§5 Introduction to hyperbolicity

Lecture 18
03/08-2022

- The horseshoe

Recall : Bernoulli shift

- $K = \mathbb{Z}_2^{\mathbb{Z}} = \{$ bi-inf sequences $a_i \in \mathbb{Z}_2 \}$ with product top/metric
  $\mathbb{Z}_2 = \{0,1\}$

  Ex. : $K \cong$ Cantor set

- $\sigma : K \longrightarrow K$ = shift to the left

$$(\sigma \vec{a})_i = a_{i+1}$$

$$\ldots \xleftarrow{} a_{-2} \xleftarrow{} a_{-1} \xleftarrow{} a_0 \xleftarrow{} a_1 \xleftarrow{} a_2 \xleftarrow{} a_3 \ldots$$

Properties :
- periodic points are dense
- $\exists$ dense orbits (residual set)
- ergodic

Rmk : Different from other examples which are manifolds

Key fact: Smale's horseshoe

M = a surface (e.g. $\mathbb{R}^2$ or S^2)

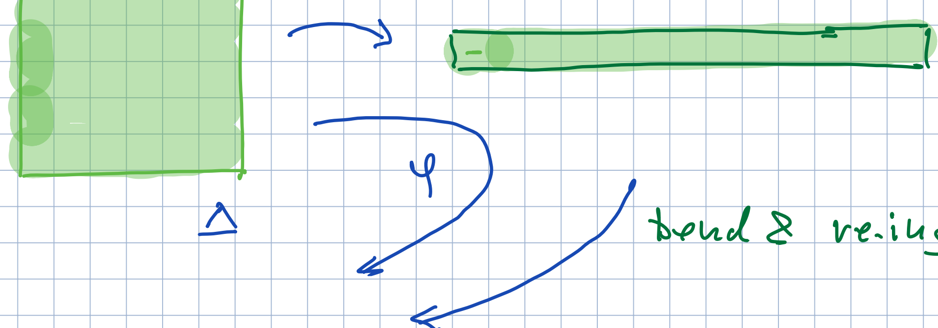
$\exists\ \varphi : M \to M$ compactly supported

and $j : K \hookrightarrow M$ s.t. $\varphi|_K = \sigma$

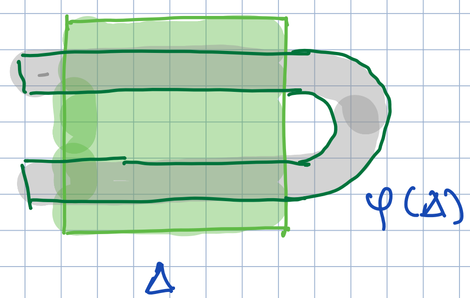
$$\begin{tikzcd} M \arrow[r, "\varphi"] & M \\ K \arrow[u, hook] \arrow[r, "\sigma"] & K \arrow[u, hook] \end{tikzcd}$$

Construction

stretch and squeeze



bend & re-insert



extend to the rest of M

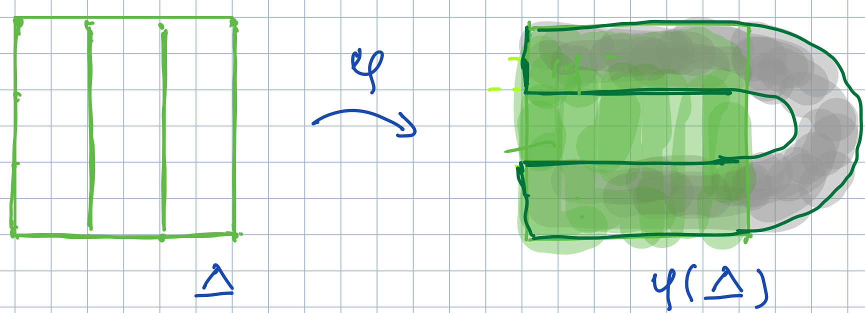
$$K = \bigcap_{k=-\infty}^{\infty} \varphi^k(\Delta) = \text{max inv subset of } \Delta$$

$\subset \operatorname{int}(\Delta)$

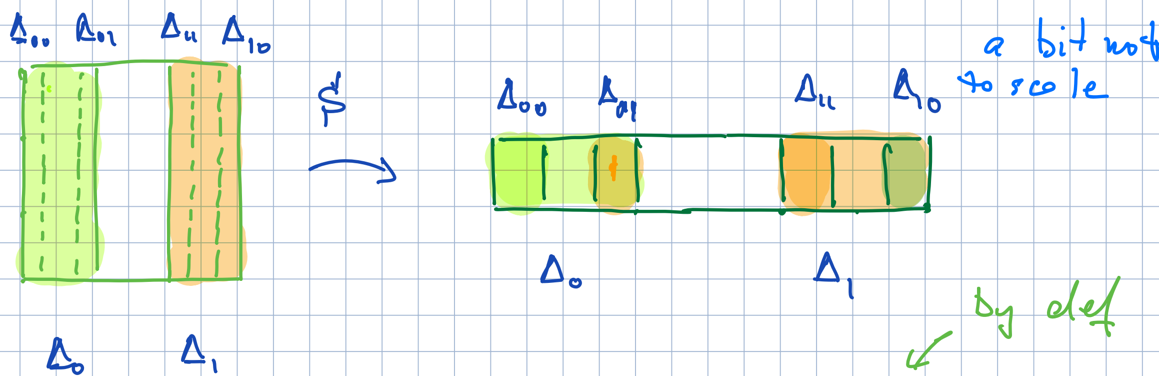
Claim: $K \cong \mathbb{Z}_2^{\mathbb{Z}}$ and $\varphi|_K = \sigma$.

Outline of the pf – symbolic dynamics

Simplification



$K \subset \Delta$ but not in $\mathrm{int}(\Delta)$ ...



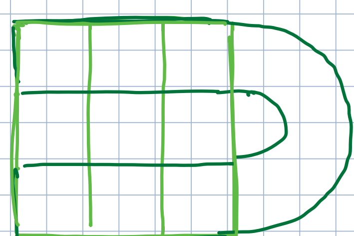
by def

$\{x \mid x \in \Delta, \varphi(x) \in \Delta\} = \Delta_0 \cup \Delta_1$

$\underbrace{\{x \mid x \in \Delta, \varphi(x) \in \Delta, \varphi^2(x) \in \Delta\}}$

$\{x \in \Delta_0 \cup \Delta_1 \mid \varphi(x) \in \Delta_0 \cup \Delta_1\}$

$= \Delta_{00} \cup \Delta_{01} \cup \Delta_{10} \cup \Delta_{11}$



etc

$\{x \mid x \in \Delta, \varphi(x) \in \Delta, \dots, \varphi^n(x) \in \Delta\}$
$= 2^{n-1}$ narrow vertical strips

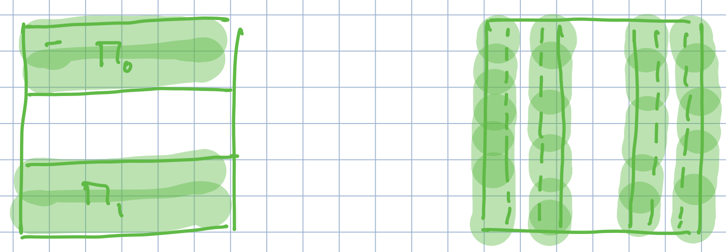
$K^+ = \{x \mid \varphi^k(x) \in \Delta \ \forall k \in \mathbb{N}\}$
$\quad = $ Cantor set $\times [0, 1]$

Coding trajectories (symbolic dynamics)

$K \to \mathbb{Z}_2^{\mathbb{N}}$

$x \mapsto a_0 a_1 a_2 \dots\dots, \quad a_k \in \mathbb{Z}_2$
if $\varphi^k(x) \in \Delta_{a_k}$

Apply the same process to φ^{-1}



$\{x \in \Delta \mid \varphi^{-1}(x) \in \Delta\} = \Pi_0 \cup \Pi_1$

$\underbrace{\{x \in \Delta \mid \varphi^{-1}(x) \in \Delta, \varphi^{-2}(x) \in \Delta\}}$

$\{x \in \Pi_0 \cup \Pi_1 \mid \varphi^{-1}(x) \in \Pi_0 \cup \Pi_1\} = \Pi_{00} \cup \Pi_{01} \cup \Pi_{10} \cup \Pi_{11}$

$\dots$

$K^- = \{x \mid \varphi^{-k}(x) \in \Delta \ \forall k \in \mathbb{N}\}$
$\quad = [0, 1] \times$ Cantor set

$K = K^+ \cap K^- = \{x \in \Delta \mid \varphi^k(x) \in \Delta \ \forall k\}$

symbolic dynamics:

<u>To summarize:</u>

$$K \longrightarrow \mathbb{Z}_2^{\mathbb{Z}}$$

$$x \longmapsto \quad \dots a_{-1} a_0 a_1 a_2 \dots \in \mathbb{Z}_2^{\mathbb{Z}}$$

$$a_k = \underset{\varphi^k(x) \in \Delta_0}{0} \qquad \underset{\varphi^k(x) \in \Delta_1}{1}, \qquad k \in \mathbb{Z}$$

- Then one shows that this map is a homeomorphism
- $\varphi|K = \sigma$ by construction $\triangleleft$

<u>Important:</u> horseshoes (or smth like it) are ubiquitous (particularly in 2D) for C^∞-infinity generic $\varphi : M^2 \circlearrowleft$

$\exists \quad K \subset M \quad$ s.t.

$$\left.\begin{aligned} &K \cong \mathbb{Z}_2^{\mathbb{Z}} \\ &\varphi^n|_K = \sigma, \text{ for some } n. \end{aligned}\right\} \text{roughly speaking}$$

(Katok, Le Calvez)

Hyperbolic maps & sets - definitions

Recall: • another example of φ with the same properties as σ is $A: \mathbb{T}^n \to \mathbb{T}^n$ linear $|\lambda| \neq 1$

• It turns out that these properties are essentially a consequence of a common feature: hyperbolicity

Def $\varphi: M \to M$ is hyperbolic if $\exists$

• a splitting

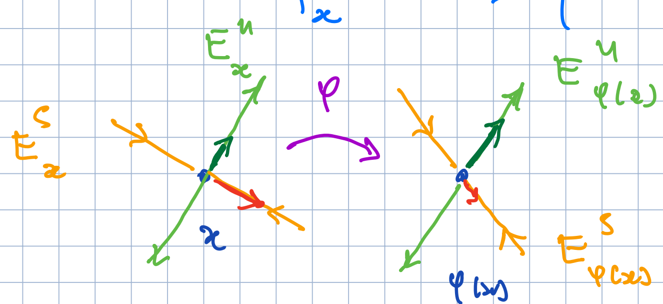
$TM = E^s \oplus E^u$: $T_x M = E^s_x \oplus E^u_x$

invariant under $D\varphi$

• $0 < \eta < 1$ ← ind of x & v

$\|D\varphi_x(v)\| \leq \eta \|v\| \quad \forall v \in E^s$

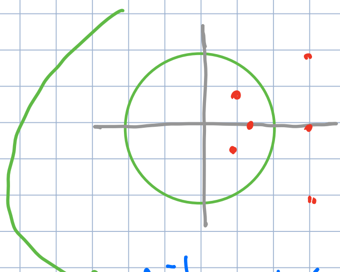
$\|D\varphi_x(v)\| \geq \eta^{-1} \|v\| \quad \forall v \in E^u$ $\quad \forall x$



don't have to be reciprocal but can chose

Shrinks vectors from E^s by $\eta < 1$
Extends vectors from E^u by $1/\eta > 1$

Ex $A \in SL(n, \mathbb{Z})$ with all $|\lambda| \neq 1$



$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\mathbb{Z}^n \longrightarrow \mathbb{Z}^n$$
$$\underbrace{A : \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n \circlearrowleft}$$

$A^{-1} \in SL(n, \mathbb{Z}) \Rightarrow$ homeo

E.g. $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ Arnold's cat map

Splitting: $T\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$

$$DA = A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$E^u =$ span(eigenvectors with $|\lambda| > 1$)

$E^s =$ span(eigenvectors with $|\lambda| < 1$)

$$\eta = \max_{|\lambda| < 1} |\lambda|$$

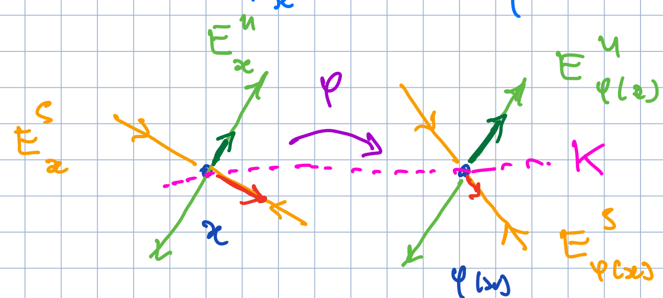
Rmk Hyperbolic maps are very rare: essentially all examples have the same alg nature as $\varphi = A$

Hyperbolic sets:

$\varphi : M \to M$; K closed invariant set

Def K is hyperbolic for φ if $\exists$

- a splitting
  $T_K M = E^s \oplus E^u$: $T_x M = E^s_x \oplus E^u_x$, $x \in K$
  invariant under $D\varphi$
- $0 < \eta < 1$ ← ind of x, v
  $\| D\varphi_x(v) \| \leq \eta \| v \| \quad \forall v \in E^s$
  $\| D\varphi_x(v) \| \geq \eta^{-1} \| v \| \quad \forall v \in E^u$, $x \in K$



Shrinks vectors from E^s by $\eta < 1$
Extends vectors from E^u by $1/\eta > 1$
but only for $x \in K$

Ex
1) $\varphi : M \to M$ hyperbolic : $K = M$
2) hyperbolic fixed or periodic pts
3) Horseshoe !

Rmk Similarly for flows but now there's also one neutral direction

$$TM = E^s \oplus E^u \oplus \underbrace{\mathrm{span}(v)}, \quad v \neq 0$$

v.f. generating the flow

allows for periodic orbits

Ex Geodesic flows on surfaces of curvature < 0 (e.g. $= -1$)

Hyperbolicity is one of the central notions in modern dynamics!

hyperbolicity + a bit more $\Rightarrow$ a lot of dynamical features

Structural stability of hyperbolic sets

Hyperbolicity $\Rightarrow$ many important dynamics features

Here we focus on str. stability

Def K = compact invariant set is locally maximal if $\exists\ U \supset K$ such that K is the maximal inv set in U:

$$\varphi^k(x) \in U\ \forall k \in \mathbb{Z} \Rightarrow x \in K$$

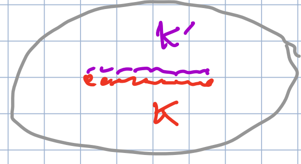
$$\Leftrightarrow \quad K = \bigcap_{k \in \mathbb{Z}} \varphi^k(U)$$

Thm K locally max & hyperbolic for φ

$\Rightarrow$ φ is str. stable near K:

$\psi \overset{C^1}{\approx} \varphi \Rightarrow \exists\ h :$ nbd of $K \xrightarrow[\text{homeo}]{}$ nbd of K

s.t. $\psi = h \varphi h^{-1}$



Ex $K =$ a hyperbolic fixed pt

Thm $\Leftrightarrow$ Hartman-Grobman

$\Leftarrow)$ $\varphi(x) = x$ $\underset{IFT}{\Rightarrow}$ $\psi(y) = y$

$D\psi|_y \approx D\varphi|_x \Rightarrow$ hyperbolic

HG $\psi \sim D\psi \sim D\varphi \sim \varphi$

$\Rightarrow)$ $\psi = D\varphi + \dots$ Thm $\Rightarrow$ $\varphi \sim D\varphi$: HG

Ex $K = M$, φ hyperbolic

$\Rightarrow \varphi$ is str. stable

This is what we will prove

Particular case

Thm (Anosov)

$\varphi = A : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ hyperbolic

$\psi \overset{C^1}{\approx} A \Rightarrow \psi$ is top conj to A :

$\exists h$ $\psi = h A h^{-1}$

Pf. For the sake of simplicity
$n = 2$: $\mathbb{T}^n = \mathbb{T}^2$: A is 2×2

- Write

C^1-small

$$\varphi = A + R \;:\; \mathbb{T}^2 \longrightarrow \mathbb{T}^2 ;$$
$$h = id + H \quad \mathbb{T}^2 \longrightarrow \mathbb{T}^2 ;$$

$R : \mathbb{T}^2 \longrightarrow \mathbb{R}^3$
$H : \mathbb{T}^2 \longrightarrow \mathbb{R}^2$

will be C^0-small

$$\begin{tikzcd} \mathbb{T}^2 \arrow[r, "\varphi"] & \mathbb{T}^2 \\ \mathbb{T}^2 \arrow[u, "h"] \arrow[r, "A"] \arrow[ur, phantom, "\circlearrowright"] & \mathbb{T}^2 \arrow[u, "h"'] \end{tikzcd} \qquad \text{lift} \qquad \begin{tikzcd} \mathbb{R}^2 \arrow[r, "\varphi"] & \mathbb{R}^2 \\ \mathbb{R}^2 \arrow[u, "h"'] \arrow[r, "A"'] & \mathbb{R}^2 \arrow[u, "h"'] \end{tikzcd}$$

not really needed

$$\varphi = h A h^{-1}$$

$$(A+R)\circ(id+H) = (id+H)\,A$$
$$(A+R)\circ(x + H(x)) = Ax + H(Ax)$$
$$\cancel{Ax} + AH(x) + R(x + H(x)) = \cancel{Ax} + H(Ax)$$
$$H(Ax) - AH(x) = R(x + H(x))$$

This the equation on H we need to solve.

Let's start with simpler equation

$$\underbrace{H(Ax) - AH(x)}_{\text{unknown}} = \underbrace{R(x)}_{\text{given}}$$

$L: H \longmapsto H\circ A - A\circ H$ Linear map in H

$C^0(\mathbb{T}^2;\mathbb{R}^2) \longrightarrow C^0(\mathbb{T}^2;\mathbb{R}^2)$

Claim L is invertible

Pf and $\|L^{-1}\| \leq \frac{1}{1-\lambda}$ ← does not matter

e_1, e_2 eigenvectors of A

$\lambda_1 \; \lambda_2$ eigenvalues

$\lambda_1 = \lambda_2^{-1} > 1 > \lambda_2 =: \lambda$

$H = H_1 e_1 + H_2 e_2$

$R = R_1 e_1 + R_2 e_2$

$\Rightarrow \quad H_1(Ax) - \lambda_1 H_1(x) = R_1(x)$

$H_2(Ax) - \lambda_2 H_2(x) = R_2(x)$

Consider $P: C^0(\mathbb{T}^2) \longrightarrow C^0(\mathbb{T}^2)$

identity

$g \longmapsto g\circ A \qquad \|P\| = 1$

$\underbrace{(P - \lambda_i I)}_{\lambda_i(\lambda_i^{-1}P - I)} H_i = R_i$

when $i=1 \qquad \lambda_1^{-1} = \lambda_2 < 1 \Rightarrow \|\lambda_2 P\| < 1$

$\Rightarrow (\lambda_2 P - I)^{-1} = -\underbrace{(I + \lambda_2 P + \lambda_2^2 P^2 + \dots)}_{\text{converges}}$

when $i=2$

$$P-\lambda_2 I = P^{-1}\underbrace{(I-\lambda_2 P)}_{\text{invertible}} \qquad \|\lambda_2 P\| = \lambda_2 < 1$$

$$\Rightarrow \quad L = \begin{pmatrix} P-\lambda_1 I \\ P-\lambda_2 I \end{pmatrix} \leftarrow \text{invertible}$$

$\triangleleft$

Back to solving

$$H\circ A - A\circ H = R(I+H)$$

<u>Recall</u>: Contraction mapping principle:

$$\underset{\text{compl. metric space}}{\Phi : X \xrightarrow{C^0} X} \qquad d(\Phi(x), \Phi(y)) \le \eta\, d(x,y)$$

$$0 < \eta < 1$$

$$\Rightarrow \exists! \text{ fixed pt } x : \underbrace{\Phi(x) = x}$$ fixed pt equation

<u>Pf</u> Take any $y \in X$ and set

$y_k = \Phi^k(y) \leftarrow$ Cauchy sequence $\leftarrow$ <u>Ex</u>

$y_k \longrightarrow x \quad \leftarrow$ Fixed pt

$$\Phi(x) = \lim \Phi(y_k)$$

$$\| \qquad \qquad \|$$

$$\lim y_{k+1} = x$$

$\triangleleft$

Take $X = C^0(\mathbb{T}^2; \mathbb{R}^2)$ with sup-norm

$$\Psi : X \longrightarrow X$$
$$\Psi(H) = R(I + H)$$

Key equation:

$$L(H) = \Psi(H)$$
$$\Leftrightarrow \quad H = L^{-1}\Psi(H)$$ ← fixed pt equation

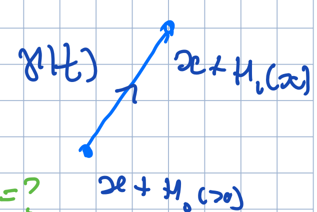
Claim $\Phi = L^{-1}\Psi : X \longrightarrow X$ is a contraction mapping

$\Downarrow$

Thm

Pf. • $\| L^{-1}\Psi(H_1) - L^{-1}\Psi(H_0) \| \le \| L^{-1} \| \, \| \Psi(H_1) - \Psi(H_0) \|$

• $\Psi(H_1) - \Psi(H_0)(x)$



$$= R(x + H_1(x)) - R(x + H_0(x)) \; =?$$

$$= \int_0^1 \frac{d}{dt} R(\underbrace{x + tH_1(x) + (1-t)R_0(x)}_{\gamma(t)}) \, dt$$

standard and useful trick

Alternatively one can apply the mean value thm to component-wise $R(\gamma(t))$

$$\sup_x |\Psi(H_1) - \Psi(H_0)(x)|$$

$$= \sup_x |R(x + H_1(x)) - R(x + H_0(x))|$$

$$\leq \sup_x \int_0^1 \left| \frac{d}{dt} R(\underbrace{x + tH_1(x) + (1-t)H_0(x)}_{\gamma(t)}) \right| dt$$

$$\leq \sup_x \int_0^1 \left| DR_{\gamma(t)} \cdot (H_1(x) - H_0(x)) \right| dt$$

$$\leq \sup_x \sup \|DR\| \cdot |H_1(x) - H_0(x)|$$

$$\Rightarrow \|\Psi(H_1) - \Psi(H_0)\| \leq \|R\|_{C^1} \cdot \|H_1 - H_0\|$$

$$\Rightarrow \|\Phi(H_1) - \Phi(H_0)\| \leq \underbrace{\|L^{-1}\| \cdot \|R\|_{C^1}}_{\eta} \cdot \|H_1 - H_0\|$$

$\|R\|_{C^1}$ small enough $\Rightarrow$ $0 < \eta < 1$

$\Rightarrow$ Φ is a contraction mapping

Also need to show that $h = id + H$ is a homeo. Not obvious but not hard

- Ex Hint use again the fact that A is hyperbolic $\lhd$

Rmk R C^1-small $\Rightarrow$ $id + R$ is a C^1-diffeo

(Nuance) $\Leftarrow$ Inv. Function theorem

But R C^0-small $\not\Rightarrow$ $id + R$ is a homeo

Rmk: A similar argument proves
(Ex) the Hartman–Grobman thm.

$\longrightarrow$ The End $\longleftarrow$