§3 Homeomorphisms of S^1

Lecture 9

02/01-2022

Generalities: Equivalence of Dynamical systems

Setting

homeo or diffeo in some reasonable class

$\varphi, \psi : M \longrightarrow M$

a closed manifold

"Def" φ & ψ are equivalent if

$$\begin{tikzcd} M \arrow[r, "\varphi"] & M \\ M \arrow[u, "h"] \arrow[r, "\psi"] & M \arrow[u, "h"'] \end{tikzcd}$$

$\psi = h \varphi h^{-1}$

ψ is "conj" to φ

- usually h is roughly of the same type as φ & ψ (e.g. volume preserving)
- But usually h is only C^0 even when φ & ψ are C^k, $1 \leq k \leq \infty$

Want h to preserve the most essential features: periodic orbits, top transitivity, ergodicity ...

Q: Any hope of "classification"?

Rmk: In some limited number of cases, Yes. Overall, No

Related notion: structural stability

"Def" φ is str. stable if

$$\psi \underset{C^k}{\approx} \varphi \Rightarrow \psi \underset{\uparrow\text{top. conj}}{\sim} \varphi$$

Rmk: rare but interesting

str stable property:

φ has it & $\psi \underset{C^k}{\approx} \varphi \Rightarrow \psi$ has it

Rmk: more reasonable

Rmk (Flows)

The true conj: $\psi^t = h\varphi^t h^{-1}$
(even when $h \in C^0$) is usually true restrictive (see below)

$\Rightarrow$ usually just want h to send orbits to orbit but not to preserve time-parametrizations

Some comments and Examples

- "Persistence" of fixed pts under small perturbations

- $\varphi: M \xrightarrow{C^1} M$ ← manifolds

- $\varphi(p) = p$

$\underbrace{D\varphi_p}$: $T_pM \longrightarrow T_pM$ invertible

the "linearization" $\varphi(x) = D\varphi_p(x) + \dots$

$p = 0$

Def p is non-deg if

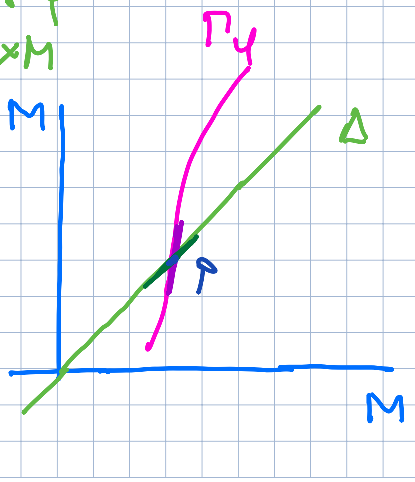
- $D\varphi_p$ does not have 1 as eigenvalue
  i.e. $I - D\varphi_p$ is invertible

Ex $\Updownarrow$

- $\Gamma_\varphi \pitchfork \Delta$ at p

graph of φ in $M \times M$

the diagonal



Prop-Ex

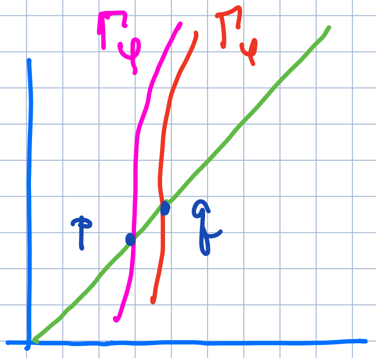
Assume that p is non-deg

and $\psi \underset{C^1}{\approx} \varphi$

$\Rightarrow$ • Near p $\exists$ a fixed pt q of ψ

• $D\psi_q \approx D\varphi_p$ $\leftarrow$ use a chart containing p & q

inverse function thm



slope of Γ_φ at p

$\approx$

slope of Γ_ψ at q

Def-Rmk x = q-periodic pt (min period) is non-deg if $x^q \in \mathrm{Fix}(\varphi^q)$ is non-deg for φ^q

Rmk $\psi = h\varphi h^{-1}$, $h \in C^1$

$p \in \mathrm{Fix}\,\varphi \Rightarrow q = h(p) \in \mathrm{Fix}(\psi)$

$$\begin{tikzcd}
T_pM \arrow[r, "D\varphi_p"] & T_pM \\
T_qM \arrow[u, "Dh"] \arrow[r, "D\psi_q"'] & T_qM \arrow[u, "Dh^{-1}"']
\end{tikzcd}$$

if $h \in C^1$

$$\Rightarrow \underbrace{\text{eigenvalues of } D\varphi_p = \text{eigenvalues of } D\varphi_q}_{\text{easy to change by a } C^1\text{-small pert}}$$

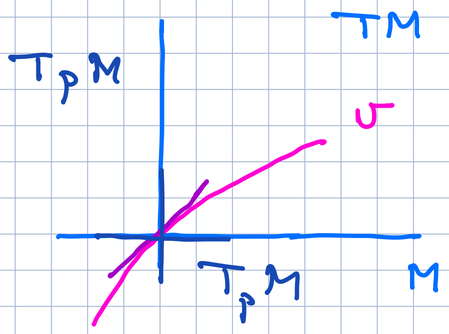
⇒ Not much hope for "classification" and str stability if $h \in C^1$

<u>Rmk</u> (Flows)

φ^t = flow of vect. field v

$v(p) = 0 \Rightarrow p \in \mathrm{Fix}(\varphi^t) \quad \forall t$

In a chart: $v(x) = Dv_p(x) + \dots$



$Dv_p : T_pM \to T_pM$

<u>Def</u> p is non-dg if

- Dv_p is "non-dg" does not have 0 as an eigenvalue
- $\mathrm{Graph}(v) \pitchfork \underbrace{M}_{\text{zero section}} \subset TM$

<u>Ex.</u> State and prove an analogue of Prop for flows

Rmk Assume that $\psi^t = h\varphi^t h^{-1}$ $\forall t$
"the conjugation preserves time"

$\Rightarrow$ γ = periodic orbit of φ^t
$\Updownarrow$ with period T

$h(\gamma)$ = periodic orbit of ψ^t
with period T

The period is very easy to change by a small perturbation:

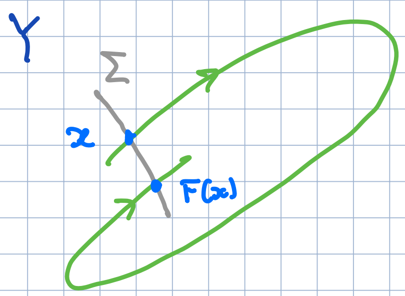
E.g. $v \rightsquigarrow (1+\varepsilon)v$

$\Rightarrow$ Not much hope for str. stability or classification when h preserves time.

- <u>Conceptually:</u>

  <u>maps in dim n</u> $\iff$ <u>flows in dim $n+1$</u>

Cross-sections :
- φ^t flow on Y^{n+1} generated by v
- $\Sigma^n \subset Y^{n+1}$, $v \pitchfork \Sigma$ and the return map F is defined

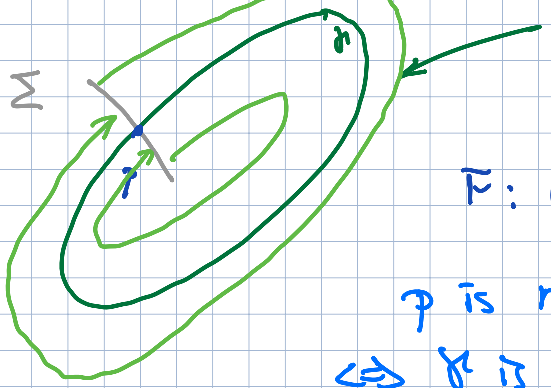


$\Rightarrow$ Dynamics of F captures a lot of dynamics of φ^t:

periodic orbit of F $\hookrightarrow$ Periodic orbits of the flow

Global crossections rarely exist

<u>Poincare return map</u>



periodic orbit γ of φ^t

$F: (\text{nbhd of } p \text{ in } \Sigma) \to \Sigma$

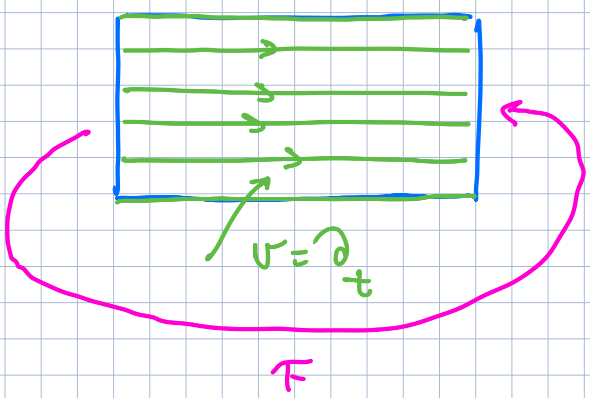
p is non-deg for F

$\underset{\text{def}}{\iff}$ γ is non-deg for v (or φ^t)

Mapping torus

$F : M \longrightarrow M$

$$Y = M \times \underset{t}{[0,1]} \,/\, \sim \qquad (x,0) \sim (F^{-1}(x), 1)$$



- v descends to Y $\Rightarrow$ flow φ^t
- $\Sigma = M \times 0$ is a closed section
- F = return map

$$\mathrm{Per}(F) = \mathrm{Per}(\varphi^t)$$

F is top transitive, ergodic, minimal etc $\Longleftrightarrow$ φ^t is top transitive, ergodic, minimal etc

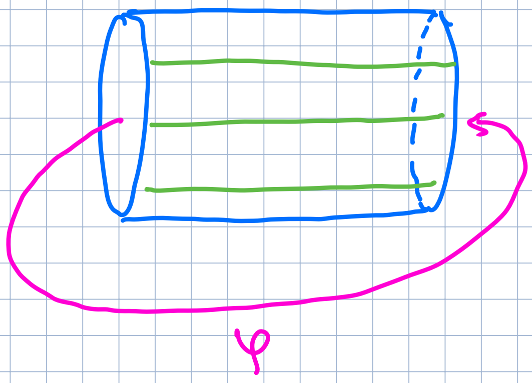
Dynamics of F $\longleftrightarrow$ Dynamics of φ^t

Ex - Ex

$M = S^1$, $\varphi : S^1 \xrightarrow{C^k} S^1$ $k = 0, 1, \dots$

orientation pres, homeo or diffeo

$Y = S^1 \times [0,1] \Big/ _{(x,0) \sim (\varphi(x),1)}$



Prove that Y is a C^k- manifold which is diffeo ($k \geq 1$) or homeo (C^0) to $\mathbb{T}^2$

Lecture 10
02/03-2022

Specializing

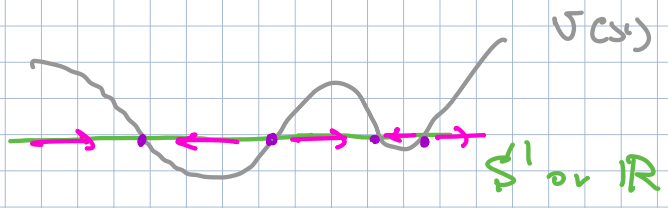
Dynamics on S^1: questions

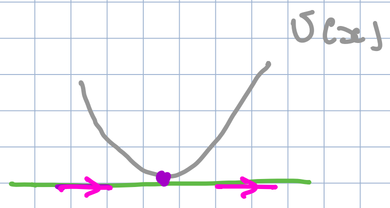
- Flows on S^1 or $\mathbb{R}$ are rather simple

→ $\dot{x} = v(x)$ can be integrated explicitly

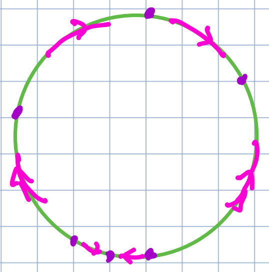
$$\frac{dx}{v(x)} = dt \quad \ldots.$$

→ Easy to visualize:





- Fixed pts = zeros of v
- no periodic orbits or interesting dynamics



But homeo or diffeo's

$\varphi: S^1 \to S^1$, $S^1 = \mathbb{R}/\mathbb{Z}$

can already be very interesting

Some questions:

- What are str. stable maps?
- R_α rotation by α

  $\theta \to \theta + \alpha$

  Is R_α equiv R_β $\alpha \neq \beta$?

  Cases: $\alpha = \frac{p}{q}$, $\beta \notin \mathbb{Q}$ or $\alpha, \beta \notin \mathbb{Q}$ or

  $\alpha = \frac{p_1}{q}$, $\beta = \frac{p_2}{q}$

- Is $\varphi(\theta) = \theta + \alpha + \varepsilon \sin(2\pi\theta)$ equivalent to R_α

- Can we have φ without periodic orbits and dense orbits?

Trying to answer $\Rightarrow$ unexpected results

Rotation number

- classification questions ⟷ invariants
- Rotation number $\rho : \underbrace{\text{Homeo}_+(S^1)}_{H} \longrightarrow S^1$

construction: $\varphi \in H = \text{Homeo}_+(S^1)$

$S^1 = \mathbb{R}/\mathbb{Z}$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{R} \arrow[d, "\pi"] \\
S^1 \arrow[r, "\varphi"'] \arrow[u, phantom, "\Uparrow \exists"] & S^1
\end{tikzcd}$$

a lift of φ to $\mathbb{R} \to \mathbb{R}$

$x = e^{2\pi i\theta}$

$\varphi(x) = e^{2\pi i F(\theta)}$

use x for S^1 & $\mathbb{R}$

Properties

- $F : \mathbb{R} \to \mathbb{R}$ is a homeo and str. monotone increasing
- $F(x+1) = F(x) + 1$
- for any two lifts F_0 & F_1 of φ

  $F_1 - F_0 = \text{const} \in \mathbb{Z}$
- $F^k = \underbrace{F \circ \dots \circ F}_{k}$ is a lift of φ^k

Set $\rho_x(F) := \lim_{k\to\infty} \frac{1}{k} F^k(x)$ can take $|k| \to \infty$

Prop
- the limit exists
- $\rho_x(F)$ is ind. of x : $\rho(F)$
- $\rho(F_1) - \rho(F_0) \in \mathbb{Z}$ for any two lifts F_1 & F_0 of φ

well-defined

Def The rotation number of φ:

$$\rho(\varphi) = (\rho(F) \bmod 1) \in S^1$$

other ways to write $\rho_x(F)$:

set $a(x) = a_\varphi(x) := F(x) - x$

$a : \mathbb{R} \to S^1 \to \mathbb{R}$ 1-periodic :

Pf. $a(x+1) = F(x+1) - (x+1) = F(x) - x$ ◁

$$F(x) = x + a(x)$$

$$F^k(x) =: x + \boxed{a_k(x)} \longleftarrow ?$$

$$F^2(x) = F(x + a(x)) = x + a(x) + a(x + a(x))$$

$$= x + \underbrace{a(x) + a(\varphi(x))}_{a_2(x)}$$

$$a_k(x) = a(x) + a(\varphi(x)) + \dots + a(\varphi^{k-1}(x))$$

$$\Rightarrow \rho_x(F) = \lim_{k\to\infty} \frac{F^k(x) \boxed{-x}}{k} \longleftarrow \text{can always add}$$

$$= \lim_{k\to\infty} \frac{a_k(x)}{k}$$

Ex. 1) $\varphi(x) = R_\alpha(x) = x+\alpha \longleftarrow$ rotation in α

$$F(x) = x + \underbrace{\alpha}_{a(x)} + \text{an integer}$$

$$F^k(x) = x + \underbrace{k\alpha}_{a_k(x)}$$

$$\Rightarrow \rho(R_\alpha) = \alpha$$

2) $\varphi(x) = x + \alpha + \varepsilon \sin(2\pi x)$

Things get complicated

$\rho(\varphi)$ depends on α & ε !

Pf of the proposition

• Independence of x

F^k = a lift of φ^k : monotone ↗

$F^k(x+1) = F^k(x) + 1$

$$x \le y \le x+1$$

$$\Rightarrow \quad F^k(x) \le F^k(y) \le F^k(x) + 1$$

$$\frac{1}{k}\left|\underbrace{F^k(x) - F^k(y)}_{\substack{\wedge\\ 1}}\right| \le \frac{1}{k} \longrightarrow 0$$

when $x \le y \le x + \overset{\mathbb{Z}}{\overset{\Downarrow}{n}}$

$$\frac{1}{k}\left|F^k(x) - F^k(y)\right| \le \frac{n}{k} \longrightarrow 0$$

• Existence

Lemma Assume a_k an arb seq

s.t. $a_{n+m} \le a_n + a_m \boxed{+L}$

e.g. $L = 0$

$a_{n+m} \le a_n + a_m$

subadditive

$\Rightarrow \lim\limits_{k\to\infty} \dfrac{a_k}{k}$ exists in $\mathbb{R} \cup \{-\infty\}$

Pf of the Lemma set $a := \liminf_{k\to\infty} \frac{a_k}{k}$

Assume $a > -\infty$

$a = -\infty$: ex

Take n so large that

$$\frac{a_n}{n} \le a + \varepsilon/3 \quad \text{and} \quad \frac{L}{n} \le \frac{\varepsilon}{3}$$

Note

$$a_{2n} \le 2a_n + L$$
$$a_{3n} \le 3a_n + 2L$$
$$\cdots$$
$$a_{\ell n} \le \ell a_n + (\ell-1)L$$

Write $k = n \cdot \ell + r$, $\quad 0 \le r \le n-1$

$$\frac{a_k}{k} = \frac{a_{n\ell+r}}{k} \le \frac{a_{n\ell}+a_r+L}{k}$$

$$\le \frac{a_{n\ell}}{k} + \frac{\overbrace{a_r+L}^{\text{bounded}}}{k} \to 0 \quad k\to\infty$$

$$\le \frac{\ell a_n + (\ell-1)L}{\ell n + r} + \frac{a_r+L}{k}$$

$$\le \frac{\ell a_n}{\ell n+r} + \frac{(\ell-1)L}{\ell n+r} + \frac{a_r+L}{k}$$

$$\le \underbrace{\frac{a_n}{n}}_{< a+\varepsilon/3} + \underbrace{\frac{L}{n}}_{< \varepsilon/3} + \underbrace{\frac{a_r+L}{k}}_{< \varepsilon/3 \;\; k\to\infty} \le a + \varepsilon$$

$\Rightarrow a < \frac{a_k}{k} < a+\varepsilon \Rightarrow$ lim exists $\triangle$

Back to the pf.

Recall $a_k(x) = F^k(x) - x =: a_k$

<u>Claim</u> $a_{n+m} \le a_n + a_m + 1$

<u>Pf</u>

$$a_{n+m} = F^{n+m}(x) - x$$

$$= \underbrace{F^n(F^m(x))}_{F^n(y)} - \underbrace{F^m(x)}_{y} + \underbrace{F^m(x) - x}_{a_m}$$

$$x + k \le y \le x + k + 1$$

$$\left.\begin{array}{l} F^n \nearrow \\ F^n(z+1) = F^n(z) + 1 \end{array}\right\} \Rightarrow$$

$$\underbrace{F^n(x+k)}_{\overset{\shortparallel}{F^n(x) + k}} \le F^n(\ y\ ) \le \underbrace{F^n(x + k + 1)}_{F^n(x) + k + 1}$$

$$F^n(\ y\ ) - \ y \le \underbrace{F^n(x) + k + 1 - x + k}_{F^n(x) - x + 1 = a_n + 1}$$

$\triangleleft$

Lemma + Claim $\Rightarrow$ lim $\in [-\infty, \infty)$ exists

<u>Ex</u>: $\frac{a_n}{n}$ bounded from below

$\Rightarrow$ lim $\in \mathbb{R}$

$\blacktriangleleft$