Lecture 13

02/17-2022

State D. thm & questions

• Denjoy's example

Recall: $\varphi: S^1 \xrightarrow{C^2} S^1 \in \mathrm{Homeo}_+$

Alternative: • $\rho(\varphi) \in \mathbb{Q} \Leftrightarrow \varphi$ has a per. orbit

• $\rho(\varphi) = \alpha \notin \mathbb{Q} \Leftrightarrow \varphi \sim R_\alpha$ does not have to be C^2

$\Rightarrow$ all orbits are dense

Q: The role of C^2?

<u>Thm</u> (The Denjoy example)

$\exists$ a C^1-diffeo $\varphi: S^1 \to S^1$

with $\rho(\varphi) = \alpha \notin \mathbb{Q}$ (Hence no per. orbits)

and no dense orbits.

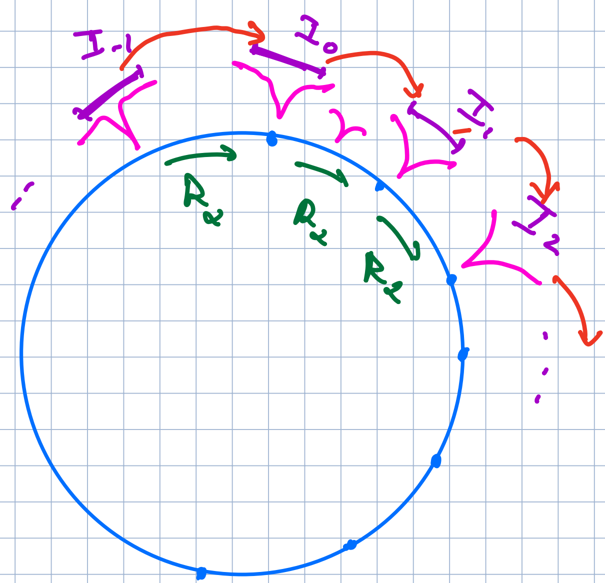
<u>Rmk</u> • Can make φ a $C^{1+\varepsilon}$ diffeo Hölder

but not a C^2-diffeo (Denjoy Thm)

• Can make φ C^∞-smooth but

not a C^∞-diffeo (φ^{-1} is not C^∞)

Idea of the construction (For C^0 φ)



- $x_n = R_\alpha^n(x)$
- $\ell_n > 0$

  $\sum_{n\in\mathbb{Z}} \ell_n < \infty$

  $|I_n| = \ell_n$

  $\varphi_n : I_n \to I_{n+1}$

- Cut S^1 at each x_n and insert I_n. Get S^1 again. To be more precise, construct $\coprod I_n \hookrightarrow S^1$ so that

  $\pi : S^1 \longrightarrow S^1$

  $I_n \to x_n$ otherwise 1-1

- Define $\varphi : S^1 \to S^1$ by

$$\varphi(x) = \begin{cases} \varphi_n(x) & \text{if } x \in I_n \\ R_\alpha(x) & \text{if } x \notin \coprod I_n \end{cases}$$

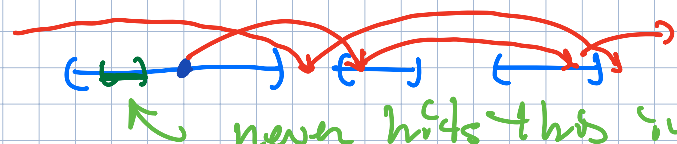
$\Rightarrow$ φ is a homeo

• φ has no periodic orbits

$\Rightarrow \rho(\varphi) \notin \mathbb{Q}$ HW: $\rho(\varphi) = \alpha$

φ has no dense orbits

if $x \in I_n$ $\varphi^k(x)$ never comes back to I_n



if $x \notin \bigsqcup I_n$, $\varphi^k(x)$ never enters any I_n.

looks like a Cantor set

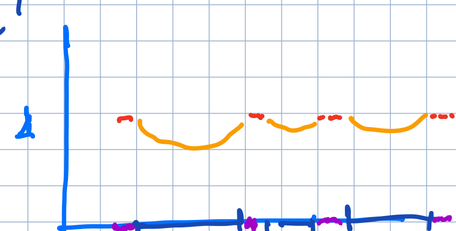
$K = S^1 \setminus U$

$U = \bigsqcup \operatorname{int}(I_n)$

$$\frac{d\varphi}{dx} = \begin{cases} 1 & \text{on } K \\ \frac{d\varphi_n}{dx_n} & \text{on } I_n \end{cases}$$

Rmk with just a bit more care of ℓ_n & φ_n (see [KH]) can make φ C^1

$\frac{d\varphi}{dx} \in C^0$

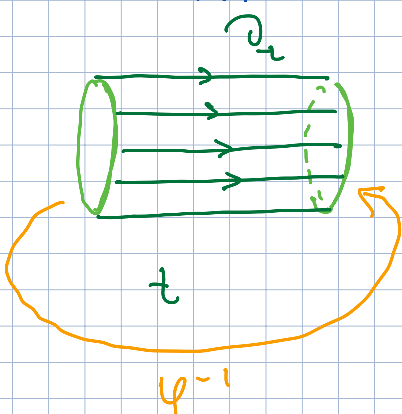


Application: v.f. on $\mathbb{T}^2$

Cor $\exists$ a C^1 v.f. on $\mathbb{T}^2$ without zeroes, or closed orbits or dense orbits.

Pf • Take $\varphi: S^1 \xrightarrow{C^1} S^1$ as in Denjoy's thm

• Form $M = S^1 \times [0,1] \Big/ {(x,0) \sim (\varphi^{-1}(x), 1)}$



the mapping torus

Ex: M is C^1 surface and $M \underset{C^1}{\cong} \mathbb{T}^2$

• $\theta_t \rightsquigarrow$ v.f. v on $M \cong \mathbb{T}^2$, C^1,
φ = time-1 flow of $v \neq 0$

• Ex: show that φ has no dense orbit or periodic orbit

$\triangleleft$

- Digression to number theory:
Diophantine vs Liouville numbers

$\alpha \notin \mathbb{Q}$ ← always

$\frac{p}{q} \in \mathbb{Q}$ relatively prime

what's the reference?

$\underline{Q}$: How fast can one approximate α by rational numbers

It turns out one should compare $|\alpha - \frac{p}{q}|$ with q!

$\underline{\text{Thm}}$ $\forall \alpha \notin \mathbb{Q}$ $\exists$ $\frac{p_i}{q_i}$ inf many s.t.

$$\left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{q_i^2} \longrightarrow 0$$

$\underline{\text{Rmk}}$ Can do $|\alpha - p_i/q_i| < \frac{1}{\sqrt{5}\,q_i^2}$
but not much better – see below:
$\exists$ α and $c > 0$ s.t.

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^2} \quad \forall \frac{p}{q} \, !$$

Math 235, Dynamical systems
Winter 2022

Lecture 1
01/04-2022

→ Go through basic info

* No exams, no hw
* Problems stated in lectures }
Up to them how much they take home
* OH: by appointment

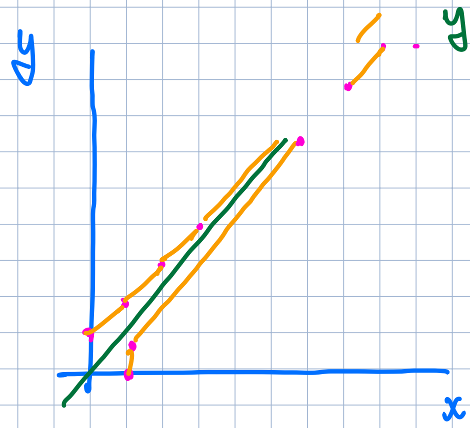
Plan:

- basic concepts and examples
- elements of ergodic theory
- maps of S^1, the Denjoy example
- local normal forms, Hartman-Grobman and local analysis of DS
- hyperbolicity, horseshoes
- topological entropy

varying degree of detail

- Not a comprehensive course
- Examples are often non-trivial and very important

"Pf" – visualization $\mathbb{Z}^2 \subset \mathbb{R}^2$



$y = \alpha x$ ⟵ no integer pts other than $(0,0)$

- Convex hull of all $(p,q) \in \mathbb{Z}^2$ below or above $y = \alpha x$ in the first quadrant

- A thread attached at ∞, first along $y = \alpha x$, put a nail at $\mathbb{Z}^2$ and pull the thread down/up
- the vertices of the hulls give the required sequence p_i/q_i

Details: [Arnold]
 – continued fractions $\triangleleft$

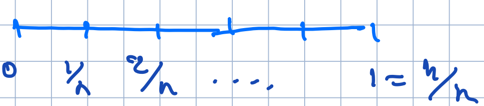
Direct pf in the spirit of the Kronecker thm:

Pf We'll prove:

$$\forall\, n \in \mathbb{N} \ \exists\ 1 \le q \le n \ \&\ p :$$

$$\left|\alpha - \frac{p}{q}\right| \le \frac{1}{nq} \le \frac{1}{q^2} \Rightarrow \text{thm}$$

- Partition $[0,1]$ into n intervals of length $1/n$



- Look at the $n+1$ pts

$$\alpha, 2\alpha, \dots, (n+1)\alpha \mod 1$$

At least two are in the same interval:

$$\exists\ 1 \le l < k \le n+1 \quad \text{s.t.}$$

$$|\underbrace{k\alpha - l\alpha}_{\text{mod } 1}| < 1/n$$

$$\Leftrightarrow \quad \exists\, p \quad |k\alpha - l\alpha - p| < 1/n$$

- Set $q = k - l$ then

$$|q\alpha - p| < 1/n$$

$$\Rightarrow \quad \left|\alpha - \frac{p}{q}\right| < \frac{1}{nq} \qquad \triangleleft$$

→ Denjoy ex ⇝ v.f. on $\mathbb{T}^2$

→ Recall the thm from Lect. 13:

Thm $\forall \alpha \notin \mathbb{Q}$ $\exists$ $\frac{p_i}{q_i}$ inf many s.t.

$$\left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{q_i^2} \longrightarrow 0$$

Q Can we do better than that?

$$|\alpha - p/q| < \varepsilon/q^2 \quad \text{or}$$

$$|\alpha - p/q| < C/q^{2+\beta} \quad ?$$

It turns out: not in general and for very few α's

Def $\alpha \notin \mathbb{Q}$ is <u>Diophantine</u> if

$\exists\, \beta \geq 0$ s.t.

$$\underbrace{\exists\, C>0 \text{ with } \left|\alpha - \frac{p}{q}\right| \geq \frac{C}{q^{2+\beta}} \quad \forall \frac{p}{q} \in \mathbb{Q}}$$

Denote the set of such α's by $\mathcal{D}_\beta$

$\mathcal{D} = \bigcup_{\beta \geq 0} \mathcal{D}_\beta$, $\quad \mathcal{D}_{\beta_1} \subset \mathcal{D}_{\beta_2} \Leftarrow \beta_1 \leq \beta_2$

Def $\alpha \notin \mathbb{Q}$ is Liouville if it is <u>not</u> Diophantine:

$\forall\, \beta \geq 0$ ~~$\forall\, C>0$~~ $\exists\, p/q \in \mathbb{Q}$ s.t.

$$\left|\alpha - \frac{p}{q}\right| < \frac{\cancel{C}\,1}{q^{2+\beta}}$$

can replace C by 1 by playing with β

Notation: $\mathcal{L} = \mathbb{R} \setminus \mathcal{D}$

In other words: sending $\beta \to \infty$ we obtain a rational approximations of α converging to α faster than any power of $1/q$!

Thm (Ex)

- $\forall \beta > 0$, $\mathcal{D}_\beta$ has a full measure

  $\Rightarrow \mathcal{D} = \bigcup_{\beta \geq 0} \mathcal{D}_\beta$ has a full measure

$\tilde{\approx}$ $\Updownarrow$

- $\mathcal{D}$ is meager: countable union of closed nowhere dense sets

Thm $\mathcal{L}$ is zero measure and second cat !! (small & large !!)

Pf – [Oxtoby]

Ex. $\alpha = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}$ is Liouville

Thm Every $\alpha \in \mathcal{L}$ is transcendental

$\Downarrow$

Cor $\alpha = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}$ is transcendental

Probably the simplest expl. transcendental number construction

Pf

α algebraic of deg n:

$f(\alpha) = 0$: f = pol of deg n (min) with coeff in $\mathbb{Z}$, no roots in $\mathbb{Q}$

Thm

$\Uparrow$

Lemma $\alpha \in \mathcal{D}_{n-2}$ $\leftarrow$ can be improved

Pf Set

$$M = \left\lceil \max_{|\alpha - x| \leq 1} |f'(x)| \right\rceil \in \mathbb{N}$$

Claim $\forall\ p/q$

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{Mq^n} \quad ; \quad C = \frac{1}{M} < 1$$

$\forall\ x$ with $|x-\alpha| \leq 1$:

$$|f(x) - \cancel{f(\alpha)}| \leq M \cdot |x-\alpha| \leq M$$

$$\overset{\shortparallel}{0}$$

Take $p/q = x$ can assume $\left|\frac{p}{q} - \alpha\right| < 1$

$\left(\text{otherwise } \left|\frac{p}{q} - \alpha\right| \geq 1 \geq \frac{1}{Mq^n}\right)$

Then

$$\left|f\left(\frac{p}{q}\right)\right| \leq M \cdot \left|\frac{p}{q} - \alpha\right|$$

$$\Rightarrow \underbrace{\left|q^n f\left(\frac{p}{q}\right)\right|}_{\in \mathbb{N}} \leq q^n \cdot M \cdot \left|\frac{p}{q} - \alpha\right|$$

$$\Rightarrow \geq 1$$

$$\Rightarrow 1 \leq q^n M \left|\frac{p}{q} - \alpha\right|$$

$$\Rightarrow \left|\frac{p}{q} - \alpha\right| \geq \frac{1}{Mq^n}$$

$\triangleleft$

Herman's theorem and small denominators

– Back to diffeo's of S^1

Recall:

Thm (Denjoy)

$\varphi : S^1 \xrightarrow{C^2} S^1$, $\rho(\varphi) \notin \mathbb{Q}$

$\Rightarrow \varphi \sim R_\alpha$: $\exists\, h$: $\varphi = h R_\alpha h^{-1}$ only C^0

Q Can we improve Denjoy's thm in a different way: make h smooth?

Not in general, but ...

Thm (M. Herman, 1979, Yoccoz 1984)

Assume $\varphi : S^1 \to S^1$ is a C^∞-diffeo and $\alpha = \rho(\varphi) \in \mathcal{D}$

$\Rightarrow \varphi = h R_\alpha h^{-1}$ with $h : S^1 \to S^1$ a C^∞-diffeo !

Rmk: builds up on work of Arnol & Moser

Deep & difficult, much more precise result

What goes wrong when $\alpha \in \mathcal{L}$?

– Small Denominators

Idea: need to solve the equation

$$f\circ R_\alpha - f = g \quad \leftarrow \text{given} \qquad g \in C^\infty$$

unknown

necessary

$\int g(x)\,dx = 0$

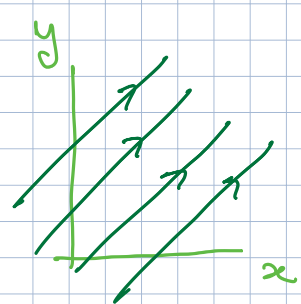
Want f to to exist and be sufficiently smooth.

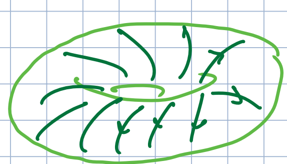
Ex. • Show that a sol $f \in C^\infty$ exists $\forall\, g \in C^\infty$ when $\alpha \in \mathcal{D}$

• But if $\alpha \in \mathcal{L}$ a sol might or might not exist depending on α

Example $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, (x,y)

$v = \alpha\partial_x + \partial_y$





minimal

all integral curves are dense

When does the equation

unknown

given, C^∞

$$L_v f = g \;:\; \alpha\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = g$$

have a sol? How smooth

Ex $\int g\,dx\,dy = 0 \leftarrow$ necessary condition

Recall the set-up

Lecture 15

02/24-2022

Prop. Assume that $\alpha \in \mathcal{D}$. Then $f \in C^\infty$ exists $\forall g$ with $\iint g = 0$

- If $\alpha \in \mathcal{L}$ a sol might or not exist, or fail to be smooth

Pf – method (small denominators)

$g \in C^\infty(\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2)$

Fourier series

$$g = \sum_{p,q} g_{p,q}\, e^{2\pi i (qx + py)}$$

$$f = \sum_{p,q} f_{p,q}\, e^{2\pi i (qx + py)}$$

$$L_v f = -\alpha \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

$$= \sum_{p,q} \underbrace{2\pi i\,(-q\alpha + p)\, f_{p,q}}_{g_{p,q}}\, e^{2\pi i (qx + py)}$$

$\Rightarrow g_{0,0} = \iint g$ must be 0

$(p,q) \neq (0,0)$

$$\boxed{f_{p,q} = \frac{1}{2\pi i}\,\frac{g_{p,q}}{p - q\alpha} = \frac{1}{2\pi i\, q}\left(-\alpha + \frac{p}{q}\right)^{-1} g_{p,q}}$$

"small denominator"

Analysis fact:

$$u = \sum u_{p,q}\, e^{2\pi i (qx+py)} \in C^\infty$$

$\Leftrightarrow$ $|u_{p,q}| \xrightarrow[(p,q)\to\infty]{} 0$ faster than any pol:

$\forall\ a,b \in \mathbb{N}$

$$|u_{p,q}| \cdot (|p|^a + |q|^b) \longrightarrow 0$$

- $g \in C^\infty \Rightarrow |g_{p,q}| \longrightarrow 0$ faster than any pol

- $\alpha \in \mathcal{D}$ $\quad \left|-\alpha + \frac{p}{q}\right| > \frac{C}{q^{2+\beta}}$ for some $C>0$ and β.

$$|f_{p,q}| = \frac{1}{2\pi} |g_{p,q}| \cdot C \cdot q^{3+\beta}$$

also $\to 0$ faster than any pol

$\Rightarrow f \in C^\infty$

- But if $\alpha \in \mathcal{L}$ and we are unlucky with g we can have

$$|f_{p,q}| \to \infty \quad \text{or} \quad \to 0 \text{ but slowly}$$

A sol may fail to exist.

$\lhd$

§4 Local Analysis of dynamical systems – Very Briefly

Two classes of questions:

→ Asymptotic & Lyapunov stability
- of applied interest

→ Local normal forms, linearization, "classification", etc
- hugely important, somewhat similar to $\mathrm{Diff}(S^1)$

→ Two types of DS:
- . discrete – maps or germs
- . continuous – flows ← focus on this

similar

a bit more intuitive

Asymptotic & Lyapunov stability

Setting: M = smooth manifold

v = smooth v.f.

$v(p) = 0$, $p \in M$

will relax $\longrightarrow$ $\left\{ \text{Assume } \varphi^t(x) \text{ is defined for all } t \in \mathbb{R} \text{ or at least } t \geq 0 \right.$

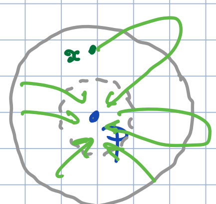
Def v (or p) is asymptotically stable if

- $\forall\, x$ close to p $\quad \varphi^t(x) \xrightarrow[t\to\infty]{} p$

Ex. (unif conv) $\Updownarrow$

- $\exists$ nbd $U \ni p$ s.t. $\forall\, V$ = nbd of p $\exists\, T > 0$ s.t. $\varphi^{t \geq T}(U) \subset V$

if φ^t is not defined for all $t \geq 0$ require it to be



Integral curves starting close to p converge to p

$\Rightarrow$ Can assume $M = \mathbb{R}^n$, $p = 0$

v is defined on some nbd of $p = 0$.

Rmk • v cannot be vol preserving

$\mathrm{vol}(\varphi^t(U)) = \mathrm{vol}(U) < \mathrm{vol}(V)$

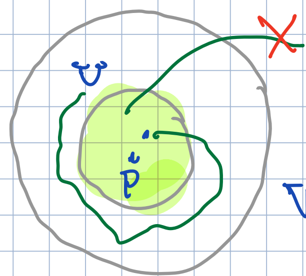
$\Rightarrow$ v cannot be Hamiltonian or symplectic

Def p is Lyapunov stable if

• $\forall\, V = $ nbd of p $\exists\, U =$ nbd of p

s.t. $\underbrace{\forall\, x \in U \quad \varphi^t(x) \in V \quad \forall\, t \geq 0}_{\varphi^t(U) \subset V}$

Trajectories starting close to p remain close to p



$V =$ arb small

Rephrasing for $p = 0 \in \mathbb{R}^n$, v defined near $p=0$

• v is asymptotically stable if

$\exists\, r > 0$ s.t. $\|x\| < r$

$\Rightarrow$ $\varphi^t(x)$ is defined for all $t \geq 0$ and

$\varphi^t(x) \longrightarrow 0$ $\underbrace{\text{(uniformly in } x)}_{\text{automatic}}$

• v is Lyapunov stable if

$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0$ s.t.

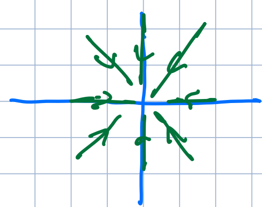
$\forall \|x\| < \delta$ $\varphi^t(x)$ is defined for all $t \geq 0$

and $\|\varphi^t(x)\| < \varepsilon$

A.S. $\Rightarrow$ L.S.

Examples for $\mathbb{R}^2$

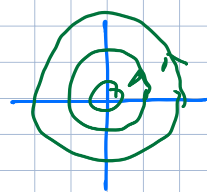
- $\dot{x} = -x$, $\dot{y} = -y$



Asympt. stable

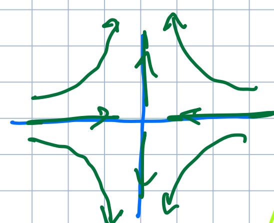
change signs $\Rightarrow$ "unstable"

- $\dot{x} = -y$, $\dot{y} = x$



L.S. but not A.S.

- $\dot{x} = -x$, $\dot{y} = y$

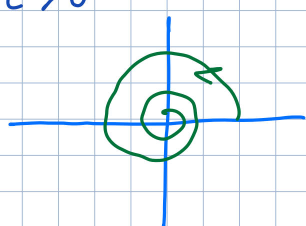


Neither L.S. (nor A.S.)

saddle

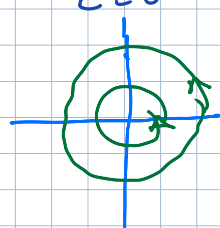
- $$\left.\begin{aligned} \dot{x} &= -y - \varepsilon x \\ \dot{y} &= x - \varepsilon y \end{aligned}\right\}$$ linear comb of the first two

$\varepsilon > 0$



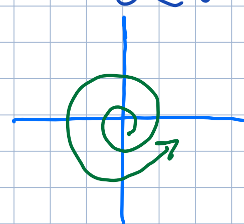
A.S.

$\varepsilon = 0$



L.S. but not A.S.

$\varepsilon < 0$



"unstable"

- etc

Interpretation:

- p: $v(p)=0$ is an equilibrium
- only "stable" equilibrian can be observed in practice

Ex

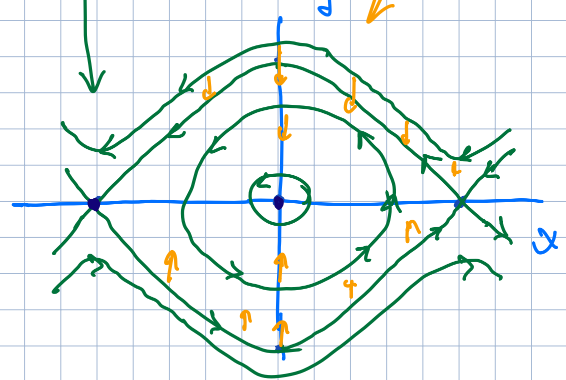
$$\dot{x} = y$$
$$\dot{y} = -\sin x - \overset{\overset{0}{\wedge}}{\varepsilon y}$$

on $\mathbb{R}^2$ or $\underset{y}{\mathbb{R}} \times \underset{x}{S^1} = TS^1$

pendulum

drag

loosing energy



$(0,0)$ A.S. $\varepsilon > 0$

L.S $\varepsilon = 0$

$(\pi,0)$ "unstable"

not A.S. or L.S

Rmk $\varepsilon < 0$: pumping in energy

$(\pi,0)$ neither

$(0,0)$ "unstable" $\Big\}$ neither is observe

stability criteria:

• Lyapunov functions

Setting:

- v defined on a nbd of $0 \in \mathbb{R}^2$, C^∞
- $f : (\text{nbd of } 0) \xrightarrow[C^\infty]{} \mathbb{R}$, $f(0) = 0$

Def. f is a Lyapunov function for v if

- f has an isolated min at 0
- $L_v f(x \neq 0) < 0$ (or $L_v f \leq 0$)

Thm Assume that v has a Lyapunov function.

- $L_v f \leq 0 \Rightarrow 0$ is L.S.
- $L_v f < 0 \Rightarrow 0$ is A.S.

Pf

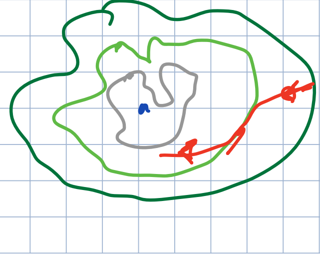
- $f(\varphi^t(x))$ decreasing (since: $L_v f < 0$)

$\Rightarrow$ $\underbrace{\{f \leq \varepsilon\}}_{\text{compact}}$ & $\underbrace{\{f < \varepsilon\}}_{\text{open}}$ are invariant

$\Rightarrow$ $\varphi^t(x)$ is defined for all $t \geq 0$

- $\{f < \varepsilon\} \leftarrow$ arb small nbds of 0

$\Rightarrow$ L.S.



• Remains to prove

$L_v f < 0 \Rightarrow$ A. S.
away from 0

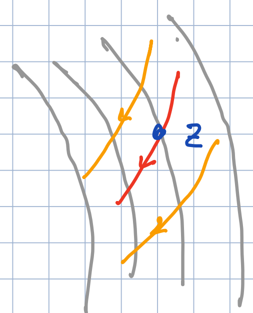
$x \in$ small nbd of 0
suffices to show:
$$\inf_{t \geq 0} f(\varphi^t(x)) = 0$$
Then $\varphi^t(x) \longrightarrow 0$

Assume not: $a = \inf_{t \geq 0} f(\varphi^t(x)) > 0$

Take z = a limit pt of $\varphi^t(x)$ as $t \to \infty$
$z \in \omega(x)$,

Then $f(z) = a > 0 \Rightarrow z \neq 0$
$L_v f(z) = b > 0$

Important argument



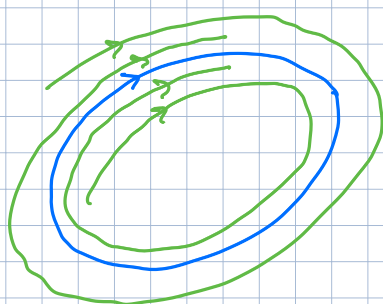
Claim $\exists\ \varepsilon > 0$ and $\tau > 0$
s.t. $f(\varphi^\tau(y)) \leq a - \varepsilon$
$\forall\ y$ near z

Pf True a z. By continuity true near z
with smaller ε & τ

Take $y = \varphi^t(x)$ near z. Then
$$f(\varphi^\tau(y)) = f(\varphi^{t+\tau}(x)) \leq a - \varepsilon$$
$\longrightarrow \leftarrow$ with $a = \inf$ $\triangleleft$

Rmk. • $\exists$ a similar criterion for diffeos ...

• Generalizes to integral curves other than equilibria: e.g. periodic orbits.



Problem with Lyapunov functions: difficult to find

Rmk $\exists$ L. function with $L_v f$ $\iff$ A.S.

Thm: essentially a necessary and sufficient condition