$

φ = the flow of v in time $t > 0$

$$\psi = R_{p/q} \circ \varphi \qquad (p,q) = 1$$

$\Rightarrow$ • ψ is non-deg all periodic orbit has period q

- ψ is str stable $\leftarrow$ tedious but not difficult
- $\rho(\psi) = \dfrac{p}{q}$

Rmk 1) Can replace $R_{1/q}$ by any $\mathbb{Z}_q$-action on S^1 and v by an inv. one
$\Rightarrow$ diffeomorphic example

2) Roughly speaking every str stable diffeo of S^1 has this form:

Thm $\varphi : S^1 \xrightarrow{C^2} S^1$ is str stable

$\iff$ $\rho(\varphi) = p/q \in \mathbb{Q}$ and all periodic orbits of φ are non-deg

Rmk: then all periodic orbits have period q

Cor str stable $\varphi : S^1 \to S^1$, $C^{k \geq 2}$ form an open and dense set in C^k topology

A very rare phenomenon

Rmk • A bit counterintuitive: $\exists$ "more" φ with $\rho(\varphi) \in \mathbb{Q}$ than with $\rho(\varphi) \notin \mathbb{Q}$

• Thm $\Leftarrow$ Denjoy's thm – next tedious – [Arnold]

2. What about irrational ρ ?

Lecture 12
02/10 - 2022

Thm (Denjoy)

$\varphi: S^1 \xrightarrow{C^2} S^1$, $\rho(\varphi) \notin \mathbb{Q}$

$\Rightarrow \quad \varphi \sim R_\alpha$

Cor $\rho(\varphi) \notin \mathbb{Q}$, φ is C^2

$\Rightarrow$ φ is uniquely ergodic
(and hence $\underbrace{\text{minimal}}_{\text{every orbit is dense}}$)

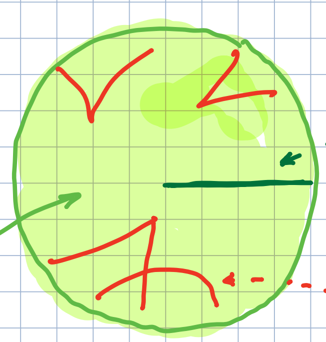
Rmk: the invariant measure is usually not the Lebesgue measure when

$$\varphi = h R_\alpha h^{-1}$$

it is $h^* \mu_{Lebesgue} \neq \mu_{Lebesgue}$ unless $\varphi = R_\alpha$

but still continuous

Overall Picture



Pf (Outline - steps)

hidden in the pf

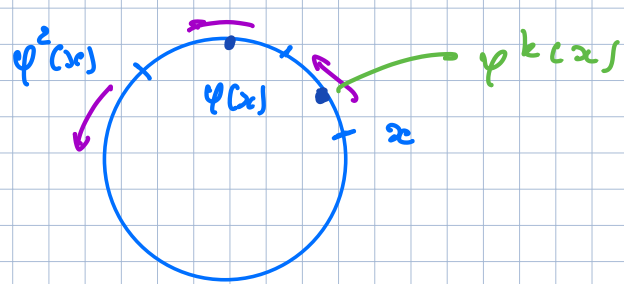
i) Pick $x \in S^1$; $\alpha = \rho(\varphi)$

~~Pf of existence of $\rho(\varphi)$~~

$\Longrightarrow$ $\forall\ n$ $\{\varphi^j(x) \mid 0 \le j \le n\} \subset S^1$

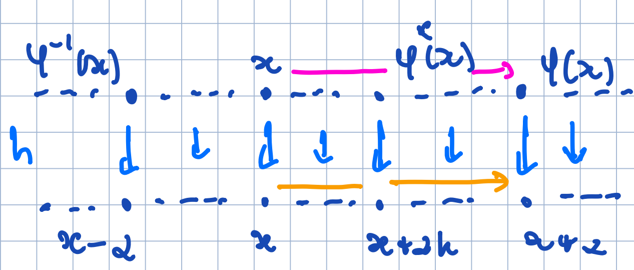
can be any finite range

has the same cyclic order as $\{R_\alpha^j(x)\} = \{x + j\alpha\}$.



$\Longrightarrow$ enough to show that an orbit ($\Longleftrightarrow$ every orbit) is dense

Then extend h by continuity



$R_\alpha h = h \varphi$

2) Assume ~~not~~ not. Pick $I \subset S^1$ s.t.

open interval

$\Theta(x) \cap I = \emptyset$

Easy

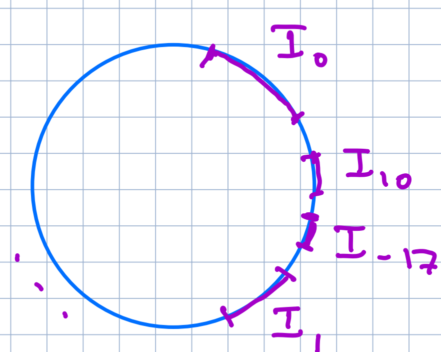
$\Rightarrow \quad \varphi^j(I),\ j \in \mathbb{Z}$ are mutually disjoint:

$$\varphi^i(I) \cap \varphi^j(I) = \emptyset \qquad i \neq j$$

set $I_j = \varphi^j(I)$

$$\Rightarrow \quad \sum_{j=-\infty}^{\infty} |I_j| < \infty$$

$$\Rightarrow \quad \boxed{|I_j| \longrightarrow 0 \qquad j \longrightarrow \pm\infty}$$



Note

$$\left.\begin{aligned} |I_1| &= \int_{I_0} \left|\frac{d\varphi}{dx}\right| dx \\ |I_2| &= \int_{I_1} \left|\frac{d\varphi}{dx}\right| dx \\ &\;\;\ldots \end{aligned}\right\}$$

connects $|I_j|$ and $\dfrac{d\varphi}{dx}$

3) ← This where the most effort goes

$\frac{d\varphi}{dx}^{\pm 1}$ bounded variation (or rather $\ln\left|\frac{d\varphi}{dx}^{\pm 1}\right|$)

$$\Rightarrow \sum_{j=-\infty}^{+\infty} |I_j| = \infty \qquad \rightarrow\leftarrow$$

◀

Rmk f bounded variation on $I = [0,1]$:

$\rightarrow$ partition $X: 0 = x_0 < x_1 < \dots < x_n = 1$

$$\rightarrow \text{var} = \sup_X \sum_j |f(x_j) - f(x_{j-1})| < \infty$$

when f is C^1

$$= \int_0^1 |f'|\,dx$$

- $C^1 \Rightarrow$ Lipshitz $\Rightarrow$ Bounded variation
- monotone $\Rightarrow$ Bounded variation $= |f(1) - f(0)|$

Q
- What happens when φ is not C^2
  Say only C^1 or C^0, $\rho(\varphi) \notin \mathbb{Q}$
- When can we have h $C^{k>0}$ in Denjoy theorem?

Focus on this question

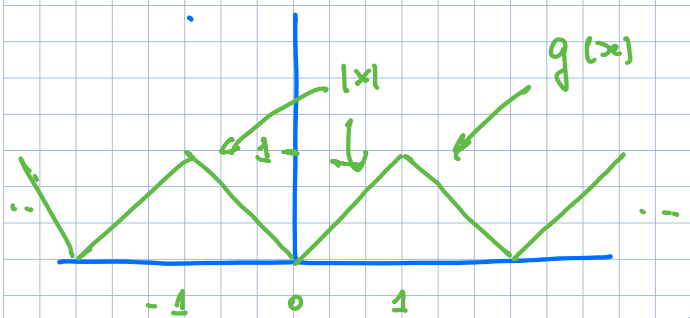
Digression: C^0 vs C^1

Set $C^1 = C^1([0,1])$, $C^0 = C^0([0,1])$

$$C^1 \subsetneq C^0$$

But how far from being differentiable a C^0-function can be

Construction:

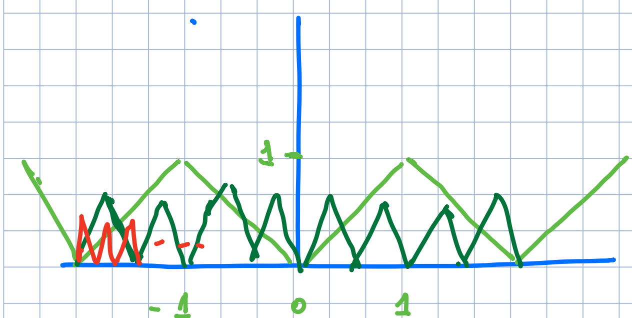


$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} g(2^{n-1}x)$$

$$f(x) = g(x) + \frac{1}{2} g(2x) + \frac{1}{4} g(4x) + \dots$$

converges uniformly

$\Rightarrow f \in C^0$



Ex Prove that f is nowhere monotone and nowhere differentiable

Ref: Gelbaum & Olmstead "Counterexamples in Analysis"

Rmk
- monotone $\Rightarrow$ almost everywhere differentiable (Lebesgue)
- But a function can be everywhere differentiable (but not C^1) but nowhere monotone (very hard)

A different approach:

Thm Nowhere differentiable functions form a second category set in C^0. Equivalently: functions diff at one pt form a meager (first category) set $\mathcal{D}$ in C^0.

countable union of nowhere dense closed sets

Ref: [Oxtoby]

Outline of the pf

Step 1

$$E_n = \{ f \in C^0 \mid |f(x+h) - f(x)| \le nh \ \exists x \ \forall h \}$$

$[0, 1-\frac{1}{n}]$, $\quad 0<h<1-x$

Clearly a function differentiable (on the right) at some pt x is in E_n for some n : $\bigcup E_n \supset \mathcal{D}$

functions differentiable at one pt

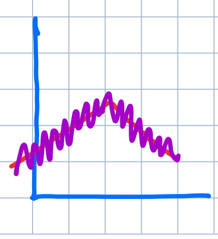
Ex Show that E_n is closed

Step 2

Ex Show that E_n is nowhere dense

Hint: • approximate $f \in C^0$ by p-wise linear functions $\hat{f}$

• approximate $\hat{f}$ by a sawtooth function.



Steps 1+2: $\mathcal{D} \subset \bigcup E_n$ is meager $\triangleleft$

Lecture 13

02/17-2022

State D. thm & questions

• **Denjoy's example**

Recall: $\varphi : S^1 \xrightarrow{C^2} S^1 \in \mathrm{Homeo}_+$

Alternative: • $\rho(\varphi) \in \mathbb{Q} \iff \varphi$ has a per. orbit

does not have to be C^2

• $\rho(\varphi) = \alpha \notin \mathbb{Q} \iff \varphi \sim R_\alpha$

$\Rightarrow$ all orbits are dense

Q: The role of C^2?

Thm (The Denjoy example)

$\exists$ a C^1-diffeo $\varphi : S^1 \to S^1$

with $\rho(\varphi) = \alpha \notin \mathbb{Q}$ (Hence no per. orbit)

and no dense orbits.

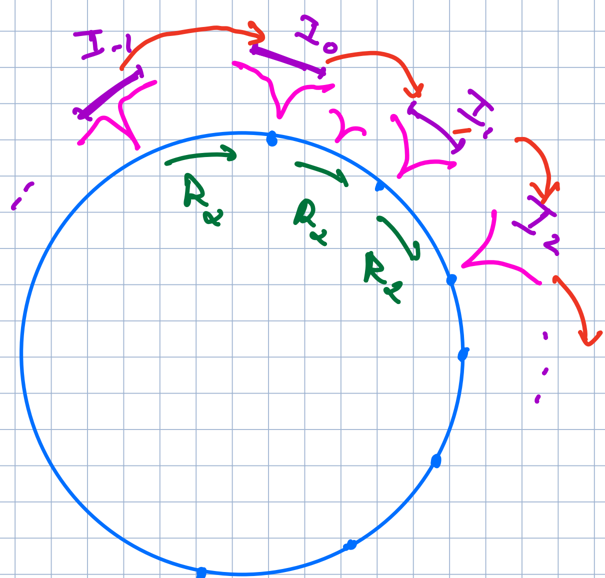
Rmk • Can make φ a $C^{1+\varepsilon}$ diffeo (Hölder)

but not a C^2-diffeo (Denjoy Thm)

• Can make φ C^∞-smooth but

not a C^∞-diffeo (φ^{-1} is not C^∞)

Idea of the construction (For C^0 φ)



- $x_n = R_\alpha^n(x)$
- $\ell_n > 0$

$\sum_{n \in \mathbb{Z}} \ell_n < \infty$

$|I_n| = \ell_n$

$\varphi_n : I_n \to I_{n+1}$

- Cut S^1 at each x_n and insert I_n. Get S^1 again. To be more precise, construct $\coprod I_n \hookrightarrow S^1$ so that

$\pi : S^1 \longrightarrow S^1$

$I_n \to x_n$ otherwise 1-1

- Define $\varphi : S^1 \to S^1$ by

$$\varphi(x) = \begin{cases} \varphi_n(x) & \text{if } x \in I_n \\ R_\alpha(x) & \text{if } x \notin \coprod I_n \end{cases}$$

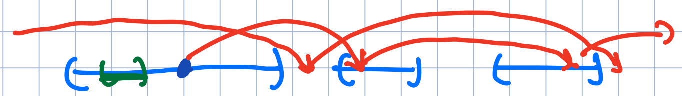
$\Rightarrow \varphi$ is a homeo

• φ has no periodic orbits

$\Rightarrow \rho(\varphi) \notin \mathbb{Q}$ HW: $\rho(\varphi) = \alpha$

φ has no dense orbits

if $x \in I_n$ $\varphi^n(x)$ never comes back to I_n



never hits this interval

if $x \notin \bigsqcup I_n$, $\varphi^k(x)$ never enters any I_n.

looks like a Cantor set

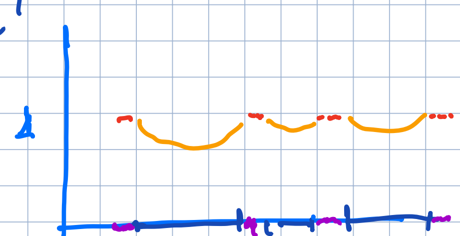
$K = S^1 \setminus U$

$U = \bigsqcup \operatorname{int}(I_n)$

$$\frac{d\varphi}{dx} = \begin{cases} 1 & \text{on } K \\ \frac{d\varphi_n}{dx_n} & \text{on } I_n \end{cases}$$

Rmk With just a bit more care of ℓ_n & φ_n (see [KH]) can make φ C^1

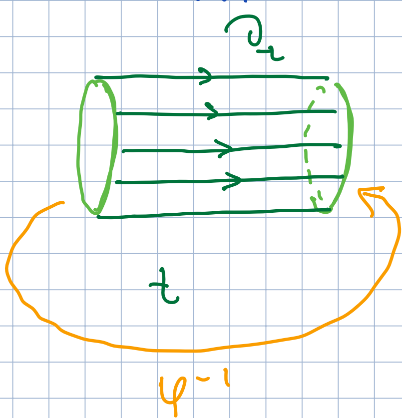
$\frac{d\varphi}{dx} \in C^0$



Application: v.f. on $\mathbb{T}^2$

Cor $\exists$ a C^1 v.f. on $\mathbb{T}^2$ without zeroes, or closed orbits or dense orbits.

Pf
- Take $\varphi : S^1 \xrightarrow{C^1} S^1$ as in Denjoy's thm
- Form $M = S^1 \times [0,1] \Big/ (x,0) \sim (\varphi^{-1}(x), 1)$



the mapping torus

Ex: M is C^1 surface and $M \underset{C^1}{\cong} \mathbb{T}^2$

- ∂_t ~> v.f. v on $M \cong \mathbb{T}^2$, C^1, $v \neq 0$
  φ = time-1 flow of v
- Ex: show that φ has no dense orbit or periodic orbits

$\lhd$

- Digression to number theory:

Diophantine vs Liouville numbers

$\alpha \notin \mathbb{Q}$ ← always

$\frac{p}{q} \in \mathbb{Q}$ relatively prime

what's the reference?

$\underline{Q}$: How fast can one approximate α by rational numbers

It turns out one should compare $|\alpha - \frac{p}{q}|$ with q!

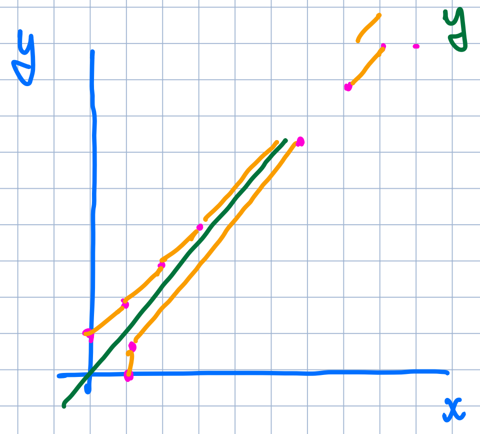
Thm $\forall \alpha \notin \mathbb{Q}$ $\exists$ $\frac{p_i}{q_i}$ inf many s.t.

$$\left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{q_i^2} \longrightarrow 0$$

Rmk Can do $|\alpha - p_i/q_i| < \frac{1}{\sqrt{5}\, q_i^2}$
but not much better – see below:
$\exists$ α and $c > 0$ s.t.

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^2} \quad \forall \frac{p}{q} \; !$$

"Pf" – visualization $\qquad \mathbb{Z}^2 \subset \mathbb{R}^2$



$y = \alpha x \leftarrow$ no integer pts other than $(0,0)$

- Convex hull of all $(p,q) \in \mathbb{Z}^2$ below or above $y = \alpha x$ in the first quadrant

- A thread attached at ∞, first along $y = \alpha x$, put a nail at $\mathbb{Z}^2$ and pull the thread down/up
- the vertices of the hulls give the required sequence p_i/q_i

Details: [Arnold]
– continued fractions $\qquad \triangleleft$

Direct pf in the spirit of the Kronecker thm:

Pf We'll prove:

$$\forall\, n \in \mathbb{N} \;\; \exists \;\; 1 \le q \le n \;\;\&\;\; p :$$

$$\left|\alpha - \frac{p}{q}\right| \le \frac{1}{nq} \le \frac{1}{q^2} \;\Rightarrow\; \text{thm}$$

- Partition $[0,1]$ into n intervals of length $1/n$

- Look at the $n+1$ pts

$$\alpha,\ 2\alpha,\ \dots,\ (n+1)\alpha \mod 1$$

At least two are in the same interval:

$$\exists\ 1 \le l < k \le n+1 \quad \text{s.t.}$$

$$\underbrace{|k\alpha - l\alpha|}_{\text{mod } 1} < 1/n$$

$$\Leftrightarrow \quad \exists\, p \quad |k\alpha - l\alpha - p| < 1/n$$

- Set $q = k - l$ then

$$|q\alpha - p| < 1/n$$

$$\Rightarrow \quad \left|\alpha - \frac{p}{q}\right| < \frac{1}{nq} \qquad \triangleleft$$

Lecture 14

02/22 - 2022

→ Denjoy ex ⇝ v.f. on $\mathbb{T}^2$

→ Recall the thm from Lect. 13:

Thm $\forall \alpha \notin \mathbb{Q}$ $\exists$ $\frac{p_i}{q_i}$ inf many s.t.

$$\left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{q_i^2} \longrightarrow 0$$

Q Can we do better than that?

$$|\alpha - p/q| < \varepsilon/q^2 \quad \text{or}$$

$$|\alpha - p/q| < C/q^{2+\beta} \quad ?$$

It turns out: not in general and for very few α's

Def $\alpha \notin \mathbb{Q}$ is **Diophantine** if

$\exists\, \beta \geq 0$ s.t.

$$\underbrace{\exists\, C > 0 \text{ with } \left|\alpha - \frac{p}{q}\right| \geq \frac{C}{q^{2+\beta}} \quad \forall\, \frac{p}{q} \in \mathbb{Q}}$$

Denote the set of such α's by $\mathcal{D}_\beta$

$\mathcal{D} = \bigcup_{\beta \geq 0} \mathcal{D}_\beta$, $\quad \mathcal{D}_{\beta_1} \subset \mathcal{D}_{\beta_2} \Leftarrow \beta_1 \leq \beta_2$

Def $\alpha \notin \mathbb{Q}$ is Liouville if it is <u>not</u> Diophantine:

$\forall\, \beta \geq 0$ ~~$\forall\, C > 0$~~ $\exists\, p/q \in \mathbb{Q}$ s.t.

$$\left|\alpha - \frac{p}{q}\right| < \frac{\cancel{C}\,1}{q^{2+\beta}}$$ can replace C by 1 by playing with β

Notation: $\mathcal{L} = \mathbb{R} \setminus \mathcal{D}$

In other words: sending $\beta \to \infty$ we obtain a rational approximation of α converging to α faster than any power of $1/q$!

Thm (Ex) • $\forall \beta > 0$, $\mathcal{D}_\beta$ has a full measure

$\Rightarrow$ $\mathcal{D} = \bigcup_{\beta \geq 0} \mathcal{D}_\beta$ has a full measure

$\approx$ $\Updownarrow$

• $\mathcal{D}$ is meager: countable union of closed nowhere dense sets

Thm $\mathcal{L}$ is zero measure and second cat !! (small & large !!)

Pf – [Oxtoby]

Ex. $\alpha = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}$ is Liouville

Thm Every $\alpha \in \mathcal{L}$ is transcendental

$\Downarrow$

Cor $\alpha = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}$ is transcendental

Probably the simplest expl. transcendental number construction

Pf

α algebraic of deg n:

$f(\alpha) = 0$: f = pol of deg n min with coeff in $\mathbb{Z}$, no roots in $\mathbb{Q}$

Thm

$\Uparrow$

Lemma $\alpha \in \mathcal{D}_{n-2}$ $\leftarrow$ can be improved

Pf Set

$$M = \left\lceil \max_{|\alpha - x| \le 1} |f'(x)| \right\rceil \in \mathbb{N}$$

Claim $\forall\ p/q$

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{Mq^n} \quad ; \quad C = \frac{1}{M} < 1$$

$\forall\ x$ with $|x-\alpha| \leq 1$:

$$|f(x) - \underbrace{\cancel{f(\alpha)}}_{=0}| \leq M \cdot |x-\alpha| \leq M$$

Take $p/q = x$ can assume $\left|\frac{p}{q} - \alpha\right| < 1$

$\left(\text{otherwise } \left|\frac{p}{q} - \alpha\right| \geq 1 \geq \frac{1}{Mq^n}\right)$

Then

$$\left|f\left(\frac{p}{q}\right)\right| \leq M \cdot \left|\frac{p}{q} - \alpha\right|$$

$$\Rightarrow \underbrace{\left|q^n f\left(\frac{p}{q}\right)\right|}_{\in \mathbb{N}} \leq q^n \cdot M \cdot \left|\frac{p}{q} - \alpha\right|$$

$$\Rightarrow \geq 1$$

$$\Rightarrow 1 \leq q^n M \left|\frac{p}{q} - \alpha\right|$$

$$\Rightarrow \left|\frac{p}{q} - \alpha\right| \geq \frac{1}{Mq^n}$$

$\triangleleft$

Herman's theorem and small denominators

– Back to diffeo's of S^1

Recall:

Thm (Denjoy)

$\varphi : S^1 \xrightarrow{C^2} S^1$, $\rho(\varphi) \notin \mathbb{Q}$ only C^0

$\Rightarrow \varphi \sim R_\alpha$: $\exists\, h$: $\varphi = h R_\alpha h^{-1}$

Q Can we improve Denjoy's thm in a different way: make h smooth?

Not in general, but ...

Thm (M. Herman, 1979, Yoccoz 1984)

Assume $\varphi : S^1 \to S^1$ is a C^∞-diffeo and $\alpha = \rho(\varphi) \in \mathcal{D}$

$\Rightarrow \varphi = h R_\alpha h^{-1}$ with $h : S^1 \to S^1$ a C^∞-diffeo!

Rmk: builds up on work of Arnol & Moser

Deep & difficult, much more precise result

What goes wrong when $\alpha \in \mathcal{L}$?

– Small Denominators

Idea: need to solve the equation

$$f \circ R_\alpha - f = g \quad \leftarrow \text{given} \qquad g \in C^\infty$$

unknown

necessary: $\int g(x)\,dx = 0$

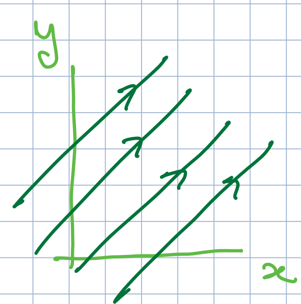
Want f to to exist and be sufficiently smooth.

Ex. • Show that a sol $f \in C^\infty$ exists $\forall g \in C^\infty$ when $\alpha \in \mathcal{D}$

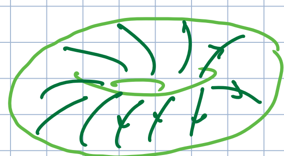
• But if $\alpha \in \mathcal{L}$ a sol might or might not exist depending on α

Example $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (x, y)

$$v = \alpha \partial_x + \partial_y$$



minimal



all integral curves are dense

When does the equation

unknown, given, C^∞

$$L_v f = g \; : \; \alpha \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = g$$

have a sol? How smooth

Ex $\int g \, dx\,dy = 0 \leftarrow$ necessary condition

Recall the set-up

Lecture 15

02/24-2022

Prop • Assume that $\alpha \in \mathcal{D}$. Then $f \in C^\infty$ exists $\forall g$ with $\iint g = 0$

• If $\alpha \in \mathcal{L}$ a sol might or not exist, or fail to be smooth

Pf - method (small denominators)

$g \in C^\infty(\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2)$

Fourier series

$$g = \sum_{p,q} g_{p,q}\, e^{2\pi i(qx+py)}$$

$$f = \sum_{p,q} f_{p,q}\, e^{2\pi i(qx+py)}$$

$$L_v f = -\alpha \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

$$= \sum_{p,q} \underbrace{2\pi i(-q\alpha + p) f_{p,q}}_{g_{p,q}}\, e^{2\pi i(qx+py)}$$

$\Rightarrow g_{0,0} = \iint g$ must be 0

$(p,q) \neq (0,0)$

$$\boxed{f_{p,q} = \frac{1}{2\pi i} \frac{g_{p,q}}{p - q\alpha} = \frac{1}{2\pi i\, q}\left(-\alpha + \frac{p}{q}\right)^{-1} g_{p,q}}$$

"small denominator"

Analysis fact:

$$u = \sum u_{p,q} e^{2\pi i (qx+py)} \in C^\infty$$

$\Leftrightarrow$ $|u_{p,q}| \underset{(p,q)\to\infty}{\longrightarrow} 0$ faster than any pol:

$\forall$ $a, b \in \mathbb{N}$

$$|u_{p,q}| \cdot (|p|^a + |q|^b) \longrightarrow 0$$

- $g \in C^\infty \Rightarrow |g_{p,q}| \longrightarrow 0$ faster than any pol

- $\alpha \in \mathcal{D}$ $\quad \left|-\alpha + \frac{p}{q}\right| > \frac{C}{q^{2+\beta}}$ for some $C>0$ and β.

$$|f_{p,q}| = \frac{1}{2\pi} |g_{p,q}| \cdot C \cdot q^{3+\beta}$$

also $\to 0$ faster than any pol

$\Rightarrow f \in C^\infty$

- But if $\alpha \in \mathcal{L}$ and we are unlucky with g we can have

$|f_{p,q}| \to \infty$ or $\to 0$ but slowly

A sol may fail to exist.

$\lhd$

§4 Local Analysis of dynamical systems - Very Briefly

Two classes of questions:

→ Asymptotic & Lyapunov stability
- of applied interest

→ Local normal forms, linearization, "classification", etc
- hugely important, somewhat similar to $\mathrm{Diff}(S^1)$

→ Two types of DS:
- discrete - maps or germs
- continuous - flows ← focus on this

similar

a bit more intuitive

Asymtotic & Lyapunov stability

Setting: M = smooth manifold

v = smooth v.f.

$v(p) = 0$, $p \in M$

will relax → $\{$ Assume $\varphi^t(x)$ is defined for all $t \in \mathbb{R}$ or at least $t \geq 0$

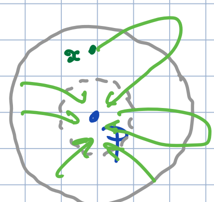
Def v (or p) is asymptotically stable if

- $\forall\, x$ close to p $\quad \varphi^t(x) \xrightarrow[t\to\infty]{} p$

Ex. (unif conv) $\Updownarrow$

- $\exists$ nbd $U \ni p$ s.t. $\forall\, V$ = nbd of p $\exists\, T > 0$ s.t. $\varphi^{t \geq T}(U) \subset V$

if φ^t is not defined for all $t \geq 0$ require it to be



Integral curves starting close to p converge to p

$\Rightarrow$ Can assume $M = \mathbb{R}^n$, $p = 0$

v is defined on some nbd of $p = 0$.

Rmk • v cannot be vol preserving

$\mathrm{vol}(\varphi^t(U)) = \mathrm{vol}(U) < \mathrm{vol}(V)$

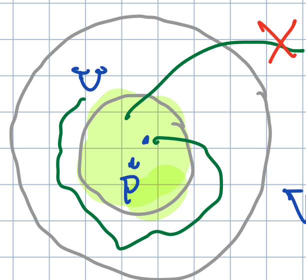
$\Rightarrow$ v cannot be Hamiltonian or symplectic

Def p is Lyapunov stable if

- $\forall\, V =$ nbd of p $\exists\, U =$ nbd of p

s.t. $\underbrace{\forall\, x \in U \quad \varphi^t(x) \in V \quad \forall\, t \geq 0}_{\varphi^t(U) \subset V}$

Trajectories starting close to p remain close to p



$V =$ orb small

Rephrasing for $p = 0 \in \mathbb{R}^n$, v defined near $p=0$

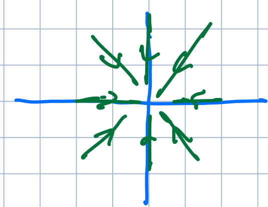
- v is asymptotically stable if
  $\exists\, r > 0$ s.t. $\|x\| < r$
  $\Rightarrow$ $\varphi^t(x)$ is defined for all $t \geq 0$ and
  $\varphi^t(x) \longrightarrow 0$ $\underbrace{\text{(uniformly in } x)}_{\text{automatic}}$

- v is Lyapunov stable if
  $\forall\, \varepsilon > 0$ $\exists\, \delta > 0$ s.t.
  $\forall \|x\| < \delta$ $\varphi^t(x)$ is defined for all $t \geq 0$
  and $\|\varphi^t(x)\| < \varepsilon$

A.S. $\Rightarrow$ L.S.

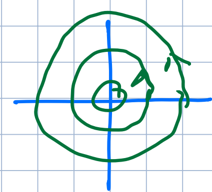
Examples for $\mathbb{R}^2$

- $\dot{x} = -x$
  $\dot{y} = -y$



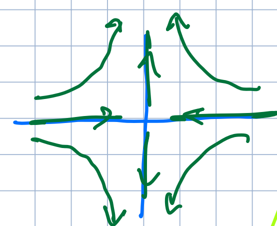
Asympt. stable
change signs ⇒ "unstable"

- $\dot{x} = -y$
  $\dot{y} = x$



L.S. but not A.S.

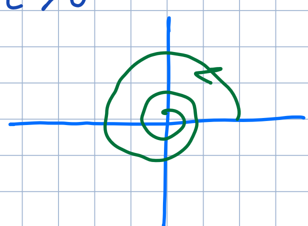
- $\dot{x} = -x$
  $\dot{y} = y$



Neither L.S. (nor A.S.)
saddle

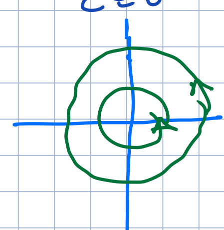
- $$\left.\begin{aligned} \dot{x} &= -y - \varepsilon x \\ \dot{y} &= x - \varepsilon y \end{aligned}\right\}$$ linear comb of the first two

$\varepsilon > 0$



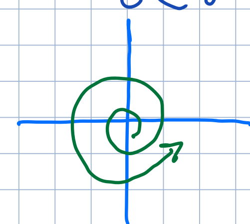
A.S.

$\varepsilon = 0$



L.S. but not A.S.

$\varepsilon < 0$



"unstable"

- etc

Interpretation:

- p: $v(p)=0$ is an equilibrium
- only "stable" equilibrian can be observed in practice

Ex.

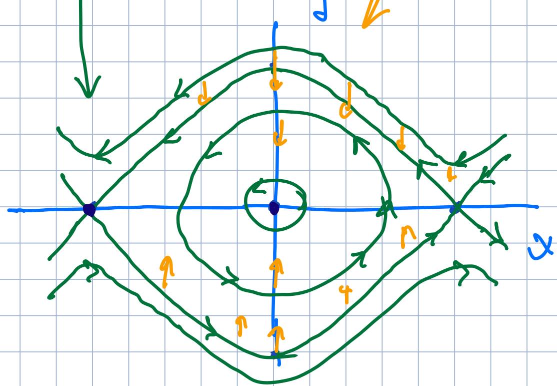
$$\dot{x} = y$$
$$\dot{y} = -\sin x - \overset{\geq 0}{\varepsilon y}$$

on $\mathbb{R}^2$ or $\mathbb{R}_y \times S^1_x = TS^1$

pendulum

drag

loosing energy

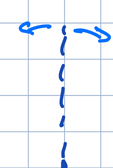


$(0,0)$

A.S. $\varepsilon > 0$

L.S $\varepsilon = 0$

$(\pi,0)$



"unstable"

not A.S. or L.S.

Rmk $\varepsilon < 0$: pumping in energy

$(\pi,0)$ neither
$(0,0)$ "unstable"
} neither is observe

stability criteria:

- **Lyapunov functions**

Setting:

- v defined on a nbd of $0 \in \mathbb{R}^2$, C^∞
- f: (nbd of 0) $\xrightarrow[C^\infty]{} \mathbb{R}$, $f(0)=0$

Def. f is a Lyapunov function for v if

- f has an isolated min at 0
- $L_v f(x \neq 0) < 0$ (or $L_v f \leq 0$)

Thm Assume that v has a Lyapunov function.

- $L_v f \leq 0 \Rightarrow 0$ is L.S.
- $L_v f < 0 \Rightarrow 0$ is A.S.

Pf

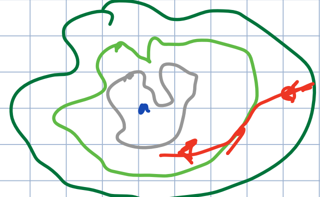
- $f(\varphi^t(x))$ decreasing (str: $L_v f < 0$)

$\Rightarrow \underbrace{\{f \leq \varepsilon\}}_{\text{compact}}$ & $\underbrace{\{f < \varepsilon\}}_{\text{open}}$ are invariant

$\Rightarrow \varphi^t(x)$ is defined for all $t \geq 0$

- $\{f < \varepsilon\} \leftarrow$ arb small nbds of 0

$\Rightarrow$ L.S.



• Remains to prove

$L_v f < 0 \Rightarrow$ A.S.
away from 0

$x \in$ small nbd of 0
suffices to show:

$$\inf_{t \geq 0} f(\varphi^t(x)) = 0$$

Then $\varphi^t(x) \longrightarrow 0$

Assume not: $a = \inf_{t \geq 0} f(\varphi^t(x)) > 0$

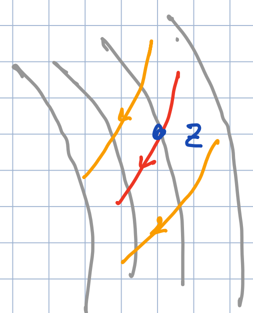
Take z = a limit pt of $\varphi^t(x)$ as $t \to \infty$

$z \in \omega(x)$,

Then $f(z) = a > 0 \Rightarrow z \neq 0$

$L_v f(z) = b > 0$

Important argument



Claim $\exists\ \varepsilon > 0$ and $\tau > 0$
s.t. $f(\varphi^\tau(y)) \leq a - \varepsilon$
$\forall\ y$ near z

Pf True a z. By continuity true near z
with smaller ε & τ

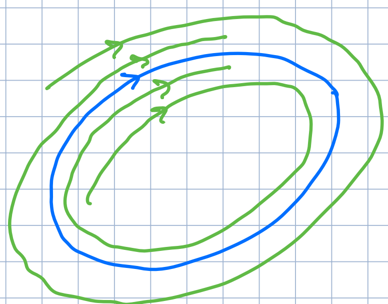
Take $y = \varphi^t(x)$ near z. Then

$$f(\varphi^\tau(y)) = f(\varphi^{t+\tau}(x)) \leq a - \varepsilon$$

$\longrightarrow \leftarrow$ with $a = \inf$ $\triangleleft$

<u>Rmk</u>. $\exists$ a similar criterion for diffeos ...

- Generalizes to integral curves other than equilibria: e.g. periodic orbits.



Problem with Lyapunov functions: difficult to find

<u>Rmk</u> $\exists$ L. function with $L_v f$ $\iff$ A.S.

Thm: essentially a necessary and sufficient condition

- Asymptotic stability via linearization

Lecture 23
03/01-2022

$v(x) = A(x) + \underbrace{R(x)}_{} = A(x) + \underbrace{\ldots}_{}$ higher order terms

Principle: the flow of A "approximates" the flow of v

$\varphi_A^t(x) = e^{At}x$

Goal: stability criterion via A

Thm $\operatorname{Re}\lambda < 0$ for all eigenvalues of A
$\Rightarrow$ 0 is A.S.

Rmk
- sufficient but not necessary
  e.g. $v = -x^2\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y}$
- much more involved for L.S.

Ex. (Ex) $v(x) = Ax$ ← linear v.f.

$\mathrm{Re}\,\lambda < 0 \Rightarrow e^{At}x \xrightarrow[t\to\infty]{} 0$

obvious when A is diagonalizable

Pf of the theorem

- For the sake of simplicity assume that all $\lambda \in \mathbb{R}$
- Goal: find a Lyapunov function
- $\exists$ linear change of variables $\Rightarrow$

$A \rightsquigarrow BAB^{-1}$

$$A = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \text{with } |*| < \varepsilon$$

Ex.

arb small

- In other words

$$A = \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}_{\Lambda} + \underbrace{\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}}_{P}$$

$= \Lambda + P, \quad \lambda_i < 0$

with $\|P\| < \varepsilon$

- Set $f(x) = \langle x, x \rangle = \sum x_i^2$

Note $L_{Kx} f = \langle (K + K^T)x, x \rangle$
for any matrix K

• $$L_v f = 2\langle \Lambda x, x\rangle + \langle (P+P^T)x, x\rangle + L_{R(x)} f$$

$\rightarrow$ $$\langle \Lambda x, x\rangle = \sum \lambda_i |x_i|^2$$
$$\leq -\eta \cdot \frac{1}{2} \sum |z_i|^2 = -\eta \cdot f$$

where $\eta = 2 \min |\lambda_i|$

$\rightarrow$ $$\langle (P+P^T)x, x\rangle \leq 2\|P\| \cdot \langle x, x\rangle = 2\|P\| \cdot f$$
$$\leq 2\varepsilon \cdot f$$

arb. small

$\rightarrow$ R involves quadratic and higher order terms $R = (R_1, \dots, R_n)$

$$\frac{\|R(x)\|}{\|x\|} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow 0$$

$$L_{R(x)} f = 2 \underbrace{\sum R_i(x) x_i}_{\text{cubic or higher order}}$$

$\Rightarrow$ $$\frac{\|R(x)\|}{\|x\|^2} \longrightarrow 0 \qquad x \longrightarrow 0$$

$\Rightarrow$ $\forall \delta > 0$

$$\|L_{R(x)} f\| \leq 2\delta \cdot f(x) \quad \text{when} \quad x \approx 0$$

• $$L_v f \leq 2(-\eta + \varepsilon + \delta) f < 0 \qquad x \neq 0$$

when $\varepsilon + \delta < \lambda$

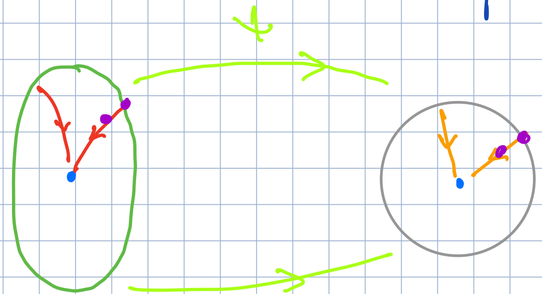
Cor In the setting of the thm
$\varphi^{t\geq 0}$ near 0 is top equivalent
to the flow $x \mapsto e^{-t}x$

$\Rightarrow$ $\exists\, h :$ nbd of $0 \to$ nbd of 0 s.t.
$h(\varphi^{t\geq 0}(x)) = e^{-t} h(x)$

Pf • We have constructed a Lyapunov function
which is a quadratic form:
$f = \langle Bx, Bx\rangle = \langle B^TBx, x\rangle > 0$
change of variables

$\Rightarrow$ $\{f = \varepsilon\}$ = ellipsoid $\cong S^{n-1}$

• Fix $\{f=\varepsilon\} \xrightarrow{\psi} S^{n-1}$ can take $\psi = B$
or the radial proj



Set $h(\varphi^t(x)) = e^{-t}\psi(x)$
Diffeo outside 0.
$\triangleleft$

Q: • Why h is not smooth? When?

Ex • Show h $C^1 \Rightarrow A = -I$

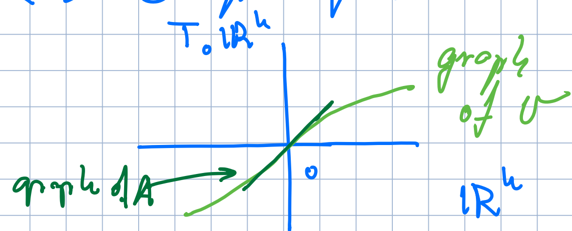
• Local normal forms and all that

Setting

• **Vector fields** $v(0)=0, \quad v(x) = Ax + \ldots$

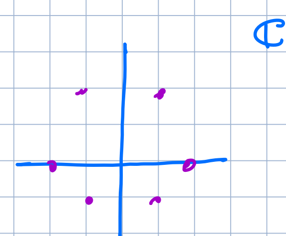
Def • 0 is a non-deg zero of v if
$\det(A) \neq 0$: no eigenvalues $\lambda = 0$

$\iff$ Graph of $v \pitchfork$ zero section at 0



Note
Graph of $A = Dv$
$= T_{(0,0)}$ graph of v

• 0 is hyperbolic if
$\mathrm{Re}\,\lambda \neq 0$



Note: hyperbolic $\overset{\not\Leftarrow}{\Rightarrow}$ non-deg

Ex $\mathrm{Re}\,\lambda < 0 \Rightarrow$ hyperbolic

• Diffeomorphisms

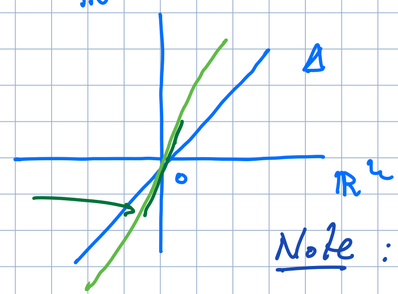
$$\varphi : \text{nbd of } 0 \longrightarrow \text{nbd of } 0$$

$$\varphi(0) = 0, \quad \varphi(x) = Px + \dots$$

$$P = D\varphi_0 : T_0\mathbb{R}^n \longrightarrow T_0\mathbb{R}^n$$

Def • 0 is a non-deg fixed pt of φ if $\det(I - P) \neq 0$: $\eta \neq 1$ ← eigenvalues

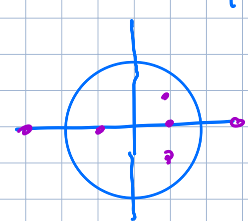
$\Longleftrightarrow$



graph$(\varphi) \pitchfork \Delta$ at 0

Note : graph$(P) = T_0\,$graph(φ)

• 0 is a hyperbolic fixed pt if P has no eigenvalues $|\eta| = 1$



Note: hyperbolic $\overset{\not\Leftarrow}{\Rightarrow}$ non-deg

Rmk : asymptotic stability $\Leftarrow \underbrace{|\eta| < 1 \ \forall \eta}$

hyperbolicity $\Leftarrow \not\Rightarrow$

Relation

Ex. $v(x) = Ax + \dots \qquad v(0) = 0$

$\varphi = \varphi^t(x) =$ flow of v in time t

$\varphi(x) = Px + \dots \qquad \varphi(0) = 0$

$\Rightarrow \quad P = e^{At} \quad : \quad \eta = e^{\lambda t}$

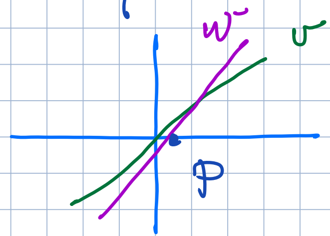
$\Rightarrow \quad v$ non-deg $\underset{t \text{ small}}{\Rightarrow}$ φ non-deg

$\not\Rightarrow$ in general : $\lambda = 2\pi i$

v hyperbolic $\Rightarrow$ $\underbrace{\varphi \text{ hyperbolic}}_{\substack{\Downarrow \\ \text{non-deg}}}$

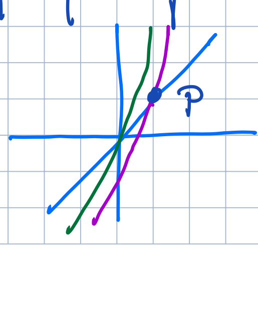
Rmk - Ex - Important

- v non-deg at 0, $\quad w \overset{C^1}{\approx} v$
  $\Rightarrow \quad w(p) = 0$ for some p close to 0



Hint: use IFT

- φ non-deg at 0, $\quad \psi \overset{C^1}{\approx} \varphi$
  $\Rightarrow \quad \psi(p) = p$ for some p close to 0



Hint: IFT

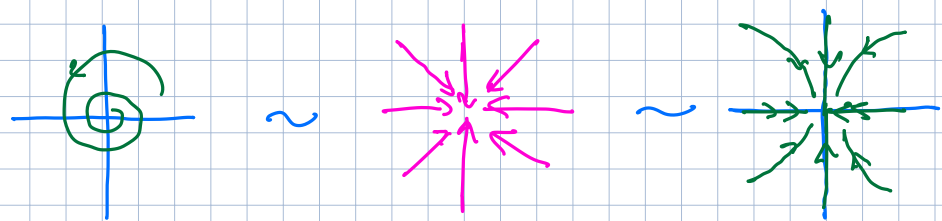
Lecture 17

03/03-2022

• Hartman - Grobman theorem

The top str of φ (or σ) near a hyperbolic pt is simple:

Ex: AS case



More generally

Thm (Hartman - Grobman)

$\varphi(0) = 0 \leftarrow$ hyperbolic :

$P = D\varphi : \mathbb{R}^n = T_0\mathbb{R}^n \to \mathbb{R}^n = T_0\mathbb{R}^n$

$\Rightarrow$ locally near 0

φ is top equiv to P

More precisely: $\exists\ h :$ nbd $0 \to$ nbd of 0

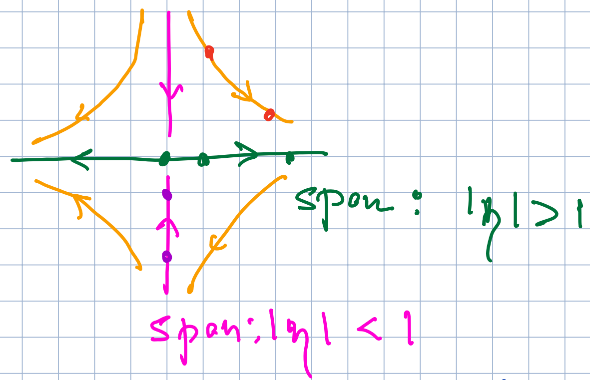
s.t. $\varphi(h(x)) = P h(x)$ as long as everything is defined

Rmk $P_1 \,\&\, P_2 : \mathbb{R}^n \to \mathbb{R}^n$ hyperbolic

$P_1 \,\&\, P_2$ are top conj

$\Leftrightarrow$ same # of eigenvalues $|\lambda| < 1$: n_-

———— . ———— . —— $|\lambda| > 1$ n_+



$$\begin{pmatrix} \varepsilon & & & & & 0 \\ & \ddots & & & & \\ & & \varepsilon & & & \\ & & & \varepsilon^{-1} & & \\ & & & & \ddots & \\ 0 & & & & & \varepsilon^{-1} \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} n_- \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} n_+ \end{array}$$

<u>Rmk</u> Similarly for vector fields (flows) but need to look at orbital equivalence: t is not preserved

<u>Pf</u> : KH, Arnold

<u>Rmk</u> The stable/unstable manifolds

$W^s(0)$: x : $\varphi^{k\to\infty}(x) \longrightarrow 0$

$W^u(0)$: x : $\varphi^{k\to-\infty}(x) \longrightarrow 0$

are actually smooth (Hadamard–Perron thm)

- **Move subtle aspects of local classification**

$v(0)=0, \quad v(x) = Ax + \dots$

$\varphi(0)=0, \quad \varphi(x) = Px + \dots$

Q **Linearization problem:**
Can we show that v is smoothly equivalent to A? Same for φ

New possibility: equivalence in formal power series: Discuss in detail

$h \leftarrow$ formal power series

$h_* v = A$

$h\varphi^t = e^{At} h$ as formal power series
or holomorphic

Sometimes: but often not

- $\exists$ obstruction even on the formal level. Resonances: linear relations

$$\lambda_j = \sum_{i \neq j} m_i \lambda_i \ , \qquad m_1 + \dots + m_n \geq 2$$

E.g. $\lambda_1 = 2\lambda_2$

$\lambda_1 = 2\lambda_1 + \lambda_2 \iff (\lambda_1 + \lambda_2 = 0)$

But not: $2\lambda_1 = 3\lambda_2$, $\lambda_1 = \lambda_1$ or $\lambda_1 = \lambda_2$

Cf. If in our $L_v f = g$ example $\alpha = \frac{p}{q} \in \mathbb{Q}$ we would have problem solving $f_{p,q} = \frac{1}{2\pi i} \frac{g_{p,q}}{-q\alpha + p}$

Similarly here

Thm (Poincaré)

Assume that A has no resonances

$\Rightarrow$ $v(x) = Ax + \dots$

is formally equivalent to A

Pf: Arnold

- Going formal $\rightsquigarrow C^\infty$ encounter further problems akin to small denominators

More details: Arnold

Pf of Poincaré's theorem

- Change of variables (smooth)

$y = h(x) = x + H(x)$ ← homogeneous pol of deg $r \geq 2$ (vector valued)

transforms $\dot{y} = Ay$ into the equation

$$\dot{x} = Ax + \underbrace{v_r(x)}_{\text{hom of deg } r} + \dots = h_*^{-1}(A)$$

- What's $v_r(x)$?

$h_*^{-1}(A) = \underbrace{A + v_r + \dots}_{\text{vector fields}}$

$$\dot{y} = Ay \rightsquigarrow \dot{x} + \frac{\partial H}{\partial x} \cdot \dot{x} = A(x + H(x))$$

matrix valued pol, $= I$ at $x = 0$

$$\left(I + \frac{\partial H}{\partial x}\right)\dot{x} = Ax + AH(x)$$

$$\left(I + \frac{\partial H}{\partial x}\right)^{-1} = I - \frac{\partial H}{\partial x} + \dots$$

Ex: $(I + B)^{-1} = I - B + \dots$ (B small)

$$\Rightarrow \quad \dot{x} = Ax + \underbrace{\left(-\frac{\partial H}{\partial x}Ax + AH(x)\right)}_{v_r(x)} + \dots$$

the Lie bracket of Ax & $H(x)$

- In other words:

$$h_*^{-1}(A\ ) = A \quad + v_r \qquad + \dots$$

- Consider V_r = hom vector valued pol of deg $r \geq 2$

$$V_r \subset \mathbb{R}[x_1, \dots, x_n] \otimes \mathbb{R}^n$$

$$L_A : V_r \longrightarrow V_r$$

$$H \longmapsto L_A H = -\frac{\partial H}{\partial x} A x + A H(x)$$

- Claim Assume A is non-resonant

$\Rightarrow L_A$ is invertible

Pf
- For the sake of simplicity assume A is linearizable
- change to the basis of eigenvectors

$e_1, \dots, e_n$

$\lambda_1, \dots, \lambda_n$

Basis in V_r: eigenvectors of L_A:

$$f_{k,j} = x_1^{k_1} \cdot \ldots \cdot x_n^{k_n} \otimes e_j \qquad \underbrace{k_1 + \dots + k_n}_{k} = r \geq 2$$

$$= \begin{pmatrix} 0 \\ \vdots \\ x_1^{k_1} \cdots x_n^{k_n} \\ \vdots \\ 0 \end{pmatrix}$$

Claim

$\Uparrow$

Subclaim:

$$L_A f_{k,j} = (\overbrace{-\sum k_i \lambda_i + \lambda_j}^{\neq 0 \ :\ \text{a resonance}}) f_{k,j}$$

$$\left(\frac{\partial f_{k,j}}{\partial x}\right)_{ij} = k_i\, x_1^{k_1} \dots x_i^{k_i-1} \dots x_n^{k_n}$$

assuming $k_i \geq 1$

and 0 otherwise

only one $\neq 0$ raw : jth

$$Ax = \lambda_1 x_1 e_1 + \dots + \lambda_n x_n e_n = \begin{pmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{pmatrix}$$

$$-\frac{\partial f_{k,j}}{\partial x} Ax = -\left(\sum k_i \lambda_i\right) f_{k,j}$$

$$A f_{k,j} = \lambda_j f_{k,j}$$

$$L_A f_{k,j} = \left(-\sum k_i \lambda_i + \lambda_j\right) f_{k,j}$$

- Finishing the proof: inductive process

$\to$ $\dot{x} = Ax + \sigma_2(x) + \dots$

find $h_2(x) = x + H_2(x) = y$ different

transforming $\dot{y} = Ay$ into

$A + \sigma_2 + \dots = h_{2*}^{-1}(A)$

$\Rightarrow$ h_2 will turn

$\dot{x} = Ax + \sigma_2(x) + \dots$ not the same as here

into $\dot{y} = Ay + \sigma_3(y)$

$\Rightarrow$ construct a sequence of smooth local diffeo's

$$h_2,\ h_3,\ h_4,\ \dots, \qquad h_i(x) = x + H_i(x)$$

deg $= i$

killing the rth order term and modifying $\geq r+1$th

$\Rightarrow$ The composition

$$\dots \circ h_4 \circ h_3 \circ h_2$$

is defined as a formal power series and sends v to A

$\lhd$

Rmk At each finite step the image is smooth

$h_r \circ \dots \circ h_2$ gives an equivalence of

$$v \quad \cdot\& \quad A \ + \text{terms of order} \geq r+1$$

§5 Introduction to hyperbolicity

Lecture 18
03/08-2022

- The horseshoe

Recall : Bernoulli shift

- $K = \mathbb{Z}_2^{\mathbb{Z}} = \{$ bi-inf sequences $a_i \in \mathbb{Z}_2 \}$
with product top/metric $\qquad = \{0,1\}$

 Ex : $K \cong$ Cantor set

- $\sigma : K \longrightarrow K$ = shift to the left

$$(\sigma \vec{a})_i = a_{i+1}$$

$$\ldots \; a_{-2} \; a_{-1} \; a_0 \; a_1 \; a_2 \; a_3 \; \ldots$$

Properties :
- periodic points are dense
- $\exists$ dense orbits (residual set)
- ergodic

Rmk : Different from other examples which are manifolds

Key fact: Smale's horseshoe

M = a surface (e.g. $\mathbb{R}^2$ or S^2)

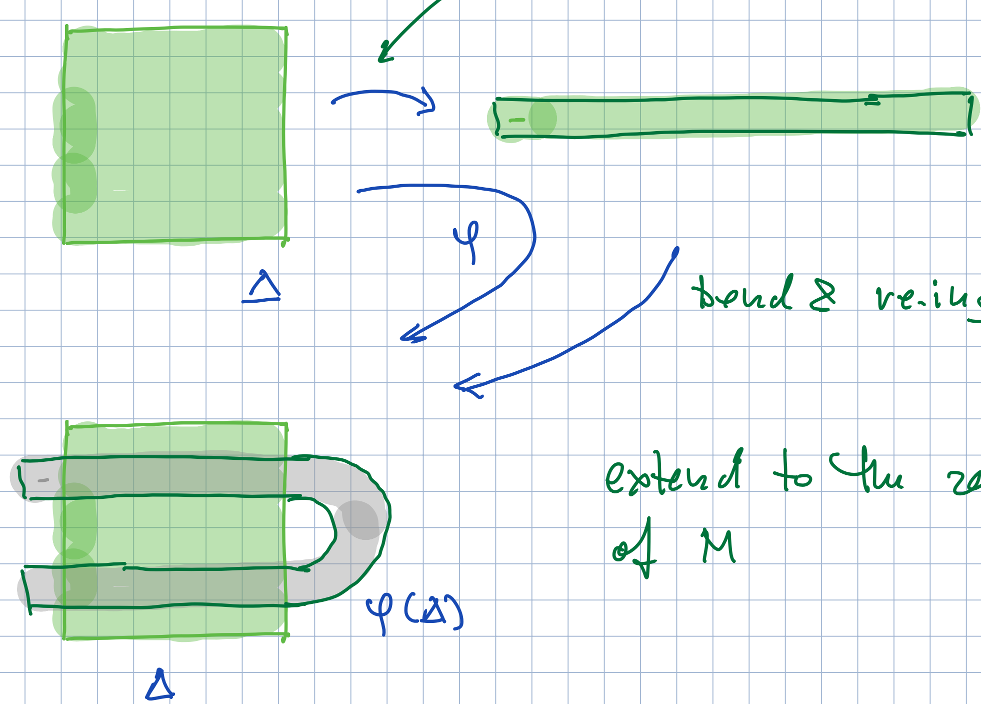
$\exists\ \varphi: M \to M$ compactly supported

and $j: K \hookrightarrow M$ s.t. $\varphi|_K = \sigma$

$$\begin{tikzcd} M \arrow[r, "\varphi"] & M \\ K \arrow[u, hook] \arrow[r, "\sigma"] & K \arrow[u, hook] \end{tikzcd}$$

Construction

stretch and squeize



bend & re-insert

extend to the rest of M

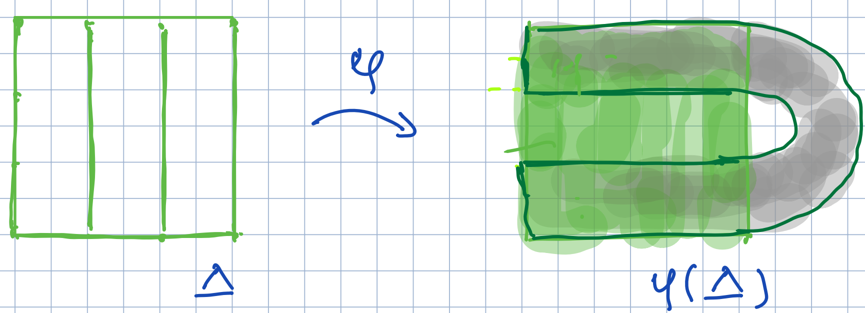
$$K = \bigcap_{k=-\infty}^{\infty} \varphi^k(\Delta) = \text{max inv subset of } \Delta$$

$\subset \operatorname{int}(\Delta)$

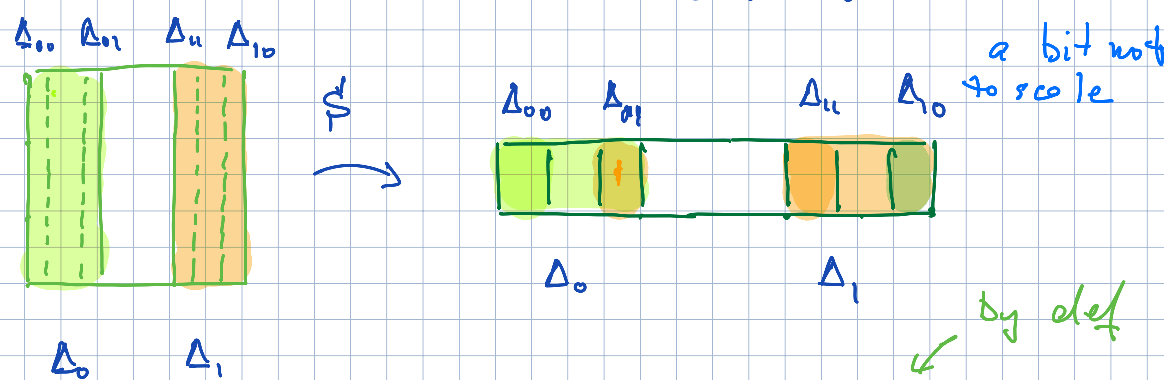
Claim: $K \cong \mathbb{Z}_2^{\mathbb{Z}}$ and $\varphi|_K = \sigma$.

Outline of the pf – symbolic dynamics

Simplification



$K \subset \Delta$ but not in $\mathrm{int}(\Delta)$ ...

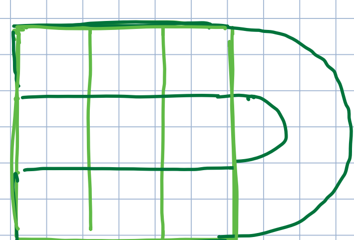


by def

$$\{x \mid x \in \Delta, \varphi(x) \in \Delta\} = \Delta_0 \cup \Delta_1$$

$$\underbrace{\{x \mid x \in \Delta, \varphi(x) \in \Delta, \varphi^2(x) \in \Delta\}}_{\{x \in \Delta_0 \cup \Delta_1 \mid \varphi(x) \in \Delta_0 \cup \Delta_1\}}$$

$$= \Delta_{00} \cup \Delta_{01} \cup \Delta_{10} \cup \Delta_{11}$$



etc

$\{x \mid x \in \Delta, \varphi(x) \in \Delta, \dots, \varphi^n(x) \in \Delta\}$
$= 2^{n-1}$ narrow vertical strips

$K^+ = \{x \mid \varphi^k(x) \in \Delta \ \forall k \in \mathbb{N}\}$
$=$ Cantor set $\times [0,1]$

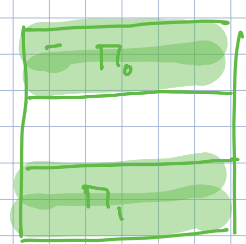
Coding trajectories (symbolic dynamics)

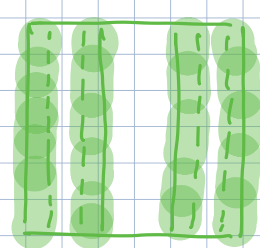
$K \longrightarrow \mathbb{Z}_2^{\mathbb{N}}$

$x \longmapsto a_0 a_1 a_2 \dots \quad , \quad a_k \in \mathbb{Z}_2$

if $\varphi^k(x) \in \Delta_{a_k}$

Apply the same process to φ^{-1}





$\{x \in \Delta \mid \varphi^{-1}(x) \in \Delta\} = \Pi_0 \cup \Pi_1$

$\underbrace{\{x \in \Delta \mid \varphi^{-1}(x) \in \Delta, \varphi^{-2}(x) \in \Delta\}}$

$\{x \in \Pi_0 \cup \Pi_1 \mid \varphi^{-1}(x) \in \Pi_0 \cup \Pi_1\} = \Pi_{00} \cup \Pi_{01} \cup \Pi_{10} \cup \Pi_{11}$

$\dots$

$K^- = \{x \mid \varphi^{-k}(x) \in \Delta \ \forall k \in \mathbb{N}\}$
$= [0,1] \times$ Cantor set

$K = K^+ \cap K^- = \{x \in \Delta \mid \varphi^k(x) \in \Delta \ \forall k\}$

symbolic dynamics:

<u>To summarize:</u>

$$K \longrightarrow \mathbb{Z}_2^{\mathbb{Z}}$$

$$x \longmapsto \quad \dots a_{-1} a_0 a_1 a_2 \dots \in \mathbb{Z}_2^{\mathbb{Z}}$$

$$a_k = \begin{cases} 0 & \varphi^k(x) \in \Delta_0 \\ 1 & \varphi^k(x) \in \Delta_1 \end{cases}, \quad k \in \mathbb{Z}$$

- Then one shows that this map is a homeomorphism
- $\varphi|K = \sigma$ by construction $\triangleleft$

<u>Important:</u> horseshoes (or smth like it) are ubiquitous (particularly in 2D) for C^∞-infinity generic $\varphi : M^2 \circlearrowleft$

$\exists \quad K \subset M$ s.t.

$K \cong \mathbb{Z}_2^{\mathbb{Z}}$

$\varphi^n|K = \sigma$, for some n. } roughly speaking

(Katok, Le Calvez)

Hyperbolic maps & sets - definitions

Recall: • another example of φ with the same properties as σ is $A : \mathbb{T}^n \to \mathbb{T}^n$ linear $|\lambda| \neq 1$

• It turns out that these properties are essentially a consequence of a common feature: hyperbolicity

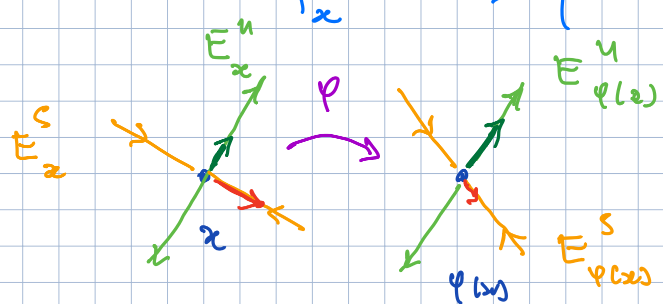
Def $\varphi : M \to M$ is hyperbolic if $\exists$

• a splitting $TM = E^s \oplus E^u$: $T_xM = E^s_x \oplus E^u_x$ invariant under $D\varphi$

• $0 < \eta < 1$ ← ind of x & v

$\|D\varphi_x(v)\| \leq \eta \|v\| \quad \forall v \in E^s$

$\|D\varphi_x(v)\| \geq \eta^{-1} \|v\| \quad \forall v \in E^u$ $\quad \forall x$

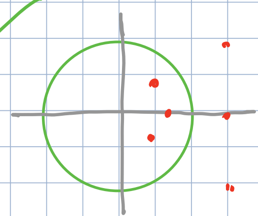


Shrinks vectors from E^s by $\eta < 1$

Extends vectors from E^u by $1/\eta > 1$

don't have to be reciprocal but can chose

Ex $A \in SL(n, \mathbb{Z})$ with all $|\lambda| \neq 1$



$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\mathbb{Z}^n \longrightarrow \mathbb{Z}^n$$
$$\underbrace{A : \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n \circlearrowleft}$$

$A^{-1} \in SL(n, \mathbb{Z}) \Rightarrow$ homeo

E.g. $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ Arnold's cat map

Splitting: $T\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$

$$DA = A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$E^u =$ span (eigenvectors with $|\lambda| > 1$)

$E^s =$ span (eigenvectors with $|\lambda| < 1$)

$$\eta = \max_{|\lambda| < 1} |\lambda|$$

Rmk Hyperbolic maps are very rare: essentially all examples have the same alg nature as $\varphi = A$

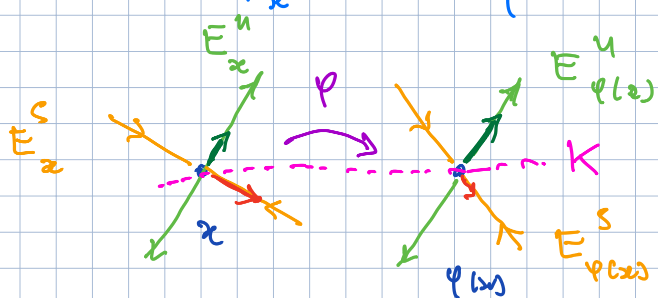
Hyperbolic sets:

$\varphi : M \to M$; K closed invariant set

Def K is hyperbolic for φ if $\exists$

- a splitting $T_K M = E^s \oplus E^u$: $T_x M = E^s_x \oplus E^u_x$, $x \in K$ invariant under $D\varphi$
- $0 < \eta < 1$ $\leftarrow$ ind of x, v

$$\|D\varphi_x(v)\| \le \eta \|v\| \quad \forall v \in E^s$$
$$\|D\varphi_x(v)\| \ge \eta^{-1} \|v\| \quad \forall v \in E^u, x \in K$$



Shrinks vectors from E^s by $\eta < 1$
Extends vectors from E^u by $1/\eta > 1$
but only for $x \in K$

Ex
1) $\varphi : M \to M$ hyperbolic : $K = M$
2) hyperbolic fixed or periodic pts
3) Horseshoe !

Rmk Similarly for flows but now there's also one neutral direction

$$TM = E^s \oplus E^u \oplus \underbrace{\mathrm{span}(v)}$$, $v \neq 0$

v.f. generating the flow

allows for periodic orbits

Ex Geodesic flows on surfaces of curvature < 0 (e.g. $= -1$)

Hyperbolicity is one of the central notions in modern dynamics!

hyperbolicity + a bit more $\Rightarrow$ a lot of dynamical features

Structural stability of hyperbolic sets

Hyperbolicity $\Rightarrow$ many important dynamics features

Here we focus on str. stability

Def K = compact invariant set is **locally maximal** if $\exists\, U \supset K$ such that K is the maximal inv set in U:

$$\varphi^k(x) \in U \ \forall k \in \mathbb{Z} \Rightarrow x \in K$$

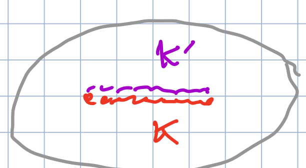
$$\Leftrightarrow \quad K = \bigcap_{k \in \mathbb{Z}} \varphi^k(U)$$

Thm K locally max & hyperbolic for φ

$\Rightarrow$ φ is str. stable near K:

$\psi \overset{C^1}{\approx} \varphi \Rightarrow \exists\, h$: nbd of $K \xrightarrow[\text{homeo}]{}$ nbd of K

s.t. $\psi = h \varphi h^{-1}$



Ex $\quad K =$ a hyperbolic fixed pt

Thm $\Leftrightarrow$ Hartman-Grobman

$\Leftarrow)$ $\varphi(x) = x \quad \underset{\text{IFT}}{\Rightarrow} \quad \psi(y) = y$

$D\psi|_y \approx D\varphi|_x \quad \Rightarrow$ hyperbolic

HG $\quad \psi \sim D\psi \sim D\varphi \sim \varphi$

$\Rightarrow)$ $\quad \psi = D\psi + \ldots$ Thm $\Rightarrow \psi \sim D\psi$: HG

Ex $\quad K = M$, φ hyperbolic

$\Rightarrow \varphi$ is str. stable

This is what we will prove

Particular case

Thm (Anosov)

$\varphi = A : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ hyperbolic

$\psi \overset{C^1}{\approx} A \Rightarrow \psi$ is top conj to A :

$\exists\, h \quad \psi = hAh^{-1}$

Pf. For the sake of simplicity

$n = 2$: $\mathbb{T}^n = \mathbb{T}^2$: A is 2×2

- Write

C^1-small

$$\psi = A + R : \mathbb{T}^2 \to \mathbb{T}^2 ;$$
$$h = id + H \quad \mathbb{T}^2 \to \mathbb{T}^2 ;$$

$R : \mathbb{T}^2 \to \mathbb{R}^2$

$H : \mathbb{T}^2 \to \mathbb{R}^2$

will be C^0-small

$$\begin{tikzcd} \mathbb{T}^2 \arrow[r, "\psi"] & \mathbb{T}^2 \\ \mathbb{T}^2 \arrow[u, "h"] \arrow[r, "A"] \arrow[ur, phantom, "\circlearrowright"] & \mathbb{T}^2 \arrow[u, "h"'] \end{tikzcd}$$

lift

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "\varphi"] & \mathbb{R}^2 \\ \mathbb{R}^2 \arrow[u, "h"] \arrow[r, "A"'] & \mathbb{R}^2 \arrow[u, "h"'] \end{tikzcd}$$

not really needed

$$\psi = h A h^{-1}$$

$$(A + R)\circ(id + H) = (id + H)A$$

$$(A+R)\circ(x + H(x)) = Ax + H(Ax)$$

$$\cancel{Ax} + AH(x) + R(x + H(x)) = \cancel{Ax} + H(Ax)$$

$$H(Ax) - AH(x) = R(x + H(x))$$

This the equation on H we need to solve.

Let's start with simpler equation

$$\underbrace{H}_{\text{unknown}}(Ax) - AH(x) = \underbrace{R}_{\text{given}}(x)$$

$L : H \longmapsto H\circ A - A\circ H$ Linear map in H

$C^0(\mathbb{T}^2;\mathbb{R}^2) \longrightarrow C^0(\mathbb{T}^2;\mathbb{R}^2)$

Claim L is invertible

Pf and $\|L^{-1}\| \le \frac{1}{1-\lambda}$ ← does not matter

e_1, e_2 eigenvectors of A

$\lambda_1 \ \lambda_2$ eigenvalues

$\lambda_1 = \lambda_2^{-1} > 1 > \lambda_2 =: \lambda$

$H = H_1 e_1 + H_2 e_2$

$R = R_1 e_1 + R_2 e_2$

$$\Rightarrow \quad H_1(Ax) - \lambda_1 H_1(x) = R_1(x)$$
$$H_2(Ax) - \lambda_2 H_2(x) = R_2(x)$$

Consider $P : C^0(\mathbb{T}^2) \longrightarrow C^0(\mathbb{T}^2)$

identity

$g \longmapsto g\circ A \qquad \|P\| = 1$

$$\underbrace{(P - \lambda_i I)}_{\lambda_i(\lambda_i^{-1}P - I)} H_i = R_i$$

when $i=1$ $\lambda_1^{-1} = \lambda_2 < 1 \Rightarrow \|\lambda_2 P\| < 1$

$$\Rightarrow (\lambda_2 P - I)^{-1} = -\underbrace{(I + \lambda_2 P + \lambda_2^2 P^2 + \dots)}_{\text{converges}}$$

when $i=2$

$$P-\lambda_2 I = P^{-1}\underbrace{(I-\lambda_2 P)}_{\text{invertible}} \qquad \|\lambda_2 P\| = \lambda_2 < 1$$

$$\Rightarrow \quad L = \begin{pmatrix} P-\lambda_1 I \\ P-\lambda_2 I \end{pmatrix} \leftarrow \text{invertible}$$

$\triangleleft$

Back to solving

$$H\circ A - A\circ H = R(I+H)$$

<u>Recall</u>: Contraction mapping principle:

$$\underset{\text{compl. metric space}}{\Phi : X} \xrightarrow{C^0} X \qquad d(\Phi(x), \Phi(y)) \leq \eta\, d(x,y)$$

$$0 < \eta < 1$$

$$\Rightarrow \exists! \text{ fixed pt } x : \underbrace{\Phi(x) = x}$$ fixed pt equation

<u>Pf</u> Take any $y \in X$ and set

$$y_k = \Phi^k(y) \leftarrow \text{Cauchy sequence} \leftarrow \underline{\text{Ex}}$$

$$y_k \longrightarrow x \leftarrow \text{Fixed pt}$$

$$\Phi(x) = \lim \Phi(y_k)$$

$$\| \qquad \qquad \|$$

$$\lim y_{k+1} = x$$

$\triangleleft$

Take $X = C^0(\mathbb{T}^2; \mathbb{R}^2)$ with sup-norm

$$\Psi : X \longrightarrow X$$

$$\Psi(H) = R(I + H)$$

Key equation:

$$L(H) = \Psi(H)$$

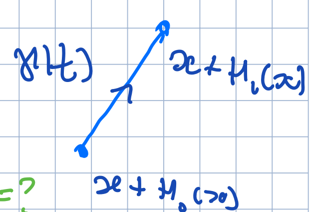
$$\Leftrightarrow \quad H = L^{-1}\Psi(H) \quad \leftarrow \text{fixed pt equation}$$

Claim $\Phi = L^{-1}\Psi : X \longrightarrow X$ is a contraction mapping

$\Downarrow$

Thm

Pf. • $\| L^{-1}\Psi(H_1) - L^{-1}\Psi(H_0) \| \le \| L^{-1} \| \, \| \Psi(H_1) - \Psi(H_0) \|$



• $\Psi(H_1) - \Psi(H_0)(x)$

$$= R(x + H_1(x)) - R(x + H_0(x)) \; =?$$

$$= \int_0^1 \frac{d}{dt} R\big(\underbrace{x + tH_1(x) + (1-t)R_0(x)}_{\gamma(t)}\big)\, dt$$

standard and useful trick

Alternatively one can apply the mean value thm to componentwise $R(\gamma(t))$

$$\sup_x |\Psi(H_1) - \Psi(H_0)(x)|$$

$$= \sup_x |R(x + H_1(x)) - R(x + H_0(x))|$$

$$\leq \sup_x \int_0^1 \Big| \frac{d}{dt} R(\underbrace{x + tH_1(x) + (1-t)H_0(x)}_{\gamma(t)}) \Big| dt$$

$$\leq \sup_x \int_0^1 \big| DR_{\gamma(t)} \cdot (H_1(x) - H_0(x)) \big| \, dt$$

$$\leq \sup_x \sup \|DR\| \cdot |H_1(x) - H_0(x)|$$

$$\Rightarrow \|\Psi(H_1) - \Psi(H_0)\| \leq \|R\|_{C^1} \cdot \|H_1 - H_0\|$$

$$\Rightarrow \|\Phi(H_1) - \Phi(H_0)\| \leq \underbrace{\|L^{-1}\| \cdot \|R\|_{C^1}}_{\eta} \cdot \|H_1 - H_0\|$$

$\|R\|_{C^1}$ small enough $\Rightarrow$ $0 < \eta < 1$

$\Rightarrow$ Φ is a contraction mapping

Also need to show that $h = id + H$ is a homeo. Not obvious but not hard - Ex Hint use again the fact that A is hyperbolic $\triangleleft$

Rmk R C^1-small $\Rightarrow$ $id + R$ is a C^1-diffeo

(Nuance) $\Leftarrow$ Inv. Function theorem

But R C^0-small $\cancel{\Rightarrow}$ $id + R$ is a homeo

Rmk: A similar argument proves
(Ex) the Hartman–Grobman thm.

→ The End ←