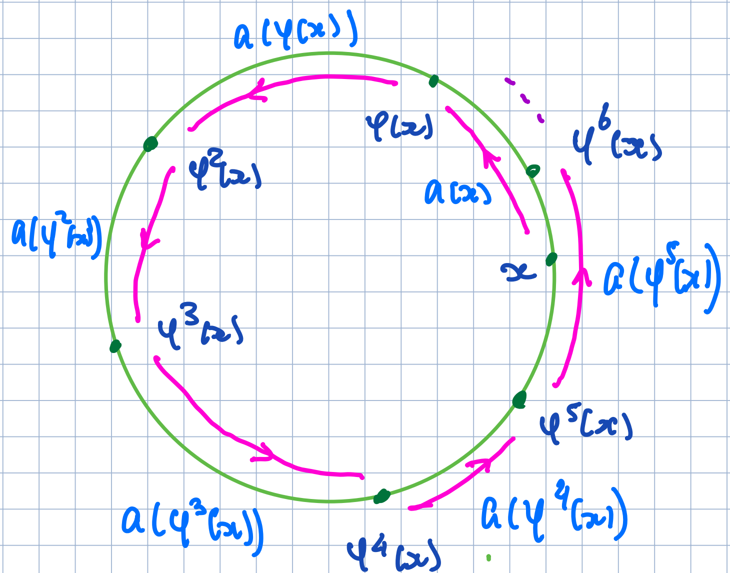
To summarize:

$\varphi : S^1 \to S^1$



$$\rho(\varphi) = \lim_{k \to \infty} \frac{1}{k} \left(a(x) + a(\varphi(x)) + \dots + a(\varphi^{k-1}(x)) \right)$$

exists and ind of x

Prop - Properties

(a) ρ is conjugation invariant

$$\rho(h \varphi h^{-1}) = \rho(\varphi)$$

(b) $\rho(\varphi) = \frac{p}{q} \in \mathbb{Q} \iff \varphi$ has a q-periodic orbit

($\Rightarrow$ all periodic orbits of φ has the same minimal period q)

(c) $\rho : H \to S^1$ is continuous with respect to the sup-norm

Pf

a) $F =$ a lift of φ

$H = \text{—} \cdot \text{—} \cdot - h \quad : \quad H(0) \in [0,1)$

$\Rightarrow \quad H^{-1} = \text{—} \cdot \text{—} \cdot - h^{-1} \quad H^{-1}(0) \in [0,1)$

$\Rightarrow \quad HFH^{-1} = \text{—} \cdot \text{—} \cdot \cdot h\varphi h^{-1}$

$0 \leq H(1) = H(0) + 1 \leq 2$

$$\Rightarrow \quad \underbrace{|H(x) - x|}_{\text{periodic}} \leq 2 \quad \forall x \in [0,1) \quad (1)$$

$\Rightarrow \forall x \in \mathbb{R}$

$\Rightarrow \quad |H^{-1}(x) - x| \leq 2 \quad$ (similar) $\quad (2)$

similarly

$|y - x| < 2 \Rightarrow |F^n(y) - F^n(x)| \leq 3 \quad (3)$

$$|HF^nH^{-1}(x) - F^n(x)|$$

$$\leq \underbrace{|H(F^nH^{-1}(x)) - F^nH^{-1}(x)|}_{\leq 2 \text{ by } (1)} + \underbrace{|F^n \underbrace{H^{-1}(x)}_{|y-x|\leq 2} - F^n(x)|}_{|F^n(y) - F^n(x)| \leq 3 \text{ by } (3)}$$

$\leq 2 + 3 = 5$

$$\Rightarrow \frac{1}{n}|HF^nH^{-1}(x) - F^n(x)| \leq \frac{5}{n} \longrightarrow 0$$

$$\Rightarrow \rho_x(HFH^{-1}) = \rho_x(F) \Rightarrow \rho(h\varphi h^{-1}) = \rho(\varphi)$$

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b)

$\Leftarrow$) $\quad x_0, \varphi(x_0), \dots, \varphi^{q-1}(x_0), \varphi^{q}(x_0) = x_0$

$$\Rightarrow \quad F^{q}(x_0) = x_0 + p$$

$$F^{kq}(x_0) = x_0 + \underbrace{kp}_{a_{kq}}$$

$$\Rightarrow \quad \rho(\varphi) = \lim_{k\to\infty} \frac{a_{kq}}{kq} = \frac{p}{q}$$

$\Rightarrow$ [KH] or [Arnold] or Ex

Hint: • Reduce to φ^{q} has a fixed pt where $\rho(\varphi) = \frac{r}{q}$: Then $\rho(\varphi) = 0$

• $F =$ lift with $F(0) \in [0, 1)$

$$0 < a(x) = F(x) - x < 1$$

$$\Rightarrow \quad \delta < a(x) < 1 - \delta \qquad \delta > 0 \;\; \forall x$$

• $\Rightarrow \delta \leq \frac{F^{n}(0)}{n} \leq 1 - \delta \Rightarrow \rho(\varphi) \neq 0$

c) [KH]

$\triangleleft$

Cor (of a) : $\rho(\varphi) \neq \rho(\psi) \Rightarrow \rho$ & ψ are not conj

$\Rightarrow$ $\underbrace{R_\alpha}_{\text{rot by } \alpha}$ is top conj $R_\beta \Leftrightarrow \alpha = \beta$ & $R_\alpha = R_\beta$

Applications

1. str stable diffeo's of S^1

<u>Q</u>: Do we have str stable diffeos of S^1?

<u>Ex 0.</u> R_α is never str stable
$R_\beta \approx R_\alpha$ $\alpha \approx \beta$ but $R_\beta \not\sim R_\alpha$ $\alpha \neq \beta$

<u>Ex 1.</u> $v: \mathbb{R} \to \mathbb{R}$ 1-periodic
Ex $\underbrace{v(x)=0}_{\text{somewhere}} \Rightarrow v'(x) \neq 0$

$x' = v(x)$ DE on $\mathbb{R}$ or S^1

$\forall\, t \neq 0 \quad \varphi = \varphi^t : S^1 \to S^1$ is non-deg

all periodic orbits are non-deg

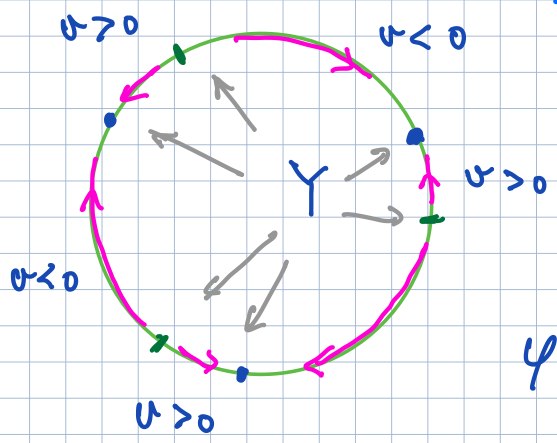
$\mathrm{Per} = \mathrm{Fix} = \{v = 0\}$

$\Rightarrow \rho(\varphi) = 0$
Prop

Pf of str stability is tedious but not difficult

Ex 2 • v = v.f. on S^1

Ex • $Y = \{v = 0\} \neq \emptyset$ invariant under $R_{1/q}$



• $v' \neq 0$ at $Y \neq \emptyset$

φ = the flow of v in time $t>0$

$$\psi = R_{p/q} \circ \varphi \qquad (p,q)=1$$

$\Rightarrow$ • ψ is non-deg all periodic orbit has period q

• ψ is str stable $\leftarrow$ tedious but not difficult

• $\rho(\psi) = \dfrac{p}{q}$

Rmk 1) Can replace $R_{1/q}$ by any $\mathbb{Z}_q$-action on S^1 and v by an inv. nbd

$\Rightarrow$ diffeomorphic example

2) Roughly speaking every str stable diffeo of S^1 has this form:

Thm $\varphi : S^1 \xrightarrow{C^2} S^1$ is str stable

$\Leftrightarrow$ $\rho(\varphi) = p/q \in \mathbb{Q}$ and all periodic orbits of φ are non-deg

Rmk: then all periodic orbits have period q

Cor str stable $\varphi : S^1 \to S^1$, $C^{k \geq 2}$ form an open and dense set in C^k topology

A very rare phenomenon

Rmk • A bit counterintuitive: $\exists$ "more" φ with $\rho(\varphi) \in \mathbb{Q}$ than with $\rho(\varphi) \notin \mathbb{Q}$

• Thm $\Leftarrow$ Denjoy's thm - next tedious - [Arnold]

Lecture 12
02/10 - 2022

2. What about irrational ρ ?

Thm (Denjoy)

$\varphi : S^1 \xrightarrow{C^2} S^1$, $\rho(\varphi) \notin \mathbb{Q}$

$\Rightarrow \quad \varphi \sim R_\alpha$

Cor $\rho(\varphi) \notin \mathbb{Q}$, φ is C^2

$\Rightarrow$ φ is uniquely ergodic

(and hence $\underbrace{\text{minimal}}_{\text{every orbit is dense}}$)

Rmk: the invariant measure is usually not the Lebesgue measure when

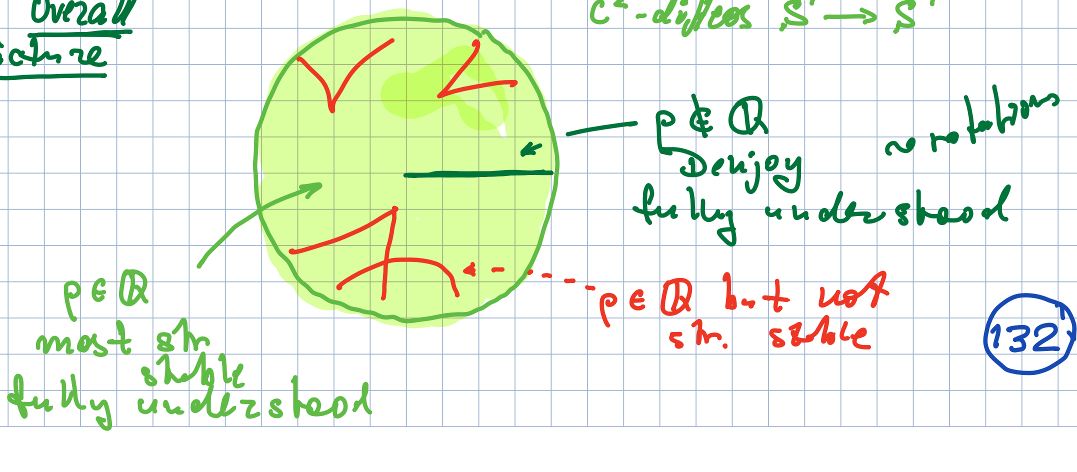
$\varphi = h R_\alpha h^{-1}$

it is $h^* \mu_{Lebesgue} \neq \mu_{Lebesgue}$

unless $\varphi = R_\alpha$

but still continuous

Overall Picture



Pf (Outline - steps)

hidden in the pf

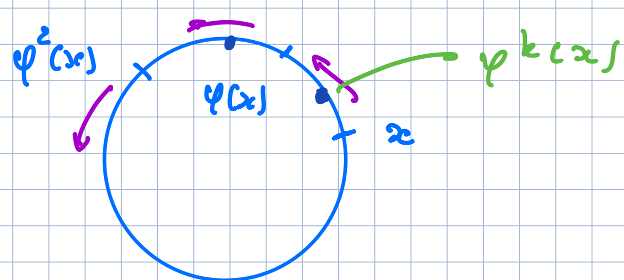
i) Pick $x \in S^1$; $\alpha = \rho(\varphi)$

Pf of existence of $\rho(\varphi)$

$\Rightarrow$ $\forall\, n$ $\{\varphi^j(x) \mid 0 \le j \le n\} \subset S^1$

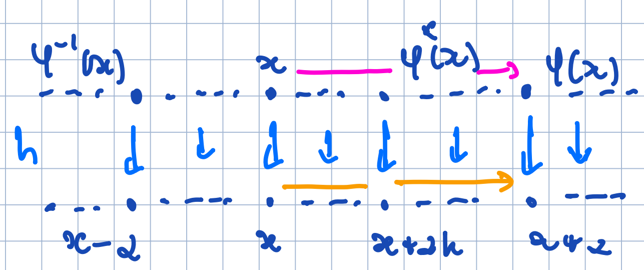
can be any finite range

has the same cyclic order as $\{R_\alpha^j(x)\} = \{x + j\alpha\}$.



$\Rightarrow$ enough to show that an orbit ($\Leftrightarrow$ every orbit) is dense

Then extend h by continuity



$R_\alpha h = h \varphi$

2) Assume not. Pick $I \in S^1$ s.t. $\Theta(x) \cap I = \emptyset$

open interval

$\Rightarrow$ $\varphi^j(I)$, $j \in \mathbb{Z}$ are mutually disjoint:

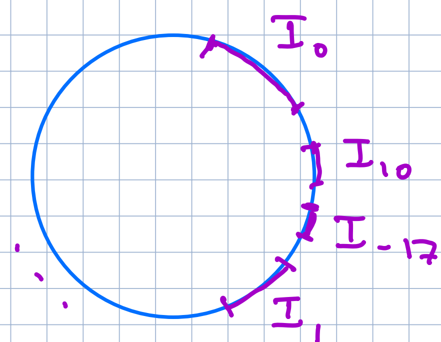
Easy

$$\varphi^i(I) \cap \varphi^j(I) = \emptyset \quad i \neq j$$

set $I_j = \varphi^j(I)$

$$\Rightarrow \sum_{j=-\infty}^{\infty} |I_j| < \infty$$

$$\Rightarrow \boxed{|I_j| \longrightarrow 0 \qquad j \longrightarrow \pm\infty}$$



Note

$$\left.\begin{aligned} |I_1| &= \int_{I_0} \left|\frac{d\varphi}{dx}\right| dx \\ |I_2| &= \int_{I_1} \left|\frac{d\varphi}{dx}\right| dx \\ &\cdots \end{aligned}\right\} \text{ connects } |I_j| \text{ and } \frac{d\varphi}{dx}$$

3) ← This where the most effort goes

$\frac{d\varphi}{dx}^{\pm 1}$ bounded variation (or rather $\ln\left|\frac{d\varphi}{dx}^{\pm 1}\right|$)

$\Rightarrow \sum_{j=-\infty}^{+\infty} |I_j| = \infty \quad \rightarrow\leftarrow$

◀

Rmk f bounded variation on $I = [0,1]$:

→ partition $X: 0 = x_0 < x_1 < \dots < x_n = 1$

→ $\mathrm{var} = \sup_X \sum_j |f(x_j) - f(x_{j-1})| \underline{< \infty}$

when f is C^1 $= \int_0^1 |f'|\, dx$

- $C^1 \Rightarrow$ Lipshitz $\Rightarrow$ Bounded variation
- monotone $\Rightarrow$ Bounded variation $= |f(1) - f(0)|$

Q
- What happens when φ is not C^2
  say only C^1 or C^0, $\rho(\varphi)$ & $\mathbb{Q}$
- When can we have h $C^{k>0}$ in Denjoy theorem?

Focus on this question

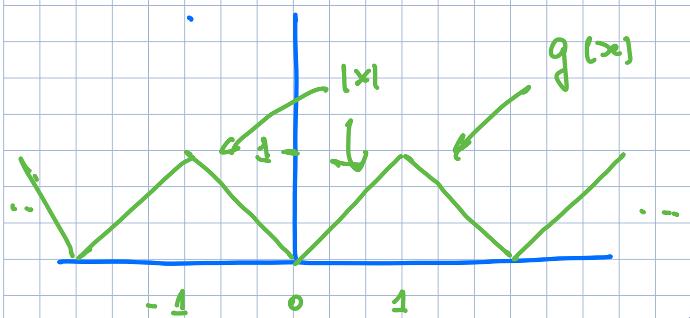
Digression: C^0 vs C^1

set $C^1 = C^1([0,1])$, $C^0 = C^0([0,1])$

$C^1 \subsetneq C^0$

But how far from being differentiable a C^0-function can be

Construction:

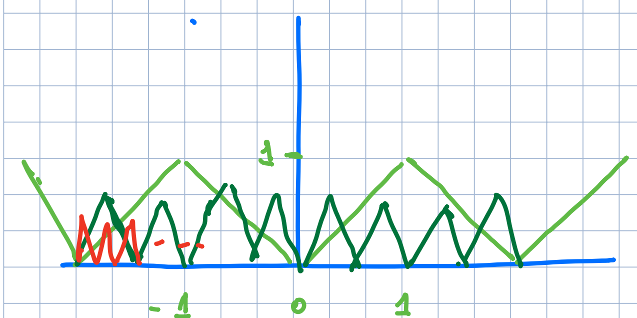


$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} g(2^{n-1}x)$$

$$f(x) = g(x) + \frac{1}{2} g(2x) + \frac{1}{4} g(4x) + \dots$$

converges uniformly

$\Rightarrow f \in C^0$



Ex Prove that f is nowhere monotone and nowhere differentiable

Ref: Gelbaum & Olmstead "Counterexamples in Analysis"

Rmk • monotone $\Rightarrow$ almost everywhere differentiable (Lebesgue)

• But a function can be everywhere differentiable (but not C^1) but nowhere monotone (very hard)

A different approach:

Thm Nowhere differentiable functions form a second category set in C^0

Equivalently: functions diff at one pt form a meager (first category) set $\mathcal{D}$ in C^0.

countable union of nowhere dense closed sets

Ref: [Oxtoby]

Outline of the pf

Step 1

$$E_n = \{ f \in C^0 \mid |f(x+h) - f(x)| \le nh \ \exists x \ \forall h \}$$

$[0, 1-\frac{1}{n}]$, $\quad 0<h<1-x$

Clearly a function differentiable (on the right) at some pt x is in E_n for some n : $\bigcup E_n \supset \mathcal{D}$

functions differentiable at one pt

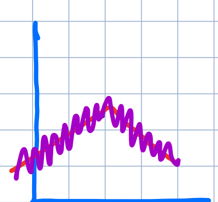
Ex Show that E_n is closed

Step 2

Ex Show that E_n is nowhere dense

Hint : • approximate $f \in C^0$ by p-wise linear functions $\hat{f}$

• approximate $\hat{f}$ by a sawtooth function.



Steps 1+2 : $\mathcal{D} \subset \bigcup E_n$ is meager $\triangleleft$