Go through the Thm
and pf again

Lecture 7

01/25 - 2022

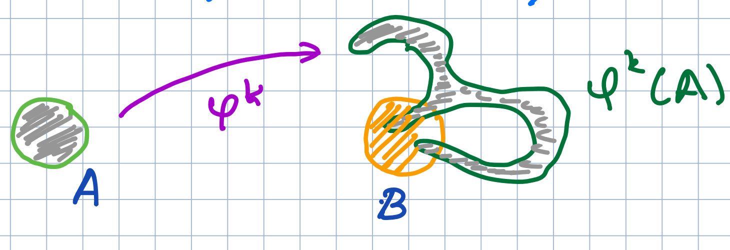
- A stronger ergodicity property: Mixing (Digression)

Def $\varphi : (M, \mu) \circlearrowleft$ is mixing if
$\forall A, B$ (measurable)
$$\mu(\varphi^{-k}(A) \cap B) \xrightarrow[k \to \infty]{} \mu(A)\mu(B)$$

Roots in probability theory

Rmk & Observations

- $\varphi^{-k}(A) = \{x \mid \varphi^k(x) \in A\}$ is defined even when φ is not invertible
- Prefer to think as
$$\mu(\varphi^k(A) \cap B) \longrightarrow \mu(A)\mu(B)$$
- $\mu(A) \neq 0$, $\mu(B) \neq 0$
$\Rightarrow$ $\varphi^k(A) \cap B \neq \emptyset$ $\quad k \gg 0$



- Topological counterpart (top mixing)
$\forall$ open sets U, V
$$\varphi^{-k}(U) \cap V \neq \emptyset \quad \forall \text{ large } k$$

- Mixing $\Rightarrow$ Top Mixing
  when $\mu(\text{open}) > 0$

- Mixing $\Rightarrow$ Ergodic

Pf • Ergodic $\Leftrightarrow$ $\forall$ inv set A

$$\underbrace{\mu(A) = 0 \text{ or } \mu(A) = 1}$$

$$\Leftrightarrow \mu(A)\,\mu(\underbrace{M \setminus A}_{B}) = 0$$

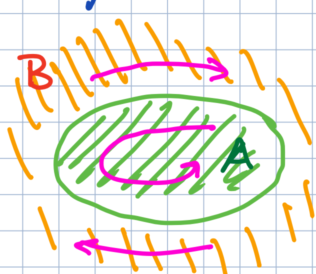
• φ mixing, $\underbrace{A \text{ invariant}}$: Need
$B = M \setminus A$

$$\underbrace{\varphi^{-k}(A)}_{A} \cap \underbrace{B}_{M\setminus A} = \emptyset$$

$$0 = \mu(\varphi^{-k}(A) \cap B) \longrightarrow \underbrace{\mu(A)\,\mu(M \setminus A) = 0}$$

$\Rightarrow$ ergodic

$\lhd$



A & B
never mix

Rmk • Similarly for flows
• Other notions of mixing
(weak, etc) ...

Examples

Ex

- isometries are not (top) mixing

$\Rightarrow$ rotations of S^1, translations of $\mathbb{T}^k$, lin flows on $\mathbb{T}^k$ are not mixing

mixing $\Rightarrow$ ergodicity

mixing $\not\Leftarrow$ unique ergodicity

- Hyperbolic $A : \mathbb{T}^2 \to \mathbb{T}^2$ are mixing [KH]

$$\begin{tikzcd}
 & \text{ergodicity} & \\
\text{mixing} \arrow[ur, Rightarrow] \arrow[rr, Rightarrow, "\times"] & & \text{unique ergodicity} \arrow[ul, Rightarrow] \arrow[ll, Rightarrow, "\times"']
\end{tikzcd}$$

- **Examples continued:**
  Shift Transformations

Setting:

- $\mathbb{Z}_n = \{0, \dots, n-1\}$ an alphabet
- $M = \mathbb{Z}_n^{\mathbb{Z}} = \{x = \dots x_{-1} x_0 x_1 x_2 \dots\}$
  $=$ bi-inf sequences
  compact metric space

- $\varphi : M \longrightarrow M$ shift to the left

$$\varphi(x)_i = x_{i+1} \quad , \quad \text{homeo}$$

$$\dots \; x_{-2} \; x_{-1} \; \boxed{x_0} \; x_1 \; x_2 \; x_3 \; \dots$$

- Rmk $\quad \mathbb{Z}_n$ a gp
  $\Rightarrow$ $M = \mathbb{Z}_n^{\mathbb{Z}}$ is a compact top. gp
  φ is a gp homomorphism

- Inv measures. (Bernoulli measures)

  - Fix $\quad 0 < p_i < 1 \quad i = 0, \dots, n-1$
    $\sum p_i = 1$ probability of $a_i \in \mathbb{Z}_n$

- Cylinders

$I = (i_1, \dots, i_s)$ multiindex

$Y = (a_1, \dots, a_s) \in \mathbb{Z}_m^s$

$$C_Y^I = \{x \mid x_{i_j} = a_{i_j} \mid \forall\, i_j \in I\}$$

These also form a base of the top of $\mathbb{Z}_m^{\mathbb{Z}}$

$$\dots\dots x_{-1} x_0 x_1 \dots \underset{\displaystyle a_{i_j}}{\overset{\displaystyle x_{i_j}}{\shortparallel}} \dots$$

<u>Def</u>

$p_i = \mu(a_i)$

$$\mu(C_Y^I) = p_{a_1} \cdot p_{a_2} \cdot \dots \cdot p_{a_s}$$

$\Downarrow$

μ is the product measure

$\Rightarrow$ Extend to a measure

$\Rightarrow$ a φ invariant measure μ on M (Probability, Borel)

<u>Ex</u> $p_i = \frac{1}{n}$: all a_i have the same prob

$\Rightarrow$ the Haar measure on $\mathbb{Z}_n^{\mathbb{Z}}$ (invariant under right or left translations, $\mu(\text{open}) > 0$

Recall top properties of φ:

- dense periodic pts = periodic seq x_i
  $p(k) = n^k \leftarrow$ # of per pts of per k
- top transitive: $\exists$ a dense orbit

On the measure theory side:

Thm φ is mixing for μ

$\Rightarrow$

Cor φ is ergodic & top mixing

Rmks: • φ is not uniquely ergodic
(per. orbits or different $\{p_i\}$)
and not minimal

- similarity with hyperbolic
  $A: \mathbb{T}^n \to \mathbb{T}^n$

Pf

- Observations: [KH] enough to check mixing when A & B are cylinders

$$\cdots\; x_{i_1} \;\cdots\; x_j \;\cdots\; x_{i_2} \;\cdots$$
$$\cdots\; \overset{\shortparallel}{a_{i_1}} \;\cdots\; \overset{\shortparallel}{a_{i_2}} \cdots = Y$$

Need $\mu(\varphi^{-k}(C_Y^I) \cap C_X^J) \to \mu(C_Y^I)\,\mu(C_X^J)$ for any two such cylinders different length

- Note . $\varphi^{-k}(C_Y^I) = C_Y^{I+k}$

$I+k = (i_1+k, \ldots, i_s+k)$

$\Rightarrow \forall\, I \,\&\, J$

$$\varphi^{-k}(C_Y^I) = C_Y^{I+k}$$

disjoint from J when k is large

- Recall $\qquad Y = (a_1, \dots, a_s)$

$$\mu(C_Y^I) = P_{a_1} \cdots P_{a_s}$$

- $L \cap J = \emptyset \qquad Y = (a_1 \dots a_s)$

$$C_Y^L \cap C_X^J = C_{Y \cup X}^{L \cup J} \qquad X = (b_1, \dots, b_r)$$

$$\Rightarrow \boxed{\underbrace{\mu(C_Y^L \cap C_I^J)}_{a_1 \dots a_s b_1 \dots b_r} = \underbrace{\mu(C_Y^L)}_{a_1 \dots a_s} \cdot \underbrace{\mu(C_X^J)}_{b_1 \dots b_n}}$$

- $L = I + k \qquad k$ is large

disjoint from J

$$C_Y^L = C_Y^{I+k} = \varphi^{-k}(C_Y^I) \qquad \text{disjoint}$$

$$\Rightarrow \quad \varphi^{-k}(C_Y^I) \cap C_X^J = C_Y^L \cap C_X^J$$

$$\Rightarrow \quad \mu(\varphi^{-k}(C_Y^I) \cap C_X^J) = \mu(\varphi^{-k}(C_Y^I))\, \mu(C_X^J)$$

$\Rightarrow$ mixing

$\triangleleft$

Probabilistic Interpretation

- $\mathbb{Z}_2 = \{0, 1\}$; $P_0 = P_1 = \frac{1}{2}$ unbiased coin
- $M =$ bi-inf sequences of 0 & 1's
- each sequence

  $=$ sequence of coin tosses

  $0 =$ heads

  $1 =$ tails

  an experiment on a trial

- $I = \{0, \dots, m\}$

  $Y = \{b_0, \dots, b_m\}$ $\qquad b_i = 0$ or 1

  $C^I_Y =$ event : the first $m+1$ tosses give outcome Y : $\mu(C^I_Y) = \frac{1}{2^{m+1}}$

$$\varphi^{-k}(C^I_Y) = C^{k+I}_Y$$

$=$ event : the tosses $k, \dots, k+m$ give outcome Y

$$\lim_{k \to \infty} \frac{1}{k} \#\{0 \le i \le k-1 \mid \varphi^k(x) \in C^I_Y\}$$

$=$ frequency with which the sequence Y occurs in x

Ergodicity $\Rightarrow$ for almost all trials x

$$\underbrace{\text{the frequency}}_{\substack{\text{something we}\\ \text{can measure}\\ \text{by experiment,}\\ \text{"statistics"}}} = \underbrace{\text{probability of } Y}_{\underbrace{\mu(C_Y^I)}_{\text{abstract notion}}}$$

Ex. Interpret mixing in terms of conditional probability.

Lecture 8
01/27 - 2022

• Existence of invariant measures

M compact metric space (separable)
$\varphi: M \longrightarrow M$ homeo or just C^0
We have used:

Fact: φ has an invariant (ergodic) measure.

Goal: justify this

Thm (Krylov - Bogolubov)
M compact, $\varphi: M \xrightarrow{C^0} M$
$\Rightarrow \exists$ an invariant Borel probability measure

Preliminaries: $\qquad M$ as above

- $C^0(M)$ = Banach space with
  $\|f\| = \sup_{x \in M} |f(x)|$

- Dual space:

$$C^0(M)^* = \{ \Phi : C^0(M) \to \mathbb{R} \mid \text{bounded} \}$$

Thm (Riesz Representation Thm)
$C^0(M)^*$ = the space of finite Borel measures μ (not necessarily pos)

$$\Phi(f) = \int_M f \, d\mu$$

Rmk
- $\mu = \mu_+ - \mu_-$ ← pos. measures
- Φ pos : $f \geq 0 \Rightarrow \Phi(f) \geq 0$
  $\Rightarrow \mu$ is pos
- $\Phi(1) = 1 \Rightarrow \mu$ is probability: $\int \mu = 1$
- $\Phi(f \circ \varphi) = \Phi(f) \quad \forall f$
  $\Leftrightarrow \mu$ is φ invariant

Pf

Idea: For $x \in M$ set

$$\mu_x(U) = \lim_{k\to\infty} \frac{1}{k} \{0 \le i \le k-1 \mid \varphi^i(x) \in U\}$$

as in Birkhoff ergodic theorem, or

$$\Phi_x(f) = \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} f(\varphi^i(x))$$

and define μ_x by

Riesz

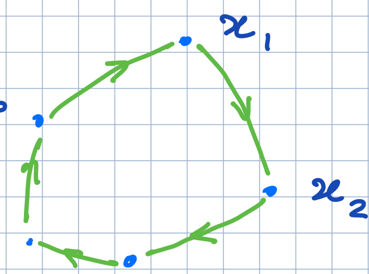
$$\Phi_x(f) = \int f \, d\mu_x$$

Then μ_x is an invariant probability measure assuming that the limits exist

Ex: x is k-periodic

$x = x_0$, $x_1 = \varphi(x)$, ... $x_i = \varphi^i(x)$, $x_k = \varphi^k(x) = x_0$

$\varphi^k(x) = x = x_0$



$\Rightarrow \quad \mu_x = \frac{1}{k} \sum \delta_{x_i}$ invariant Borel probability measure

Implementation

- Let $f_j \in C^0(M)$, $j = 1, 2, \dots$ be a countable collection dense in C^0 (with respect to the sup-norm.)

- Pick x and consider

$$a_k^j = \frac{1}{k} \sum_{i=0}^{k-1} f_j(\varphi^i(x)) \longleftarrow \text{bounded } \forall j.$$

diagonal process

$\Rightarrow$ $k_s(1) \xrightarrow[s\to\infty]{} \infty$ $\quad a^1_{k_s(1)}$ converges as $s \to \infty$

$\Rightarrow$ $k_s(1)$ contains a subsequence

$k_s(2) \xrightarrow[s\to\infty]{} \infty$ $\quad a^2_{k_s(2)}$ also converges

$\dots$

Set $k_s = k_s(s)$ subsequence in all of them

$\Rightarrow$ $a^j_{k_s} \xrightarrow[k_s \to \infty]{} a^j$ $\quad \forall j$:

$$\exists \lim_{k_s \to \infty} \frac{1}{k_s} \sum_{i=0}^{k_s - 1} f_j(\varphi^i(x)) = a^j \quad \forall j$$

$\Rightarrow$

$$\exists \lim_{k_s \to \infty} \frac{1}{k_s} \sum_{i=0}^{k_s - 1} f(\varphi^i(x)) =: \Phi_x(f)$$

exists

$\{f_j\}$ dense in $C^0(M)$

Riesz Representation theorem

$\Rightarrow \exists \; \mu_x$ s.t.

$$\Phi_x(f) = \int f \, d\mu_x$$

check (Ex):

- $f \geq 0 \Rightarrow \Phi_x(f) \geq 0$
- $\Phi_x(1) = 1$

$\Big\}$ clear

- $\Phi_x(f \circ \varphi) = \Phi_x(f)$ $\}$ calculation

$\Rightarrow \mu_x$ is positive, probability and invariant

$\triangleleft$

<u>Rmk</u> $\quad \operatorname{supp} \mu_x \subset \overline{\mathcal{O}(x)}$

Rmk (Ex) A short cut with more functional analysis:

Set
$$\Phi_x^{(k)}(f) := \frac{1}{k}\sum_{i=0}^{k-1} f(\varphi^i(x))$$
$$= \Big(\frac{1}{k}\sum_{i=0}^{k-1} \delta_{\varphi^i(x)}\Big)(f)$$

$$|\Phi_x^{(k)}(f)| \overset{=}{\leq} \|f\|$$

$$\Rightarrow \quad \|\Phi_x^{(k)}\| \underset{=}{\leq} 1 :$$

$$\Phi_x^{(k)} \in \underbrace{\text{unit ball in } C^0(M)^*}_{\text{week* compact (sequentially)}}$$

Alaoglu thm $\Rightarrow$ $\Phi_x^{(k)}$ contains a pt-wise converging subsequence:

Alaoglu's thm

$$\Phi_x^{k_i}(f) \longrightarrow \Phi_x(f) \quad \forall f$$

This is essentially the def of Φ_x

Now finish the pf as above. $\triangleleft$

Rmk Can also take
$$\lim \frac{1}{2k+1}\sum_{i=-k}^{k} f(\varphi^i(x))$$
when φ is invertible

Rmk How often does the lim exist?

$$\underbrace{\lim_{k\to\infty} \frac{1}{k} \underbrace{\sum_{i=0}^{k-1} f(\varphi^i(x))}_{\sum_{i=0}^{k-1} \delta_{\varphi^i(x)} f}}_{\Phi_x^k(f)}$$

Ex or [KH]

Answer: for any φ-inv μ the limit exist for μ-a.e. x $\forall f$

Hint: combine the Birkhof ergodic theorem with the pf of Krylov-Bogolubov thm

- How do ergodic measures enter this picture?

Notation: $\mathcal{M}_\varphi = \{ \varphi\text{-inv. prob. Borel measures} \}$

$$\mathcal{M} = \mathcal{M}_\varphi \hookrightarrow C^0(M)^*$$

$$\mu \longmapsto \Phi_\mu := \left(f \longmapsto \int_M f \, d\mu \right)$$

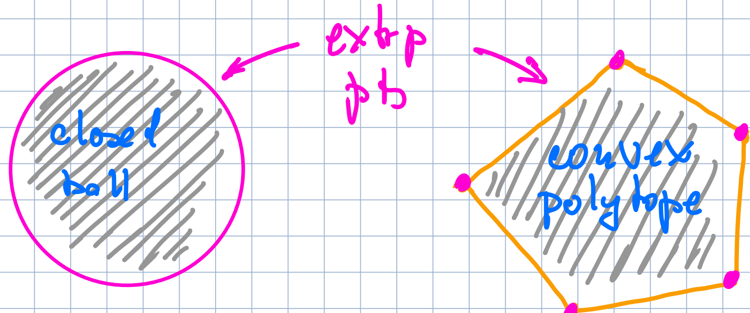
- The image is in the unit sphere $\|\Phi\| = 1$ and weak* compact & $\Phi_\mu(\mathbb{1}) = 1$
- $\mathcal{M}_\varphi$ is convex:

$$0 \leq t \leq 1 \quad , \quad \underset{\mathcal{M}_\varphi}{\underset{\cap}{\mu}} = (1-t)\underset{\mathcal{M}_\varphi}{\underset{\cap}{\mu_0}} + t \underset{\mathcal{M}_\varphi}{\underset{\cap}{\mu_1}}$$

$$\Leftarrow$$

<u>Def</u> μ is an <u>extreme</u> pt of $\mathcal{M}$ if for any such decomposition $t = 0$ or $t = 1$

can be any convex set



Notation: $\mathrm{Ext}(\mathcal{M})$

Cor $\underbrace{\mathrm{Ext}(\mathcal{M})}_{\text{ext pts}} \neq \emptyset$

Rmk In general:

$$\overline{\text{convex Hull}(\underbrace{\mathrm{Ext}(\mathcal{M})}_{\text{ext. pts}})} = \mathcal{M}$$

closure

convex

In general, when $\dim = \infty$, even the fact that

$$\mathrm{Ext}(\mathcal{M}) \neq \emptyset$$

is not obvious

Thm $\left\{\begin{matrix}\text{Ergodic} \\ \text{measures}\end{matrix}\right\} = \left\{\begin{matrix}\text{Extreme} \\ \text{pts of } \mathcal{M}_\varphi\end{matrix}\right\}$

Pf

"$\supset$" $\mu \in \mathcal{M}_\varphi$, not ergodic:

$\exists A$ with $0 < \mu(A) < 1$

Set $\mu_X(Y) = \dfrac{\mu(X \cap Y)}{\mu(X)}$ $\Big\}$ restriction to X measure

$\mu_0 = \mu_A$, $\mu_1 = \mu_{M \setminus A}$

$\Rightarrow$ $\mu = \mu(A)\mu_A + (1 - \mu(A))\mu_{M \setminus A}$

$\Rightarrow$ μ is not an extreme pt

"$\subset$" <u>Idea</u>

- μ_0, μ_1 = extreme pts, ($\Rightarrow$ ergodic)

enough to have μ_0 extr

- $\mu_0 \neq \mu_1$: $\exists A$ $\mu_0(A) \neq \mu_1(A)$

Form:

$\mu = (1-t)\mu_0 + t\mu_1$ not extreme : $t \neq 0, 1$

<u>Want to show not ergodic</u>

(A particular case)

Assume it is

Birkhoff.

$$\frac{1}{k}\sum_{i=0}^{k-1} \chi_A(\varphi^i(x)) \longrightarrow \mu(A) = (1-t)\mu_0(A) + t\mu_1(A)$$

$$\frac{1}{k}\sum_{i=0}^{k-1} \chi_A(\varphi^i(x)) \longrightarrow \mu_0(A)$$

$\Rightarrow \quad \mu_0(A) = \mu_1(A) \quad \rightarrow\leftarrow$

A catch: - need x to "a.a." for μ_0 & μ_1
- might not exist
- But then $\operatorname{supp}\mu_0 \cap \operatorname{supp}\mu_1 = \emptyset$ and μ is again not ergodic

- A more serious problem: not every non-extreme pt can be decomposed as $\mu = (1-t)\mu_0 + t\mu_1$
- Not literally, but ...

Need some functional analysis: Choquet's thm

Cor Every $\varphi : M \to M$ has an ergodic measure

Cor The following def of unique ergodicity are equivalent:

- ergodic and an ergodic measure is unique
- ergodic and inv measure is unique.

How common is ergodicity?

Setting:

- M a compact manifold (perhaps with boundary or corners)
- μ = smooth measure (Lebesgue)

E.g. M = closed ball or I^n = cube

- $H = \{\varphi : M \to M \mid \mu\text{-pres homeo}\}$ with sup-topology:

$$d(\varphi, \psi) = \sup_{x \in M} (\varphi(x), \psi(x))$$

H has the Baire property: a countable intersection of open & dense sets is dense

a residual set: dense G_δ or more generally containing a dense G_δ

Thm (Oxtoby – Ulam)

Ergodic φ form a residual subset of $\mathcal{H}$, dim ≥ 2

Ex. Show that not true when dim = 1

Cor Top transitive φ (i.e. with a dense orbit) form a residual subset of $\mathcal{H}$

Rmk • Nothing like that is true for C^∞-diffeos pres μ!

KAM

At least when dim $M = 2$ or in the Ham case or...

...

• C^1 or C^k – more subtle

Avila – Crovisier – Wilkinson

ArXiv 1408.4252

• Not easy to construct $\varphi : D^2 \to D^2$ with a dense orbit

Direct pf of Cor - Outline

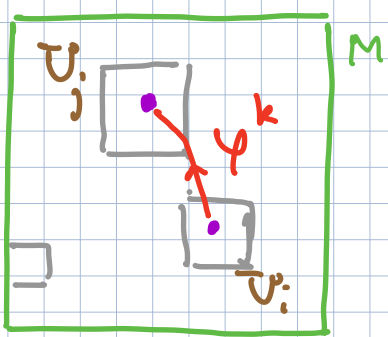
Following [Oxtoby]

- $M =$ square $[0,1]\times[0,1]$

  $\mu =$ Lebesgue measure

  $\{\varphi : M \to M \ \ \mu\text{-pres. homeo}\} = H$

- $\{U_i\} =$ collection of open squares in M with rational vertices (top base)



$$E_{ij} = \{\varphi \mid \exists k \geq 1 : \varphi^{-k}(U_j) \cap U_i \neq \emptyset\}$$

clear

key pt

<u>Claim</u> $\forall\, i,j$ E_{ij} is <u>open</u> and <u>dense</u>

Thm $\Leftarrow$ Claim:

$$\bigcap_{i,j} E_{i,j} =: E \leftarrow \text{residual}$$
$$\begin{gathered}\Psi \\ \varphi\end{gathered}$$

$G_j := \bigcup_{k=1}^{\infty} \varphi^{-k}(U_j)$ is open & dense:

$$U_i \cap \bigcup_{k=1}^{\infty} \varphi^{-k}(U_j) \neq \emptyset \Leftarrow \varphi \in E$$

Baire: $G = \bigcap G_j$ is residual in M

$\Rightarrow G \neq \emptyset$

$x \in G \qquad \forall j: \quad x \in \bigcup_{k=1}^{\infty} \varphi^{-k}(U_j)$

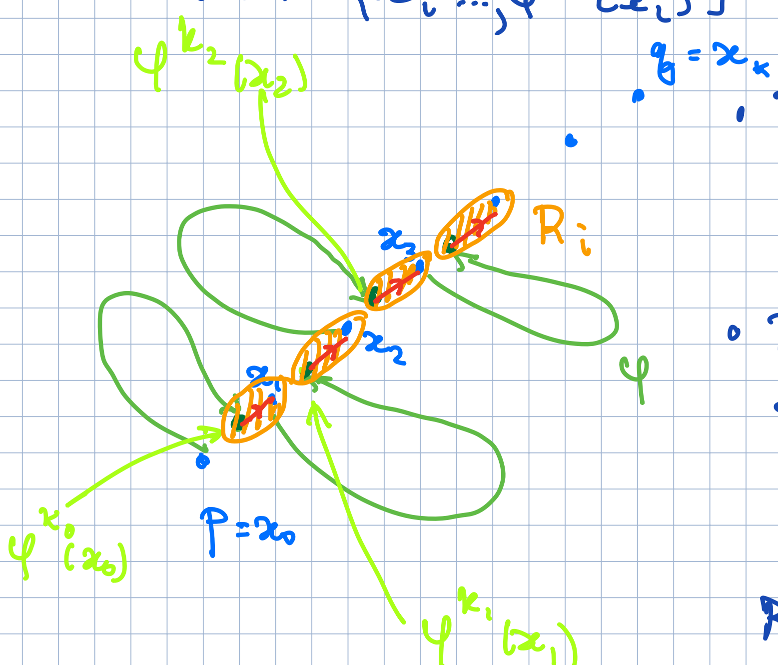
$\Rightarrow \exists k: \quad \varphi^k(x) \in U_j$

$\Rightarrow \Theta(x)$ is dense.

Rmk We have shown that for a residual set of φ's the set of x with $\Theta(x)$ dense is residual.

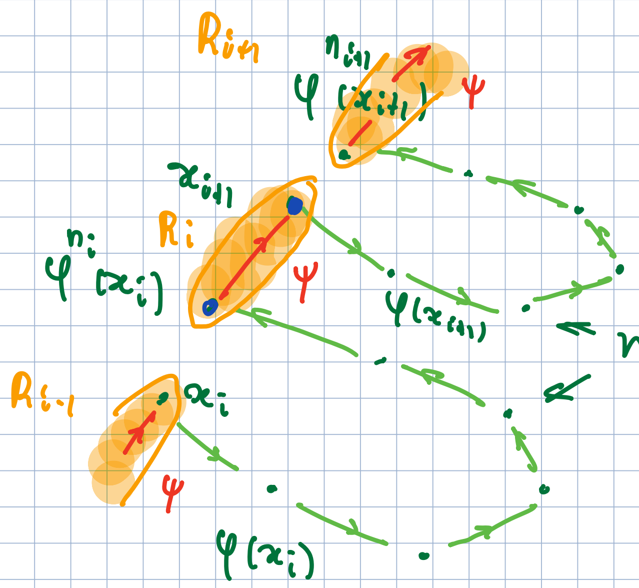
Idea of the pf of the Claim

- Given i, j and φ need to find an arbitrarily small ψ and $p \in U_i$ s.t. $(\psi\varphi)^k(p)\ U_j$ for some k
  $\Rightarrow \psi\varphi \in E_{ij}$ & $\psi\varphi \approx \varphi$
- Can assume that periodic pts of φ form a meager set
  – such φ's form a residual set
- Pick $p \in U_i$ & $q \in U_j$ and "connect" them by a seg. disj orbit $\{x_i, \dots, \varphi^{n_i}(x_i)\}$



- F = union of these orbits
- R_i's : disjoint small open sets
  $F \cap R_i = \{\varphi^{n_i}(x_i), x_{i+1}\}$

$R = \coprod R_i$



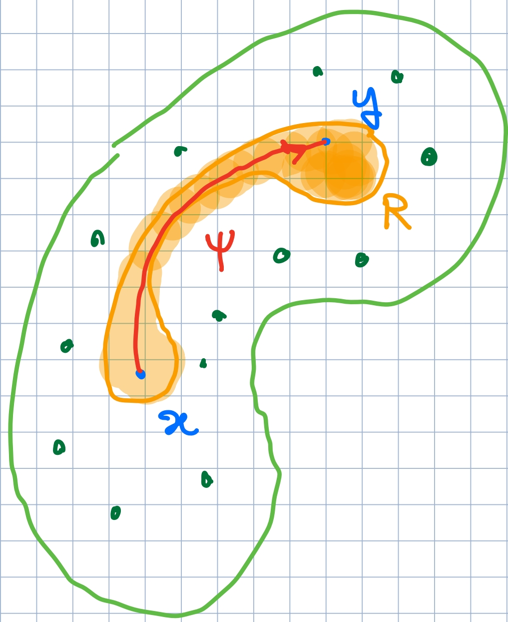
Local pic enlarged

none of these other pts are in R

- ψ : . $\operatorname{supp}\psi \subset \coprod R_i$
  . $\psi(\varphi^{n_i}(x_i)) = x_{i+1}$
  . $\|\psi\|_{C^0}$ is small

- $\Rightarrow \quad (\varphi\psi)^{n_0+\dots+n_k}(p) = q$

Rmk $\quad x_i, \varphi^{n_i}(x_i), x_{i+1}, \varphi^{n_{i+1}}(x_{i+1})$
are quite close
$\Rightarrow$ size $(R_i) \sim d(x_i, \varphi^{n_i}(x_i))$
$\Rightarrow$ cannot make
$\|\psi\|_{C^1}$ small $\quad \triangleleft$

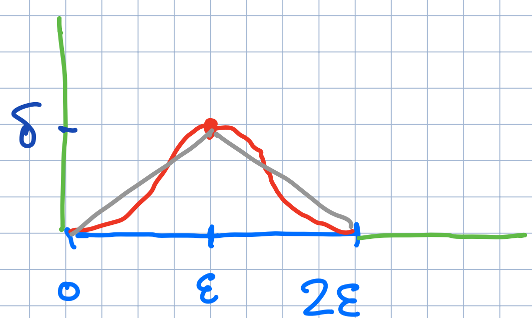


$p \approx q$, $\operatorname{supp} \psi \subset R$

$\Rightarrow \psi \approx id$

$\not\Rightarrow \psi \overset{C^1}{\approx} id$

$f(\varepsilon) = \delta$



• if δ is small

$\Rightarrow f$ C^0 small

$\max |f| = \delta$

• f C^1-small?

$f' \sim \dfrac{\delta}{\varepsilon}$

$\Rightarrow$ If $\delta \ll \varepsilon$ can make f C^1-small

If $\delta \approx \varepsilon$ cannot