Lecture 5
01/18-2022

Pf: Poincaré Recurrence

Observation: $\mu(X \cap \varphi^k(X)) = 0 \quad \forall k$

$$\Rightarrow \mu(X) = 0$$

Pf: $\mu(X \cap \varphi^k(X)) = 0 \Leftrightarrow \mu(\varphi^i(X) \cap \varphi^j(X)) = 0$

Apply φ^{-i}

$$\mu(\varphi^i(X) \cap \varphi^j(X)) = \mu(X \cap \varphi^{j-i}(X)) = 0 \qquad k = j-i$$

all disjoint

$$\mu\Big(\coprod_{k=0}^{\infty} \varphi^k(X)\Big) = \sum \mu(\varphi^k(X)) = \infty \cdot \mu(X) \le 1$$

$$\Rightarrow \mu(X) = 0$$

Given $U \subset M$, need to show a.a. $x \in U$ come back to U:

$$X = \{x \in U \mid \varphi^k(x) \notin U \ \forall k = 1, 2, \dots\}$$

Claim $X \cap \varphi^k(X) = \emptyset$

Indeed: $\varphi^k(X) \cap \underset{\underset{X}{\cup}}{U} = \emptyset$

Observation $\Rightarrow \mu(X) = 0$. $\triangleleft$

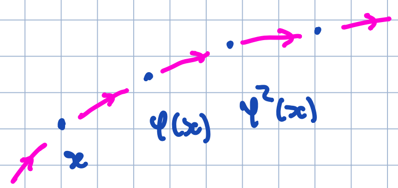
• Birkhoff Ergodic Theorem

Setting: as before

- $\varphi : M \to M$, μ φ-invariant, prob
- $f \in L^1(M)$ — $\int |f| d\mu < \infty$

might or not exist

- $\bar{f}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\varphi^i(x))$ ⟵ time average



Averaging f along the orbit of x

<u>Ex.</u> x is periodic with min period k:

$$x = \varphi^k(x)$$

$$\Rightarrow \quad \bar{f}(x) = \frac{1}{k}\left(f(x) + f(\varphi(x)) + \dots + f(\varphi^{k-1}(x)) \right)$$

<u>Thm</u> (Birkhoff Ergodic Thm)

- $\bar{f}(x)$ exists for a.a. x,
- $\Downarrow$
- $\bar{f}$ is φ-invariant: $\bar{f} \circ \varphi = \bar{f}$
- $\bar{f}$ is L^1 & $\int \bar{f} d\mu = \int f d\mu$

Non-trivial; see e.g. [KH] § 4.1 (e)
[FKS] § 1.2

Rmk: Two perspectives:

1. μ is natural and fixed; φ varies within the class of μ-preserving maps, or a smaller class.

Ex. Hamiltonian systems:
a symplectic form preserved by φ and more...
$\Rightarrow \omega^n$ a natural inv measure, etc

2. φ is given and we are interested in all invariant measures...

Pf of implications:

- invariance: $\bar{f}(\varphi(x)) = \bar{f}(x)$ for a.a. x

$$\bar{f}(x) = \lim_{n\to\infty} \frac{1}{n}\left(f(x) + \dots + f(\varphi^{n-1}(x))\right)$$

both $\exists$ for a.a. x

$$\bar{f}(\varphi(x)) = \lim_{n\to\infty} \frac{1}{n}\left(f(\varphi(x)) + \dots + f(\varphi^{n}(x))\right)$$

$$\bar{f}(\varphi(x)) - \bar{f}(x) = \lim_{n\to\infty} \frac{1}{n}\left(f(\varphi^{n}(x)) - f(x)\right)$$

$f \in L^1$

φ pres μ

$$= 0 \quad \text{for a.a. } x$$

- integral:

$$\int \bar{f}\, d\mu = \int \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\varphi^{i}(x))\, d\mu(x)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \int f\circ\varphi^{i}\, d\mu$$

all equal to $\int f\, d\mu$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot n \cdot \int f\, d\mu$$

$$= \int f\, d\mu$$

$\triangleleft$

Variants

- $$\bar{f}(x) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{i=-n}^{n} f(\varphi^i(x))$$

a.e. equal to the previous one

- For flows:

$$\bar{f}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\varphi^t(x))\,dt$$

$$\overset{a.e.}{=} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(\varphi^t(x))\,dt$$

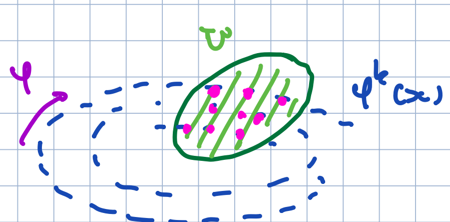
Similar statement: $\bar{f}$ exists a.e. and

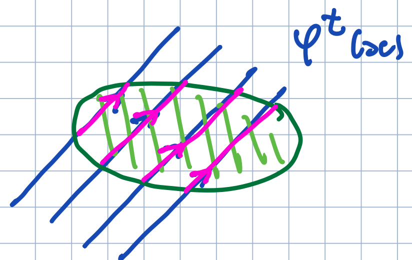
$$\int \bar{f}\,d\mu = \int f\,d\mu$$

Ex-Interpretation

$U \subset M$, $f = \chi_U$ characteristic function

$\Rightarrow \bar{f} =$ average time $\varphi^i(x)$ or $\varphi^t(x)$ spends in U





$$\bar{\varphi}(x) = \lim \frac{1}{n}\left(f(x) + f(\varphi(x)) + \ldots + f(\varphi^i(x)) + \ldots\right)$$

• Ergodicity and unique ergodicity

Def φ on μ is ergodic if for every measurable inv set A either $\mu(A) = 1 = \mu(M)$ or $\mu(A) = 0$

Rmk: no "proper" inv. subsets

measure theoretic

Prop: (φ, μ) is ergodic

$\Leftrightarrow$ $\forall f \in L^1$ $\bar{f} = \text{const} = \int f \, d\mu$:

$$\bar{f}(x) = \underbrace{\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\varphi^i(x))}_{\text{time average}} = \underbrace{\int_M f \, d\mu}_{\text{space average}}$$

for a.a. x

$\Leftrightarrow$ The only inv L^1-functions are constant functions

Equivalent, clearly.

$\Leftrightarrow$ $\forall$ a.a. x the average time $\underbrace{\varphi^k(x) \text{ spends}}_{\text{observed probability}}$ in $U = \mu(U)$

probability of U

Pf. • φ is ergodic:

characteristic function

every invariant χ_U is (a.e.) constant

or every inv U is a.e. M or $\emptyset$.

look at $f \geq$ const

use the fact that χ_U's are L^1-dense

• every $f \in L^1$ is constant (a.e.)

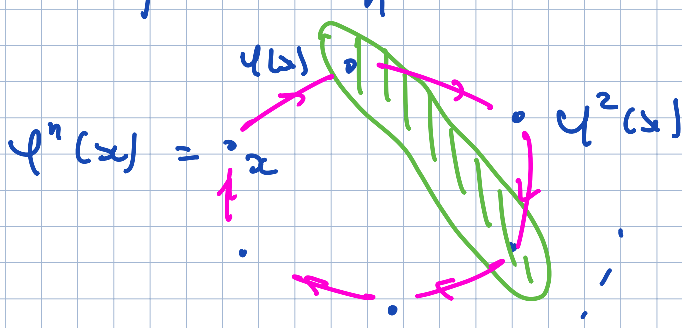
$\triangleleft$

Ex. • $x, \varphi(x), \dots, x = \varphi^n(x)$ a periodic orbit

• μ the associated measure

$$\mu = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\varphi^i(x)}$$

$$\Leftrightarrow \quad \mu(U) = \frac{1}{n} \left| \varphi^i(x)\text{'s in } U \right|$$



$\Rightarrow$ μ is ergodic

• **Def** φ is <u>uniquely</u> (or strongly) ergodic if $\exists$ exactly one invariant measure μ for which φ is ergodic.

natural measure

<u>Rmk</u> φ is ergodic with respect to μ, look at all inv. measures, none of them is ergodic ... More later when we study the space of inv. measures

Examples — wait!

<u>But an important pt:</u>

φ uniquely ergodic for μ $\leftarrow$ <u>"continuous"</u>

open sets: $\mu > 0$

$\Rightarrow$ no periodic orbits (e.g. fixed pts)

every per orbit gives rise to an ergodic measure

Thm M is compact, $\mu(M)=1$, $\varphi \in C^0$, μ is invariant

1) • φ ergodic $\Rightarrow$ $\exists$ an orbit dense in supp φ

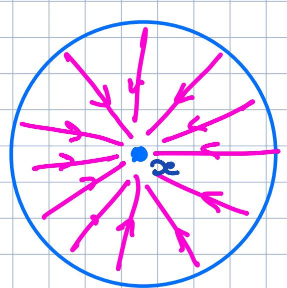
2) • φ uniquely ergodic
$\Rightarrow$ every orbit is dense in supp φ

Cor Assume that $\mu(\text{open}) > 0$ $\forall$ open:
($\Rightarrow$ supp $\varphi = M$)

• φ ergodic $\Rightarrow$ $\exists$ a dense orbit: top transitive
• φ uniquely ergodic
$\Rightarrow$ all orbits are dense: minimal

Rmk: Similar for flows

Rmk - Counterexample to Cor



$\exists$ φ uniquely ergodic without dense orbits

The only inv measure is δ_x but no dense orbits

The reason: δ_x is not "continuous"

Rmk Ic i) we'll show that $\mathcal{O}(x)$ is dense in supp μ for μ-a.a. $x \in$ supp μ

Parallel between measure and topology

Measure		Topology
inv measure μ	$\longrightarrow$ not 1-1 but "onto" later	inv. set supp μ
ergodic	$\Longrightarrow$ $\not\Longleftarrow$	top transitive: $\exists$ dense orbit
uniquely ergodic	$\Longrightarrow$ $\not\Longleftarrow$	minimal: every orbit is dense
...	...	...
more info because a lot of measures are involved		less info more robust

Rmk Similar for flows

On the pf of the Thm

Fact: M compact metric space
$\varphi : M \to M$ homeo (or just C^0)
$\Rightarrow$ φ has an (ergodic) inv. measure

Borel, probability

To be discussed later

Pf of Thm

1) μ ergodic $\Rightarrow$ $\exists$ an orbit dense in $\operatorname{supp}\mu$

- Can assume $M = \operatorname{supp}\mu$ ← compact metric space
- Let $\{U_j\}$ be a base of the induced top on $M = \operatorname{supp}\mu$ $\Rightarrow$ $\mu(U_j) > 0$

for any open V $\exists$ $U_j \subset V$

$\mathcal{O}(x)$ is dense $\Leftrightarrow$ $\forall j$: $\mathcal{O}(x) \cap U_j \neq \emptyset$

- ergodicity $\Rightarrow$

characteristic function of U_j

$\forall j$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{U_j}(\varphi^i(x)) = \mu(U_j) > 0$$

for a.e. x : $x \in X_j$: $\mu(X_j) = 1$

$\Rightarrow \quad \chi_{U_i}(\varphi^i(x)) > 0$ for some i

$\Rightarrow \quad \varphi^i(x) \in U_j$: $\mathcal{O}(x)$ enters U_j

$X = \bigcap X_j$ full measure

$\forall x \in X \quad \mathcal{O}(x)$ intersects $U_j \; \forall j$

$\triangleleft$

Rmk We have proved that $\mathcal{O}(x)$ is dense in supp μ for μ-a.a. $x \in$ supp μ

2) μ uniquely ergodic $\Rightarrow \varphi$ is minimal in supp μ

Assume $\mathcal{O}(x)$ is not dense in $M =$ supp μ

$\overline{\mathcal{O}(x)} \subsetneq M = \operatorname{supp} \mu$ compact metric space

$\Rightarrow \exists$ an ergodic inv measure ν with supp $\nu \subset \overline{\mathcal{O}(x)} \subsetneq M$

Fact

$\Rightarrow \nu \neq \mu \quad \rightarrow\leftarrow$ with unique ergodicity

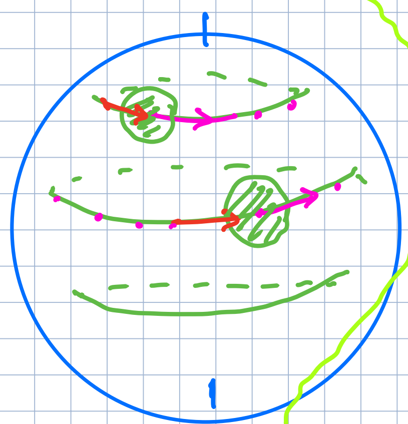
$\triangleleft$

Lecture 6
01/20-2022

Discussion:
Recurrence vs Ergodicity

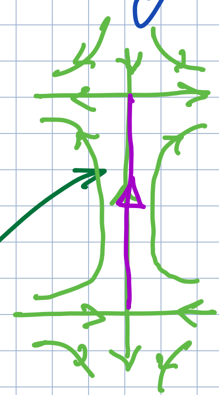
(PR): a.a. pts are recurrent: come back arbit. close to themselves

holds unconditionally (μ is inv)



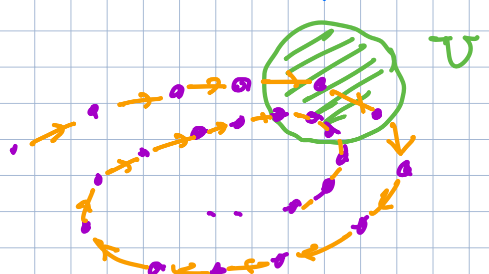
- Rot of S^2 in α
- Regardless of α, $\varphi^k(x)$ gets arbitrarily close to x

Rmk a.a. is essential: pts here do not come back



Ergodicity: does not hold unconditionally

a.a. pts enter every set U with frequency $\mu(U)$



Rmk:

ergodic $\Downarrow$ transitivity

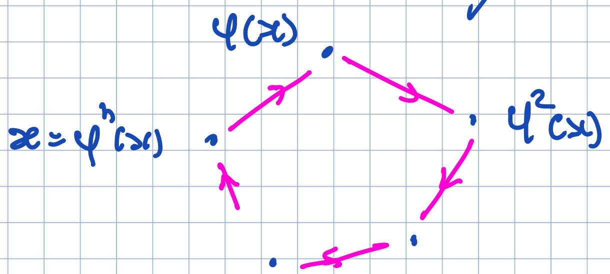
uniq erg $\Downarrow$ minimal

- **Examples**

- **Gradient flows:**
  nothing interesting

  For any μ : $\operatorname{supp} \mu \subset \operatorname{Crit}(f) = \operatorname{Fix}(\varphi)$
  and μ is ergodic $\Leftrightarrow$ $\mu = \delta_x$, $x \in \operatorname{Crit}(f)$

Rmk - Ex: a discr inv measure μ
ergodic $\Leftrightarrow$ $\mu = \delta_{O(x)}$ $\leftarrow$ periodic

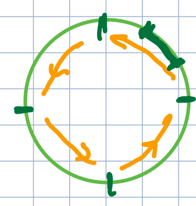


$$\mu = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\varphi^i(x)}$$

- **Rotations of $S^1 = \mathbb{R}/\mathbb{Z}$**

$\varphi : \theta \mapsto \theta + \alpha \qquad \alpha \in \mathbb{R}/\mathbb{Z}$

preserves $\mu = d\theta$



$q = 4$

- $\alpha \in \mathbb{Q} \Rightarrow$ periodic : $\varphi^q = id$, $\alpha = \frac{p}{q}$
  $\Rightarrow$ every orbit is periodic
  $\Rightarrow$ not ergodic

- $\alpha \notin \mathbb{Q}$

same condition as minimality

minimality

<u>Thm</u> $\alpha \notin \mathbb{Q} \Rightarrow \varphi$ is uniquely ergodic

<u>Pf.</u>: Focus on ergodicity (not unique)

Need to show: $\forall g \in L^1(S^1)$

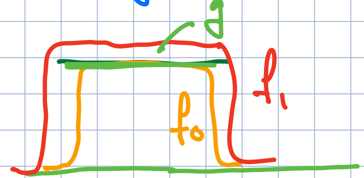
$$* \quad \frac{1}{n} \sum_{k=0}^{n-1} g(\theta + k\alpha) \longrightarrow \int g \, d\theta$$

$\Updownarrow$

For a.a. θ : in fact for all $\theta \in S^1$

$\Updownarrow$

Enough to do this for $g = \chi_U$ open



$f_0 \leq g \leq f_1$, $0 \leq f_1 - f_0 \leq \varepsilon$

$\Updownarrow$

Enough to do this for $C^0(S^1)$

Lemma (Weyl)

$\forall f \in C^0(S^1)$

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\theta + k\alpha) \xrightarrow[\text{uniformly}]{} \int_{S^1} f\, d\theta$$

Pf. Trig polynomials are C^0-dense in $C^0(S^1)$

$\Rightarrow$ Enough to prove this for trig polynomials

$$\sum_{l=n}^{m} a_l e^{2\pi i l\theta}$$

Compare with the pf of minimality for translations of $\mathbb{T}^2$

$\Rightarrow$ Enough to prove this for $f = e^{2\pi i l\theta}$:

$$\boxed{\frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi i(\theta+k\alpha)\cdot l} \xrightarrow[\text{unif}]{} 0}$$

$l \neq 0$

$$\frac{1}{n}\left|\sum_{k=0}^{n-1} e^{2\pi i l(\theta+k\alpha)}\right|$$

$$= \frac{1}{n}\underbrace{\left|e^{2\pi i l\theta}\right|}_{1}\cdot\left|\sum_{k=0}^{n-1} e^{2\pi i l k\alpha}\right|$$

$$= \frac{1}{n}\frac{\left|1-e^{2\pi i l n\alpha}\right|}{\underbrace{\left|1-e^{2\pi i l\alpha}\right|}_{\neq 0\ \forall l:\ \alpha\notin\mathbb{Q}}}$$

bounded by $2 = 1+1$

discuss later

$$\leq \frac{1}{n}\frac{2}{\left|1-e^{2\pi i l\alpha}\right|} \longrightarrow 0$$

$\triangleleft$

Rmk For unique hyperbolicity

Criterion: — exactly what we proved!

Assume that $\frac{1}{n}\sum_{k=0}^{n-1} f(\varphi^k(x)) \xrightarrow{\text{unif}} \int f\,d\mu$

for every $f\in C^0$

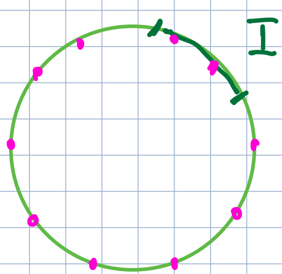
$\Rightarrow$ φ is uniquely ergodic $\triangleleft$

Digression to number theory:

uniform distribution

Def. A seq $x_k \in \mathbb{R}$ or $\mathbb{R}/\mathbb{Z} = S^1$ is uniformly distributed (mod 1) if $\forall$ $I \subset S^1$ (an interval)

$$\underbrace{\lim_{n\to\infty} \frac{1}{n} \left| \{0 \le k \le n-1 \mid x_k \in I\} \right|}_{\text{frequency with which } x_k \text{ enters } I} = \underbrace{\mu(I)}_{\text{length of } I}$$



$$\frac{\mu(I)}{\mu(S^1)=1}$$

Rmk x_k is uniformly distributed $\iff$ $x_k + \text{const}$ is

Pf: replace I by $I + \text{const}$

Q Which sequences are uniformly distributed?

Important in number theory: books and books ...

By def:

$\varphi : S^1 \circlearrowleft$ is ergodic

$\iff \forall$ a.a. $x \in S^1$, $\varphi^k(x)$ is uniformly distributed

Cor $\alpha \notin \mathbb{Q} \iff k\alpha$ is uniformly distr. mod 1

More involved dynamical systems arguments (see e.g. [Walkden])

Thm (Weyl)

$x_k = \alpha_n k^n + .. + \alpha_1 k + \alpha_0$ is uniformly distr if at least one of $\alpha_1, \dots, \alpha_n \notin \mathbb{Q}$

etc.

Ex. $n=1$ $x_k = \alpha_1 k + \alpha_0$

unif distr $\iff \alpha_1 \notin \mathbb{Q}$

- Translations of $\mathbb{T}^n$ and flows on $\mathbb{T}^n$

Very similar

$$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = \overbrace{S^1 \times \dots \times S^1}^{n}$$

$$\alpha = (\alpha_1, \dots, \alpha_n)$$

$$\varphi: \theta = (\theta_1, \dots, \theta_n) \longmapsto \theta + \alpha, \quad \underbrace{\alpha = (\alpha_1, \dots, \alpha_n)}_{\text{fixed}}$$

$$\mathbb{T}^n \longrightarrow \mathbb{T}^n$$

$\mu = d\theta_1 \dots d\theta_n$ is φ-invariant

Thm - Ex

same condition as minimality

φ is (uniquely) ergodic $\iff 1, \alpha_1, \dots, \alpha_n$ are lin. ind over $\mathbb{Q}$

Pf - Ex : reduce to $e^{2\pi i \langle k, \theta \rangle} = f$

$k = (k_1, \dots, k_n)$

Likewise - uniform distributions ...

For flows on $\mathbb{T}^n$

$$\varphi^t(\theta) = \theta + t \cdot \alpha$$
$$= (\theta_1 + t\alpha_1, \dots, \theta_n + t\alpha_n)$$

<u>Thm - Ex</u>

φ^t is (uniquely) ergodic

$$\iff \underbrace{\alpha_1, \dots, \alpha_n \text{ are lin ind over } \mathbb{Q}}_{\text{same condition as minimality}}$$

Toral Endomorphisms

a new class of examples

- $\underbrace{A \in SL(n, \mathbb{Z})}_{\substack{\text{integer entries} \\ \det A = 1}}$ $\iff A^{-1} \in SL(n, \mathbb{Z})$

- $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ $\qquad A : \mathbb{Z}^n \circlearrowleft$

$$\Rightarrow \quad A : \mathbb{T}^n \longrightarrow \mathbb{T}^n$$
$$x \longmapsto Ax$$

- $\mu = dx_1 \ldots dx_n$ is invariant : $\det A = 1$
- what are fixed/periodic pts

$\to$ $A0 = 0 \Rightarrow 0 \in Fix(A)$

$\Rightarrow \delta_0$ is invariant

$\Rightarrow$ A is not uniquely ergodic, not minimal

$\to$ Other fixed or periodic pts

x is k-periodic

$A^k x = x$ in $\mathbb{T}^n$

Lifting to $\mathbb{R}^n$: $A^k x = x +$ integer vector

Def A is hyperbolic if
A has no eigenvalues
with abs value 1

Ex $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ ← Arnold's cat map is hyperbolic

Ex A is hyperbolic $\Rightarrow$ per. pts are dense

↑ [KH]?

e.g. A is hyperbolic

Thm A has no eigenvalue which is a root of unity $\Longleftrightarrow$ A is ergodic

$\Downarrow$

Cor A has no eigenvalue which is a root of unity $\Rightarrow$ A has a dense orbit

In fact $\mathcal{O}(x)$ is dense for a.a. $x \in \mathbb{T}^n$

Rmk: • when $n=2$: no root of unity $\Longleftrightarrow$ hyperbolic

• $n>2$: Probably not?

Pf:

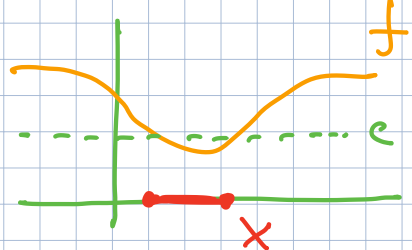
Recall:

- A ergodic: every invariant set has $\mu = 0$ or $\mu = 1$
- A not ergodic: $\exists$ inv. set X with $0 < \mu(X) < 1$

can have L^1 or L^2

$\Longleftrightarrow$ $\exists$ an inv $f \in L^\infty$ s.t. $f \neq \text{const}$ (a.e.)

- Take $f = \chi_X$
- In the other direction take $X = \{x \mid f(x) \le c\}$ for a suitable c.



$\Rightarrow$ Need to show

A hyperbolic $\Longleftrightarrow$ every invariant $f \in L^\infty$ is constant

$\Leftarrow$) Assume that A has an eigenvalue which is a root of unity.

Goal: construct an invariant function $f \in L^\infty$

- The same is true for $B := A^T$:

$\exists\ q \geq 1$ s.t. $B^q v = v$, $v \in \mathbb{R}^n$, $v \neq 0$

$\Leftrightarrow (B^q - I) v = 0$ in $M(n, \mathbb{Z})$

Ex. Prove that $v \in \mathbb{Q}^n$ and hence can assume $v \in \mathbb{Z}^n$

- Take the smallest q with this property

$$\Rightarrow \quad e^{2\pi i \langle v, A^q x \rangle} = e^{2\pi i \langle B^q v, x \rangle} = e^{2\pi i \langle v, x \rangle}$$

Set $f(x) = \sum_{j=0}^{q-1} e^{2\pi i \langle v, A^j x \rangle} \in L^2(\mathbb{T}^n)$, $C^\infty(\mathbb{T}^n)$

$$= \sum_{j=0}^{q-1} e^{2\pi i \langle B^j v, x \rangle}$$

Ex. $f \neq \text{const} \Leftarrow v \neq 0$

Use the fact that q is minimal deg

Claim: $f(Ax) = f(x)$ invariant

Pf $$f(Ax) = \sum_{j=0}^{q-1} e^{2\pi i \langle B^{j+1} v,\, x\rangle}$$

$$= \sum_{j=1}^{q-1} e^{2\pi i \langle B^{j} v,\, x\rangle} + e^{2\pi i \langle B^{q} v,\, x\rangle}$$

$$= f(x)$$

$\triangleleft$

$\Rightarrow$) No roots of unity

$\Rightarrow$ No inv. functions

(L^2 or L^∞ or L^1 ...)

Assume f is invariant

$f \in L^2$ and $f(A^m x) = f(x)$ $\quad \forall\, m \in \mathbb{Z}$

a.a. x

Fourier expansions

$$f(x) = \sum_{\ell \in \mathbb{Z}^n} f_\ell \exp(2\pi i \langle \ell, x \rangle)$$

$$\|$$

Need $f_\ell = 0$

$\ell \neq 0$

$$f(A^m x) = \sum_{k \in \mathbb{Z}^n} f_k \exp(2\pi i \langle k, A^m x \rangle)$$

$$= \sum_{k \in \mathbb{Z}^n} f_{k} \exp(2\pi i \langle \underbrace{B^m k}_{\ell \in \mathbb{Z}^n}, x \rangle)$$

$k = B^{-m}\ell$ $\qquad B = A^T$

$$f(x) = \sum_\ell f_\ell \exp(2\pi i \langle \ell, x \rangle)$$

$$\| \qquad \Rightarrow \qquad \|$$

$$f(A^m x) = \sum_\ell f_{B^{-m}\ell} \exp(2\pi i \langle \ell, x \rangle)$$

$\forall\, m \in \mathbb{Z}$

$\forall \ell \in \mathbb{Z}^n$

$$\dots = f_{B^{-1}\ell} = f_\ell = f_{B\ell} = f_{B^2\ell} = \dots$$

i.e. $\boxed{f_\ell = f_{B^m\ell}}$ $\quad \forall m \in \mathbb{Z} \quad \forall \ell \in \mathbb{Z}^n$

Goal: $\ell \neq 0 \Rightarrow f_\ell = 0$

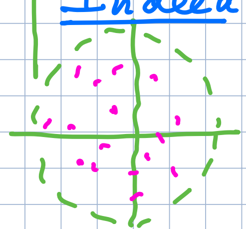
Hence $f = \text{const}$

Note: $\underbrace{|f_\ell| \to 0 \text{ as } |\ell| \to \infty}_{\Leftarrow\, f \in L^2}$

Assume $f_\ell \neq 0$, $\ell \neq 0$

$\Rightarrow$ $B^m\ell$, $m \in \mathbb{Z}$ can take only finitely many values

Indeed



Assume not: then $B^{m_s}\ell \to \infty$ for some $m_s \to \pm\infty$

$\Rightarrow f_\ell = f_{B^{m_s}\ell} \to 0$

$\Rightarrow f_\ell = 0$

only finitely many $\mathbb{Z}^n$ pts in a ball

$$\Rightarrow \quad B^{m_1} \ell = B^{m_0} \ell \qquad m_1 > m_0$$

$$\Rightarrow \quad B^{\underbrace{m_1 - m_0}_{q}} \ell = \ell$$

$$\Rightarrow \quad B^{q} \ell = \ell$$ eigenvector

$\Rightarrow$ an eigenvalue which is a root of unity!

$\rightarrow\leftarrow$

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