Ex 6 Geodesic flows: surfaces of neg curvature

Lecture 3
01/11-2022

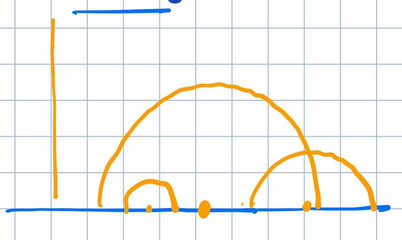
Hyperbolic plane

- $\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$ upper half plane, $z = x + iy$

Riemannian metric of const curv $= -1$ $\leftarrow$ hyperbolic metric

$$\frac{dx^2 + dy^2}{y^2}$$

- Geodesics: circles with centers on the x-axis including vertical lines:



Return to this a bit later

Ex After knowing that $PSL(2, \mathbb{R})$ are isometries:

- check that a vertical line is a geodesic
- check that for any circle $\exists\, g$ sending a vertical line to that circle

- **Isometries:**

$$SL(2,\mathbb{R}) \longrightarrow \underbrace{PSL(2,\mathbb{R})}_{SL(2,\mathbb{R})/\{\pm I\}} \longrightarrow Iso(\mathbb{H})$$

orientation preserving isometries

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xmapsto{\text{"}2 \to 1\text{"}} \underbrace{z \mapsto \frac{az+b}{cz+d}}_{\text{fractional-linear transformation}}$$

$ad - bc = 1$

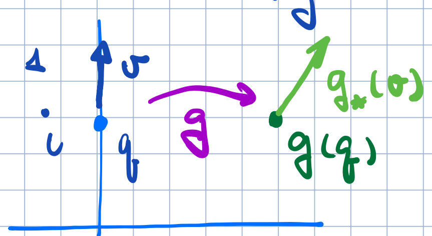
not hard

Ex • Check that this is an action by isometries

- Check that $PSL(2,\mathbb{R}) \longrightarrow Iso(\mathbb{H})$ is an isomorphism: every isometry has this form

Hint • $g \in Iso(\mathbb{H})$ is completely determined by $g(q, v)$ (q,v) - fixed

$v \in T_q\mathbb{H}$, $\|v\|$, $q \in \mathbb{H}$



$q = i$
$v = 1$

- For any $(p, w) \in ST\mathbb{H}$
  $\exists!\, g$ s.t. $g(q,v) = (p,w)$

- $PSL(2,\mathbb{R}) \longrightarrow ST\mathbb{H}$
  $g \longmapsto g(q,v)$
  is an diffeo

- **Compact surfaces with hyperbolic metrics**

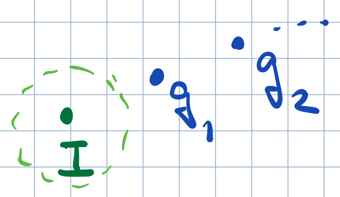
Fact (not obvious)

$\Sigma_{g\geq 2}$ $\quad \exists\ \Gamma \subset PSL(2,\mathbb{R})$ (or $SL(2,\mathbb{R})$)

$\uparrow$ a discrete subgroup

$\to \exists$ a nbhd U of I s.t. $U \cap \Gamma = \{I\}$

s.t. $\Sigma_g \underset{\text{diffeo}}{\cong} \mathbb{H}/\Gamma$



Cor Σ_g admits a metric of constant curvature -1 (a hyperbolic metric)

Cor • $\mathbb{H} \underset{\text{diffeo}}{\cong} \mathbb{R}^2$ is the universal covering of Σ_g

• $\Rightarrow \pi_{n\geq 2}(\Sigma_g) = 0$

Cor $\quad ST\Sigma_g \cong \underbrace{\Gamma \backslash SL(2,\mathbb{R})}_{\text{an algebraic model for } ST\Sigma_g}$

Rmk Γ is not unique: different metrics on $\Sigma_{g\geq 2}$ with curv $= -1$.

- Algebraic construction:
  - $\Gamma \subset SL(2,\mathbb{R})$ a discrete subgroup s.t. $M = \Gamma \backslash SL(2,\mathbb{R})$ is compact and smooth
  - $g(t) : \mathbb{R} \to SL(2,\mathbb{R})$ a one parameter subgroup

$\Rightarrow$ a flow on M

$$\varphi^t(x) = x \cdot g(t)$$

Ex. Taking Γ as before we get a flow on $ST\Sigma_{g\geq 2}$

Specific examples

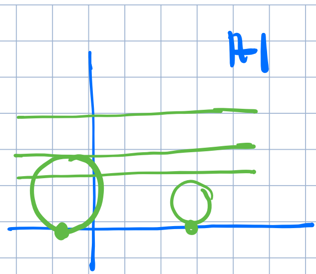
- $g(t) = \begin{pmatrix} \cos t, & \sin t \\ -\sin t, & \cos t \end{pmatrix} \in SO(2) \subset SL(2,\mathbb{R})$ elliptic

$\Rightarrow$ all orbits are periodic

- $g(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ Horocycle flow parabolic

the projection of any orbit to Σ_g (or $\mathbb{H}$) is a geodesic circle of curvature $k = 1$

Ex: what are these?



e.g. horizontal lines

Ex : Prove that the horocycle flow in $ST\Sigma_g$ has no closed orbits

(Need to show that no orbit can close up as $\mathbb{H} \longrightarrow \Sigma_g$

- $g(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in SL(2, \mathbb{R})$ hyperbolic subgroup

the geodesic flow :

some orbits are closed (closed geodesics)

but some are dense !

Two non-obvious facts :

- The union of periodic orbits is dense
- $\exists$ dense orbits : top transitive

Rmk Generalizes to groups other than $SL(2,\mathbb{R})$; important !

Further reading

- [CFS] : § 4.4
- [KH] : § 5.4 ←

Ex 7 • Shift transformations

"Symbolic Dynamics"

• Preliminaries – pt set topology

• A = compact metric space

• $A^{\mathbb{Z}} = \quad \cdots \times \overset{-1}{A} \times \overset{0}{A} \times \overset{1}{A} \times \cdots$

Elements: (bi)infinite sequences

$x = \{x_i \in A \mid i \in \mathbb{Z}\}$

• With product topology:

open sets $\cdots \times U_{-1} \times U_0 \times U_1 \times \cdots = \prod U_i$

where all but a finite number $U_i = A$

Fact $A^{\mathbb{Z}}$ is compact

• metric

$$d(x,y) = \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} d(x_i, y_i)$$

can put any conv. series

Rmk

$$\dim A^{\mathbb{Z}} = \Big(1 + 2\sum_{i=1}^{\infty} \frac{1}{2^i}\Big) \operatorname{diam} A$$

$$= 1 + \cancel{\tfrac{1}{2}} \frac{\cancel{2}}{1 - \frac{1}{2}}$$

$$= 3 \operatorname{diam} A$$

Observation:

$x_i = y_i$ for $|i| \le N$

$$\Rightarrow \quad d(x,y) \le \underbrace{2 \sum_{i=N+1}^{\infty} \frac{1}{2^i} \cdot \operatorname{diam} A}_{2 \cdot \frac{1}{2^{N+1}} \frac{1}{1-\frac{1}{2}} \cdot \operatorname{diam} A}$$

$$\Rightarrow \quad d(x,y) \le \frac{\operatorname{diam} A}{2^{N-1}}$$

$$\begin{array}{ccccccccccc}
\cdots & x_{-N} & \cdots & x_{-1} & x_0 & x_1 & \cdots & x_N & \cdots \\
\neq & \parallel & & \parallel & \parallel & \parallel & & \parallel & \neq \\
\cdots & y_{-N} & \cdots & y_{-1} & y_0 & y_1 & \cdots & y_N & \cdots
\end{array}$$

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{agree}}$

$\Rightarrow$ $d(x,y)$ is small

• Shift Transformation

Set • $A = \{0, 1\}$ $\quad d(0,1) = 1$

$M = A^{\mathbb{Z}}$

$=$ sequences of 0 & 1's

• $\varphi : M \to M$ shift to the left

$\varphi(x)_i = x_{i+1}$, a homeo

$$\cdots\ x_{-2}\ x_{-1}\ x_0\ x_1\ x_2\ \cdots$$

$$\cdots \leftarrow \ \leftarrow \ \leftarrow \ \leftarrow \ \cdots$$

Rmk: variants

• Replace $A = \{0, 1\}$ by the alphabet $A = \{1, \dots, n\}$. Similar properties

• Replace $A^{\mathbb{Z}}$ by

$M = A^{\mathbb{N}} = A \times A \times \dots$

$= $ one sided, infinite seq

$\varphi : M \to M$, left shift

$\varphi(x_0 x_1 x_2 \dots) = x_1 x_2 x_3 \dots$

C^0, but not invertible

Interpretation:

$A = $ collection of states

$x \in A^{\mathbb{Z}}$ a process

$x_0 = $ state at $t = 0$

$x_1 = $ — " — " $t = 1$

Properties

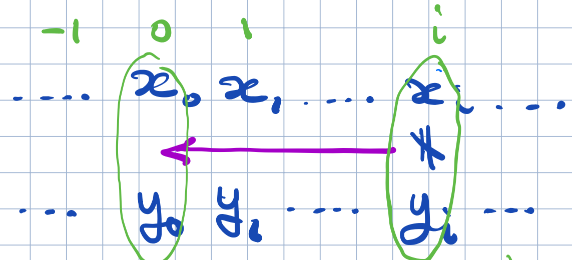
- φ is very far from an isometry:
  φ is expansive
  $\exists\ \varepsilon > 0$ s.t $\forall\ x \neq y\ \exists\ k$ with
  $$d(\varphi^k(x), \varphi^k(y)) > \varepsilon$$

<u>Pf</u> $\varepsilon = 1$, $\quad x \neq y \Rightarrow \exists\ i : x_i \neq y_i$

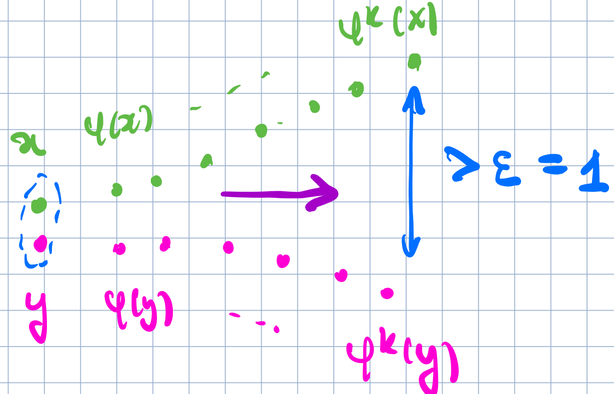
$k = -i$

$$\varphi^k(x)_0 = x_i$$
$$\varphi^k(y)_0 = y_i$$

$$d(\varphi^k(x), \varphi^k(y)) \geq d(\varphi^k(x)_0, \varphi^k(y)_0) = 1$$



contribute with weight 1 $\qquad$ contribute with weight $\frac{1}{2^{|i|}}$



$\triangleleft$

- Periodic pts = periodic sequences

$\Rightarrow$ $p(k) = |\, k\text{-periodic pts} \,|$

$= 2^k$

If $|A| = n$, $p(k) = n^k$

similar to the geodesic flow of a hyperbolic surface

- Periodic pts are dense

Pf Given x and $\varepsilon > 0$ take N so that

$$\frac{1}{2^{N-1}} < \varepsilon$$

set $\hat{x} = \overbrace{x_{-N}\, x_{-N+1} \ldots x_0 \ldots x_N}^{\text{length } 2N+1 = k}$

$y = \ldots \hat{x}\, \hat{x}\, \hat{x} \ldots$

$\Rightarrow$ $y_i = x_i \qquad |i| \leq N$

$\Rightarrow$ $d(x,y) \leq \dfrac{\operatorname{diam} A}{2^{N-1}} < \varepsilon$ $\qquad$ $\operatorname{diam} A = 1$

$$\ldots \overset{-N-1}{x_{-N-1}} \,|\, \underbrace{\overset{-N}{x_{-N}} \ldots \overset{-1}{x_{-1}}\, \overset{0}{x_0}\, \overset{1}{x_1} \ldots \overset{N}{x_N}}_{\hat{x}} \,|\, \overset{N+1}{x_{N+1}} = x$$

$$\ldots x_N \,|\, \overbrace{x_{-N} \ldots \ldots x_0\, x_1 \ldots x_N}^{\hat{x}} \,|\, x_{-N} \ldots = y \qquad \triangleleft$$

- φ is top. transitive : $\exists$ a dense orbit.

similar to the geodesic flow of a hyperbolic metric

Pf: • $M = A^{\mathbb{Z}}$ is separable : $\exists$ a countable dense set (e.g. can take periodic pts)

Denote these set by

$\{x^0, x^1, x^2, \dots\}$

each of these is a bi-inf sequence

$\forall\ y \in M \quad \exists\ x^{i_s}$ s.t.

$d(y, x^{i_s}) \to 0, \quad i_s \to \infty$

• let $\hat{x}^i$ be the finite sequence

$x_{-i} \dots x_0 \dots x_i$

and

$$\begin{array}{ccccccc} \dots & -1 & 0 & 1\,2\,3 & 4\dots 8 & \dots & \\ z = \dots 0 \dots & 0 & \hat{x}^0 & \hat{x}^1 & \hat{x}^2 & \hat{x}^3 & \dots \end{array}$$

Claim $\{\varphi^k(z)\}$ is dense

Pf • Given y & $\varepsilon > 0$

Pick $i = i_s$ so large that

- $d(y, x^i) < \frac{\varepsilon}{2}$
- $\frac{1}{2^{i-1}} < \frac{\varepsilon}{2}$

- pick k so that $\hat{x}^i$ is centered at 0 in $\varphi^k(z)$

$$\begin{array}{l} \ldots\ldots \; -1 \;\; 0 \;\; 1\,2\,3 \;\; 4 \ldots 8 \;\; \ldots \\ z = \ldots 0 \ldots 0 \; \hat{x}^0 \; \hat{x}^1 \; \hat{x}^2 \; \hat{x}^3 \ldots \hat{x}^i \end{array}$$

$$\xleftarrow{\quad \varphi^k \quad}$$

$$\Rightarrow \quad d(x^i, \varphi^k(z)) \leq \frac{1}{2^{i-1}} < \varepsilon/2$$

- $$d(y, \varphi^k(z)) \leq \underbrace{d(y, x^i)}_{<\ \varepsilon/2} + \underbrace{d(x^i, \varphi^k(z))}_{<\ \varepsilon/2} < \varepsilon$$

$\triangleleft$

Ex. Show that $M = A^{\mathbb{Z}}$ is homeo to the Cantor set

Rmk $\exists$ C^∞ φ: surface $\circlearrowright$ on
φ: disk $\circlearrowright$ on manifold $\circlearrowright$
s.t. $\exists$ K $\leftarrow$ invariant subset
with $\varphi|_K \cong (A^{\mathbb{Z}}, \text{shift})$

These are "horseshoes"
very common & important

Further Reading:
[KH] § 1.9

We will keep returning to shifts...

§2 Elements of Ergodic Theory

Lecture 4
01/13-2022

Setup

- Let now (M, μ) be a measure space:

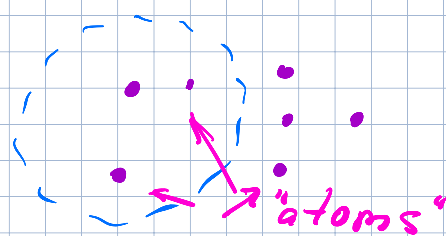
  Usually assume:
  - μ is a probability measure: $\mu(M) = 1$
  - If M is a metric space, then μ is a Borel measure: μ is defined on all open sets ($\Rightarrow$ on all Borel sets)

Ex: smooth measure
- M^n closed orientable manifold
- $\omega \in \Omega^n(M)$ $\quad \omega > 0$
- $\mu(U) = \int_U \omega$

Ex "measures supported on finite sets"

$X \subset M$ finite

$$\mu(U) = \frac{1}{|X|} \, |X \cap U|$$



Ex. linear combination:

μ_0 & μ_1 as above $\Rightarrow$ so is $\lambda\mu_1 + (1-\lambda)\mu_0$

$\forall \quad 0 \leq \lambda \leq 1$

Rmk: supp μ

closure

$$\operatorname{supp}\mu = \overline{\{x \mid \forall\ U = \text{nbhd of } x,\ \mu(U) > 0\}}$$

$$\operatorname{supp}(\lambda\mu_1 + (1-\lambda)\mu_0) = \operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1$$

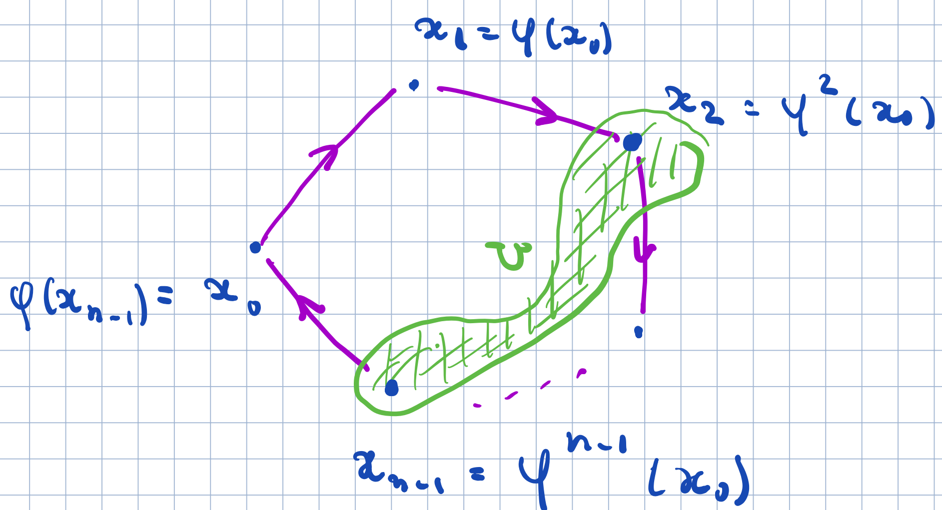
- $\varphi : M \to M$ is measure preserving and C^0 or homeo when M is also a metric space

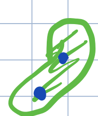
Ex. $\varphi : M \to M$

$X = \{x_0, \dots, x_{n-1}\}$ a periodic orbit

$\Rightarrow$ an invariant measure

$$\mu(U) = \frac{1}{n} |X \cap U|$$



$\mu($  $) = \dfrac{\#\{x_i \text{ in } U\}}{n}$

= frequency of entering U

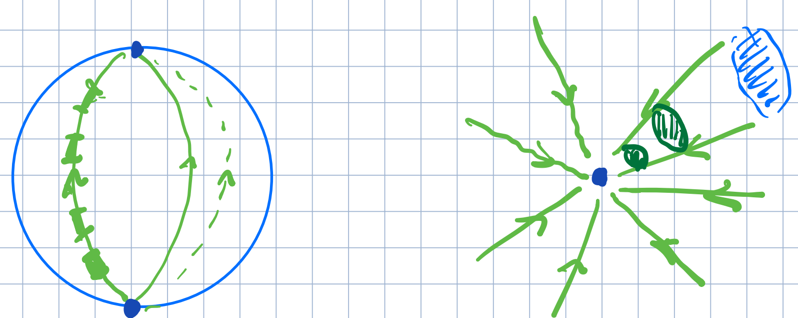
Revisiting our main examples from the measure theory perspective

Ex: Gradient flows:

Ex • For any invariant Borel measure

$$\operatorname{supp}\mu \subset \operatorname{Crit}(f) = \operatorname{Fix}(\varphi)$$

In particular when $\operatorname{Crit}(f)$ are isolated, the only inv measures come from fixed pts



Ex. Rotations of S^1 or translations of $\mathbb{T}^n$

- The standard measure $d\theta$ or $d\theta_1 \wedge \dots \wedge d\theta_n$ is obviously invariant

Rmk An isometry of a Riemannian manifold always preserves the Riemannian vol

- Depending on α, there could be other invariant measures
e.g. $\alpha \in \mathbb{Q}$ then $\theta \mapsto \theta + \alpha$ has periodic orbits, etc

We'll look into these maps some more later

- $\alpha = \frac{p}{q}$, $\varphi^q = id$
$\Rightarrow$ $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ - action on S^1

$S^1 \xrightarrow{\pi} S^1/\mathbb{Z}_k \leftarrow$ circle

every invariant measure has the form $= \pi^*($ a measure on $S^1/\mathbb{Z}_k)$

<u>Ex.</u> Geodesic flows have a natural invariant measure
Three ways to see:

1) Q^n R. manifold

$$TQ \overset{\cong}{\longleftrightarrow} T^*Q \longleftarrow \text{symplect}$$

$$v \longleftrightarrow \langle v, \cdot \rangle$$

$\Rightarrow$ TQ also gets a sympl. str ω
Geodesic flow is the Ham flow of $H(v) = \frac{1}{2}\langle v, v\rangle$

$M := STQ = \{H = 1/2\} \leftarrow$ regular level

<u>Ex</u> $\exists$ $\nu \in \Omega^{2n-1}(TQ)$ s.t.

- $\nu \wedge dH = \omega^n$ near $\{H = 1/2\}$
- $\nu|_M$ is unique & $\nu|_M \neq 0$

Could use the notation:

$$\nu|_M = \frac{\omega^n}{dH}$$

Invariant by construction?
(energy & ω conservation)

A variant & Example : a vol form & $\{H=c\}$ $\Rightarrow$ a vol form on $H=c$

- $\mathbb{R}^3$ $dx\wedge dy\wedge dz = \eta$
- $H(x,y,z) = x^2+y^2+z^2$

$S^2 = \{H=1\}$ regular level

- $\exists$ ϑ s.t.

$$\vartheta\wedge dH = \eta \quad \text{near } S^2$$

$$\vartheta = \frac{x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy}{6(x^2+y^2+z^2)}$$

$$dH = 2(x\,dx + y\,dy + z\,dz)$$

$$\vartheta\wedge dH = \frac{1}{6(x^2+y^2+z^2)}(2x^2\,dx\wedge dy\wedge dz + 2y^2\,dx\wedge dy\wedge dz + 2z^2\,dx\wedge dy\wedge dz)$$

$$= dx\wedge dy\wedge dz = \eta$$

- $\vartheta|_{S^2}$ is unique and

$$= \frac{1}{6}(x\,dy\wedge dz - \dots) = \frac{1}{6} \text{ area form on } S^2$$

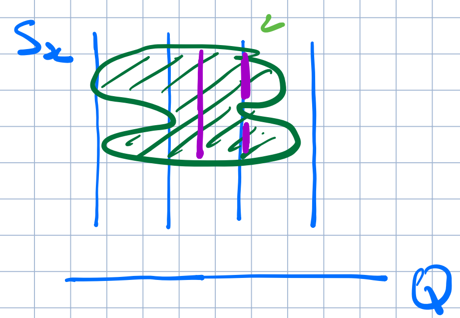
2) STQ has a natural measure

$$\begin{array}{c} STQ \\ \downarrow \pi \\ Q \end{array} \qquad \mu(U) = \int_{\pi(U)} \mu_{S_x}(S_x \cap U)\, dx$$

fibre $S_x \subset T_x Q$

natural measure on S_x

vol form on Q



Invariance not-obvious

3) For $ST\Sigma_g = \Gamma\backslash SL(2,\mathbb{R})$, i.e.
$Q = \Sigma_g$ with a hyperbolic metric

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc = 1 \right\}$$

$$\nu = \frac{da \wedge db \wedge dc}{|d|}$$

bi-invariant Haar measure

$\Rightarrow$ descends to an invariant measure on $\Gamma\backslash SL(2,\mathbb{R})$

Rmks • ν is preserved by all 1-parameter subgroups of $SL(2,\mathbb{R})$

• Other invariant measures (e.g. from periodic orbits)

Ex. <u>Shift transformations</u>

μ_A = a measure on A

$\Rightarrow$ • a Borel measure on $A^{\mathbb{Z}} = M$

$U = \quad \ldots \times U_{-1} \times U_0 \times U_1 \times \ldots$

all but a finite number $= A$

"a cylinder"

$\mu(U) = \prod \mu(U_i)$, then extend

• μ is shift-invariant

Sub-Ex $\quad A = \{1, \ldots, n\}$

$1 \geq p_i \geq 0$ s.t. $\sum p_i = 1$

probability of i

$\mu(\{i\}) = p_i$

E.g. $p_i = 1/n$

Rmk : • Thus we have many invariant measures on $A^{\mathbb{Z}}$

• $\exists$ other invariant measures: e.g. periodic orbits

• Poincaré Recurrence Thm

Simple and very important

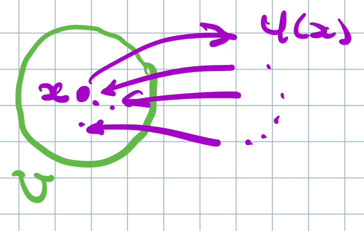
Thm (PR) ← different versions

- $\varphi: M \circlearrowleft$, μ = invariant Borel measure
- $U \subset M$, measurable (e.g. open), $\mu(U) > 0$

$\Rightarrow$ for a.a. $x \in U$ the orbit $\{\varphi^k(x) \mid k \in \mathbb{N}\}$ visits U again (can set visit time $k \geq$ any n)

Cor Assume μ is such that $\mu(\text{open}) > 0$, $\varphi: M \circlearrowleft$ μ-preserving

$\Rightarrow$ a.a. pts are recurrent: $\varphi^k(x)$ comes back and close to x

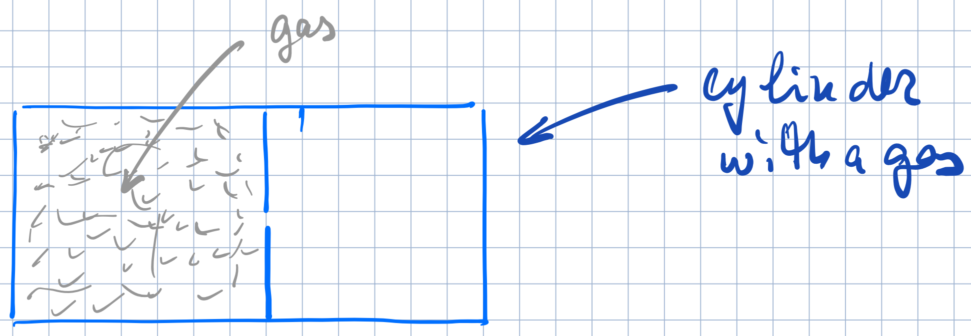


Interpretation: U = event, $\mu(U)$ = probability (no matter how small)

$x, \varphi(x), \varphi^2(x), \ldots$ a process

$\Rightarrow$ every possible event will eventually happen again if it happens once

Ex



- Initial condition: gas in one half of the cylinder
- this is a positive (but close to 0) probability event
- $\exists$ time $T > 0$ such that the gas on its own will again concentrate in one half of the cylinder
- Why don't we observe this? The reason is that T is huge! Longer than the existence of the universe!