- Asymptotic stability via linearization

Lecture 26
03/01-2022

$v(x) = A(x) + \underbrace{R(x)} = A(x) + \underbrace{\ldots}$ higher order terms

Principle: the flow of A "approximates" the flow of v

$\varphi_A^t(x) = e^{At}x$

Goal: stability criterion via A

Thm $\mathrm{Re}\,\lambda < 0$ for all eigenvalues of A $\Rightarrow$ 0 is A.S.

Rmk
- sufficient but not necessary
  e.g. $v = -x^2\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y}$
- much more involved for L.S.

Ex. (Ex) $v(x) = Ax$ ← linear v.f.

$\mathrm{Re}\,\lambda < 0 \Rightarrow e^{At}x \xrightarrow[t\to\infty]{} 0$

obvious when A is diagonalizable

Pf of the theorem

- For the sake of simplicity assume that all $\lambda \in \mathbb{R}$
- Goal: find a Lyapunov function
- $\exists$ linear change of variables $\Rightarrow$

$A \rightsquigarrow BAB^{-1}$

$$A = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \text{with } |*| < \varepsilon$$

Ex.

arb small

- In other words

$$A = \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}_{} + \underbrace{\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}}_{}$$

$$= \Lambda + P, \quad \lambda_i < 0$$

with $\|P\| < \varepsilon$

- Set $f(x) = \langle x, x\rangle = \sum x_i^2$

Note $L_{Kx} f = \langle (K+K^T)x, x\rangle$ for any matrix K

- $L_v f = 2\langle \Lambda x, x\rangle + \langle (P+P^T)x, x\rangle + L_{R(x)} f$

$\rightarrow \langle \Lambda x, x\rangle = \sum \lambda_i |x_i|^2$

$$\le -\eta \cdot \frac{1}{2} \sum |x_i|^2 = -\eta \cdot f$$

where $\eta = 2 \min |\lambda_i|$

$\rightarrow \langle (P+P^T)x, x\rangle \le 2\|P\| \cdot \langle x, x\rangle = 2\|P\| \cdot f$

$$\le 2\varepsilon \cdot f$$

arb. small

$\rightarrow$ R involves quadratic and higher order terms $R = (R_1, \dots, R_n)$

$\frac{\|R(x)\|}{\|x\|} \longrightarrow 0$ as $x \longrightarrow 0$

$L_{R(x)} f = 2 \underbrace{\sum R_i(x) x_i}_{\text{cubic or higher order}}$

$\Rightarrow \frac{\|R(x)\|}{\|x\|^2} \longrightarrow 0 \quad x \longrightarrow 0$

$\Rightarrow \forall \delta > 0$

$\|L_{R(x)} f\| \le 2\delta \cdot f(x)$ when $x \approx 0$

- $L_v f \le 2(-\eta + \varepsilon + \delta) f < 0 \quad x \ne 0$

when $\varepsilon + \delta < \lambda$

$\triangleleft$ 16 S

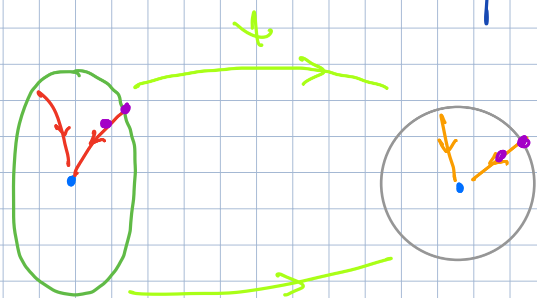
Cor In the setting of the thm
$\varphi^{t\geq 0}$ near 0 is top equivalent
to the flow $x \mapsto e^{-t}x$

$\Rightarrow \exists\ h :$ nbd of $0 \to$ nbd of 0 s.t.
$h(\varphi^{t\geq 0}(x)) = e^{-t} h(x)$

Pf • We have constructed a Lyapunov function
which is a quadratic form:
$f = \langle Bx, Bx \rangle = \langle B^T B x, x \rangle > 0$
change of variables

$\Rightarrow \{f = \varepsilon\} =$ ellipsoid $\cong S^{n-1}$

• Fix $\{f=\varepsilon\} \xrightarrow{\psi} S^{n-1}$ can take $\psi = B$
or the radial proj



Set $h(\varphi^t(x)) = e^{-t}\psi(x)$
Diffeo outside 0. $\triangleleft$

Q: • Why h is not smooth? When?

Ex • Show h $C^1 \Rightarrow A = -I$

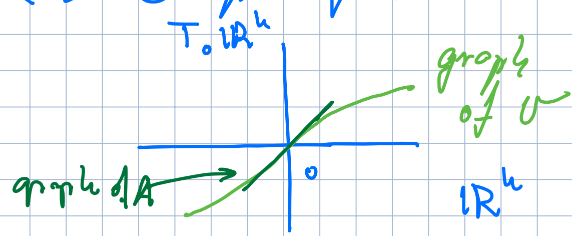
• Local normal forms and all that

Setting

• Vector fields $\qquad v(0)=0, \quad v(x) = Ax + \dots$

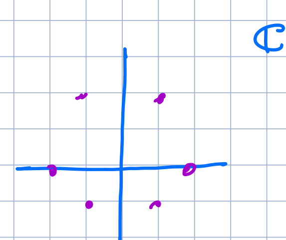
Def • 0 is a non-deg zero of v if
$\det(A) \neq 0$: no eigenvalues $\lambda = 0$

$\iff$ Graph of $v \pitchfork$ zero section at 0



Note
Graph of $A = Dv$
$= T_{(0,0)}$ graph of v

• 0 is hyperbolic if
$\operatorname{Re}\lambda \neq 0$



Note: hyperbolic $\not\Rightarrow$ non-deg

Ex $\qquad \operatorname{Re}\lambda < 0 \Rightarrow$ hyperbolic

• Diffeomorphisms

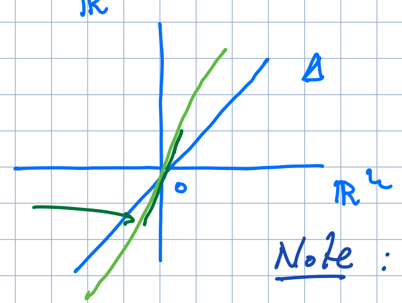
$$\varphi : \text{nbd of } 0 \longrightarrow \text{nbd of } 0$$

$$\varphi(0) = 0, \quad \varphi(x) = Px + \dots$$

$$P = D\varphi_0 : T_0\mathbb{R}^n \longrightarrow T_0\mathbb{R}^n$$

Def • 0 is a non-deg fixed pt of φ if $\det(I - P) \neq 0$: $\eta \neq 1$ ← eigenvalues

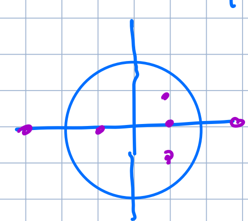
$\Longleftrightarrow$



graph$(\varphi) \pitchfork \Delta$ at 0

Note : graph$(P) = T_0$ graph(φ)

• 0 is a hyperbolic fixed pt if P has no eigenvalues $|\eta| = 1$



Note: hyperbolic $\overset{\not\Leftarrow}{\Rightarrow}$ non-deg

Rmk : asymptotic stability $\Leftarrow \underbrace{|\eta| < 1 \ \forall \eta}$

hyperbolicity $\Leftarrow \not\Rightarrow$

Relation

Ex. $v(x) = Ax + \dots \qquad v(0) = 0$

$\varphi = \varphi^t(x) =$ flow of v in time t

$\varphi(x) = Px + \dots \qquad \varphi(0) = 0$

$\Rightarrow \quad P = e^{At} \; : \; \eta = e^{\lambda t}$

$\Rightarrow \quad v$ non-deg $\underset{t \text{ small}}{\Rightarrow} \varphi$ non-deg

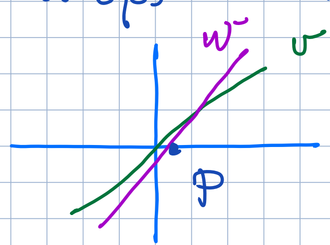
$\not\Rightarrow$ in general : $\lambda = 2\pi i$

v hyperbolic $\Rightarrow \underbrace{\varphi \text{ hyperbolic}}_{\Downarrow}$

non-deg

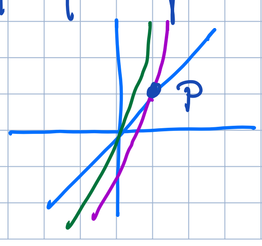
Rmk - Ex - Important

- v non-deg at 0, $w \overset{C^1}{\approx} v$
  $\Rightarrow \quad w(p) = 0$ for some p close to 0



Hint: use IFT

- φ non-deg at 0, $\psi \overset{C^1}{\approx} \varphi$
  $\Rightarrow \quad \psi(p) = p$ for some p close to 0



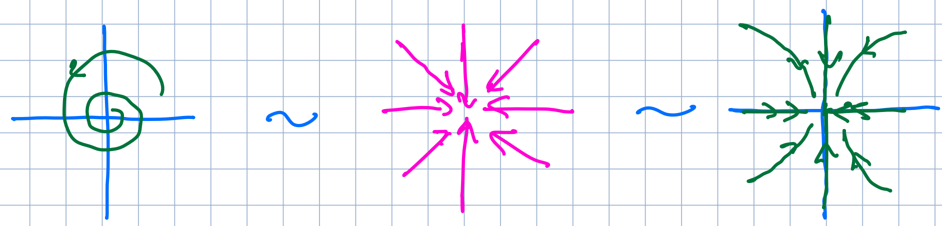
Hint: IFT

- Hartman – Grobman theorem

Lecture 17
03/03-2022

The top str of φ (or σ) near a hyperbolic pt is simple:

Ex: AS case



More generally

Thm (Hartman–Grobman)

$\varphi(0)=0 \leftarrow$ hyperbolic:

$P = D\varphi : \mathbb{R}^n = T_0\mathbb{R}^n \to \mathbb{R}^n = T_0\mathbb{R}^n$

$\Rightarrow$ locally near 0

φ is top equiv to P

More precisely: $\exists\ h$: nbd $0 \longrightarrow$ nbd of 0

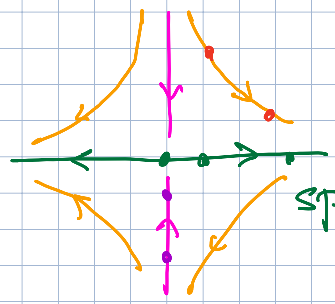
s.t. $\varphi(h(x)) = P h(x)$ as long as everything is defined

Rmk P_1 & $P_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ hyperbolic

P_1 & P_2 are top conj

$\Leftrightarrow$ same # of eigenvalues $|\lambda|<1$: n_-

$\underline{\qquad\qquad}$. $\underline{\qquad\qquad}$. $\underline{\quad}$ $|\lambda|>1$ n_+



$$\begin{array}{cc} \begin{pmatrix} \varepsilon & & & & & 0 \\ & \ddots & & & & \\ & & \varepsilon & & & \\ & & & \varepsilon^{-1} & & \\ & & & & \ddots & \\ 0 & & & & & \varepsilon^{-1} \end{pmatrix} & \begin{array}{l} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} n_- \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} n_+ \end{array} \end{array}$$

Rmk Similarly for vector fields (flows) but need to look at orbital equivalence: t is not preserved

Pf: KH, Arnold

Rmk The stable/unstable manifolds

$W^s(0)$: x : $\varphi^{k\to\infty}(x) \longrightarrow 0$

$W^u(0)$: x : $\varphi^{k\to-\infty}(x) \longrightarrow 0$

are actually smooth (Hadamard-Perron thm)

- More subtle aspects of local classification

$v(0)=0, \qquad v(x) = Ax + \dots$

$\varphi(0)=0, \qquad \varphi(x) = Px + \dots$

Q Linearization problem:
Can we show that v is smoothly equivalent to A? Same for φ

New possibility: equivalence in formal power series: Discuss in detail

$h \leftarrow$ formal power series

$h_* v = A$

$h\varphi^t = e^{At} h$ as formal power series

Or holomorphic

Sometimes; but often not

- $\exists$ obstruction even on the formal level. Resonances: linear relations

$$\lambda_j = \sum_{\substack{i \\ m_i \geq 0}} m_i \lambda_i \quad , \qquad m_1 + \dots + m_n \geq 2$$

E.g. $\lambda_1 = 2\lambda_2$

$\lambda_1 = 2\lambda_1 + \lambda_2 \iff (\lambda_1 + \lambda_2 = 0)$

But not: $2\lambda_1 = 3\lambda_2$, $\lambda_1 = \lambda_1$ or $\lambda_1 = \lambda_2$

Cf. If in our $L_v f = g$ example $\alpha = \frac{p}{q} \in \mathbb{Q}$ we would have problem solving $f_{p,q} = \frac{1}{2\pi i} \frac{g_{p,q}}{-qp2 + p}$

Similarly here

Thm (Poincaré)

Assume that A has no resonances
$\Rightarrow$ $v(x) = Ax + \ldots$
is formally equivalent to A

Pf: Arnold

- Going formal $\rightsquigarrow C^\infty$ encounters further problems akin to small denominators

More details: Arnold

Pf of Poincaré's theorem

- Change of variables (smooth)

$y = h(x) = x + H(x)$ ← homogeneous pol of deg $r \geq 2$ (vector valued)

transforms $\dot{y} = Ay$ into the equation

$$\dot{x} = Ax + \underbrace{v_r(x)}_{\text{hom of deg } r} + \dots \quad = h_*^{-1}(A\ )$$

- What's $v_r(x)$? $\quad h_*^{-1}(A) = \underbrace{A + v_r + \dots}_{\text{vector fields}}$

$\dot{y} = Ay \rightsquigarrow \dot{x} + \frac{\partial H}{\partial x} \cdot \dot{x} = A(x + H(x))$

matrix valued pol, $= I$ at $x = 0$

$$\left(I + \frac{\partial H}{\partial x}\right)\dot{x} = Ax + AH(x)$$

$$\left(I + \frac{\partial H}{\partial x}\right)^{-1} = I - \frac{\partial H}{\partial x} + \dots$$

Ex: $(I + B)^{-1} = I - B + \dots$ (B small)

$$\Rightarrow \quad \dot{x} = Ax + \underbrace{\left(-\frac{\partial H}{\partial x}Ax + AH(x)\right)}_{v_r(x)} + \dots$$

the Lie bracket of Ax & $H(x)$

• In other words:

$$h_*^{-1}(A) = A + v_r + \dots$$

• Consider V_r = hom vector valued pol of deg $r \geq 2$

$$V_r \subset \mathbb{R}[x_1, \dots, x_n] \otimes \mathbb{R}^n$$

$$L_A : V_r \longrightarrow V_r$$

$$H \longmapsto L_A H = -\frac{\partial H}{\partial x} A x + A H(x)$$

• Claim Assume A is non-resonant $\Rightarrow$ L_A is invertible

Pf • For the sake of simplicity assume A is linearizable

• change to the basis of eigenvectors

$e_1, \dots, e_n$

$\lambda_1, \dots, \lambda_n$

Basis in V_r: eigenvectors of L_A:

$$f_{k,j} = x_1^{k_1} \cdot \ldots \cdot x_n^{k_n} \otimes e_j \qquad \underbrace{k_1 + \dots + k_n}_{k} = r \geq 2$$

$$= \begin{pmatrix} 0 \\ \vdots \\ x_1^{k_1} \cdots x_n^{k_n} \\ \vdots \\ 0 \end{pmatrix}$$

Claim

$\Uparrow$

Subclaim:

$$L_A f_{k,j} = (\overbrace{-\sum k_i \lambda_i + \lambda_j}^{\neq 0 \,:\, \text{a resonance}}) f_{k,j}$$

$$\left(\frac{\partial f_{k,j}}{\partial x}\right)_{ij} = k_i\, x_1^{k_1} \dots x_i^{k_i - 1} \dots x_n^{k_n}$$

assuming $k_i \geq 1$

and 0 otherwise

only one $\neq 0$ raw : j^{th}

$$Ax = \lambda_1 x_1 e_1 + \dots + \lambda_n x_n e_n = \begin{pmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{pmatrix}$$

$$-\frac{\partial f_{k,j}}{\partial x} Ax = -\left(\sum k_i \lambda_i\right) f_{k,j}$$

$$A f_{k,j} = \lambda_j f_{k,j}$$

$$L_A f_{k,j} = \left(-\sum k_i \lambda_i + \lambda_j\right) f_{k,j}$$

- Finishing the proof: inductive process

$\rightarrow$ $\dot{x} = Ax + v_2(x) + \dots$

find $h_2(x) = x + H_2(x) = y$ different

transforming $\dot{y} = Ay$ into

$A + v_2 + \dots = h_{2*}^{-1}(A)$

$\Rightarrow$ h_2 will turn

$\dot{x} = Ax + v_2(x) + \dots$ not the same as here

into $\dot{y} = Ay + v_3(y)$

$\Rightarrow$ construct a sequence of smooth local diffeo's

$$h_2, h_3, h_4, \ldots, \quad h_i(x) = x + H_i(x)$$

deg = i

killing the r-th order term and modifying $\geq r+1$ th

$\Rightarrow$ The composition

$$\ldots \circ h_4 \circ h_3 \circ h_2$$

is defined as a formal power series and sends v to A

$\triangleleft$

Rmk At each finite step the image is smooth

$h_r \circ \ldots \circ h_2$ gives an equivalence of

$v \quad \cdot\!\!\sim \quad A$ + terms of order $\geq r+1$