Math 235, Dynamical systems
Winter 2022

Lecture 1
01/04-2022

→ Go through basic info

* No exams, no hw
* Problems stated in lectures }

Up to them how much they take home

* OH: by appointment

Plan:

- basic concepts and examples
- elements of ergodic theory
- maps of S^1, the Denjoy example
- local normal forms, Hartman-Grobman and local analysis of DS
- hyperbolicity, horseshoes
- topological entropy

varying degree of detail

- Not a comprehensive course
- Examples are often non-trivial and very important

Math 235, Dynamical Systems, Winter 2022

- **Lectures:** TTh 1:30 - 3:05 PM, McHenry Clrm 1279 (the first two weeks remotely)
- **Instructor:** Viktor Ginzburg; office: McHenry 4124
  email: ginzburg(at)ucsc.edu
- **Office Hours:** TBA or by appointment
- **Text:** There will be no "official" textbook in this course. Some suggested reading and references:
  - *Introduction to the Modern Theory of Dynamical Systems* by A. Katok and B. Hasselblatt;
  - *Geometrical Methods in the Theory of Ordinary Differential Equations* by V.I. Arnold;
  - *Lectures on Dynamical Systems* by E. Zehnder;
  - *Measure and Category* by J.C. Oxtoby; more analysis than dynamics
  - *Ergodic Theory* by I.P. Cornfeld, S.V. Fomin and Y.G. Sinai;
  - *Lecture Notes on Ergodic Theory* by C. Walkden;
  - *Dynamical Systems* by C. Robinson.
- **Tentative Syllabus:** This course will be a potpourri of dynamical systems, focusing on examples and main concepts and notions rather than technical proofs of general theorems. I plan to discuss or at least briefly touch upon some of the following topics and concepts:
  - elements of ergodic theory,
  - topological entropy,
  - structural stability,
  - maps of the circle and the Denjoy example,
  - local analysis and local normal forms,
  - hyperbolic dynamical systems.

  This will not be a comprehensive course in dynamical systems, but rather a non-technical overview of central notions and ideas. Examples are particularly important in dynamics and I will devote a lot of attention to them.

  COVID-19 Information: Please take care to comply with all university guidelines about masking in indoor settings, performing daily symptom and badge checks, testing as required by the campus vaccine policy, self-isolating in the event of exposure, and respecting others' comfort with distancing. Please do not come to class if your badge is not green. If you are ill or suspect you may have been exposed to someone who is ill, or if you have symptoms that are in any way similar to those of COVID-19, please err on the side of caution and stay home until you are well or have tested negative after an exposure.
- **Lecture notes (pdf files): To be posted here**

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§1. Introduction

What is a dynamical system?

- M a topological space (reasonably good), usually compact, e.g. a manifold

time

discrete

- $\to \varphi : M \to M$ a map, continuous or smooth, often but not always invertible. Interested in φ^k, $k \in \mathbb{N}$ or $\mathbb{Z}$

$$\underbrace{\mathbb{N} \text{ or } \mathbb{Z} \longrightarrow C^0(M,M) \text{ or } C^\infty(M,M)}_{\text{(semi) group homomorphism}}$$

or

continuous

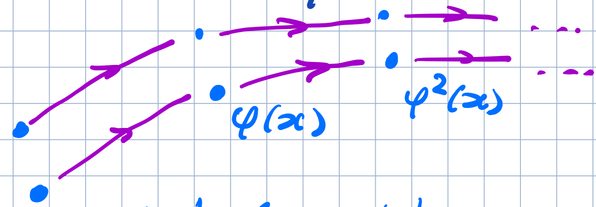
$\to \varphi^t$ a flow on M

$$\underbrace{\mathbb{R} \longrightarrow \mathrm{Homeo}(M) \text{ or } \mathrm{Diffeo}(M)}_{\text{group homomorphism}}$$

$$t \mapsto \varphi^t, \quad \varphi^{t_1+t_2} = \varphi^{t_1}\varphi^{t_2}$$

This set up models:

- M = the set of states of a (deterministic) system
- t, $t = k$ = time
- φ^t, φ^k = the evolution of the system



x = intial condition

Rmk can set $\varphi = \varphi^T$

③

- **Basic source: ODE = v.f.**

  - M a manifold (e.g. a domain in $\mathbb{R}^n$)
  - given a v.f. = ODE, complete
  - φ^t the flow of it:

    $\varphi^t(x)$ = sol with the initial condition x

  or $\varphi = \varphi^T$

- Dynamical systems $\supset$ ODE's
  But focus is different.

  In DS we are interested in
  - "global", geometrical features of φ, qualitative properties of φ
  - **not in** determining $\varphi^t(x)$ explicitly

  E.g. • The behavior of $\varphi^t(x)$ as $t \to \infty$ for x in a certain subset

  • Does $\varphi^t(x)$ comes back to x? How close? For "how many" x?

A variant: • M is a probability measure space
• φ measure preserving
...

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- Basic Definitions and Terminology

Need some language to talk about qualitative properties

- $\{\varphi^k(x) \mid k \in \mathbb{N} \text{ or } \mathbb{Z}\}$, $\{\varphi^t(x) \mid t \in \mathbb{R}\}$ — the orbit of x

  $\mathbb{N}$: positive semi-orbit

Notation: $\mathcal{O}(x)$

- x is a fixed pt if

  $\underbrace{\varphi(x) = x}_{\Rightarrow\ \varphi^k(x) = x\ \forall k}$ or $\varphi^t(x) = x \ \forall t$

- x is a periodic point if $\exists\ \underbrace{n, T}_{>0}$

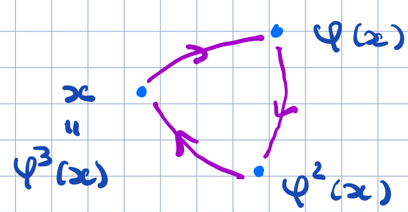
  $\varphi^n(x) = x$ or $\varphi^T(x) = x$

  n, T: a period

Rmk
- n is a period $\Rightarrow$ $2n$, $3n$, etc are also periods
- $\Rightarrow$ minimal period

Ex a fixed pt is also a periodic pt with minimal period 1

- The orbit through a periodic pt is a periodic orbit:



a periodic orbit with min period 3

- $X \subset M$ is an *invariant set* if $\underbrace{\varphi(X) \subset X}_{}$ or $\varphi^t(X) \subset X \;\; \forall t$

  $(\Rightarrow \varphi^k(x) \subset X, \; k \in \mathbb{N})$ <u>usually closed</u>

<u>Ex</u> An orbit $\mathcal{O}$ is an invariant subset

$\mathcal{O}$ is periodic $\Longleftrightarrow$ $\mathcal{O}$ is closed

M is compact

- φ (or φ^t) is <u>minimal</u> if M has no <u>closed</u> invariant subsets

  $\Longleftrightarrow$ every orbit is dense

<u>Ex.</u> φ minimal $\Rightarrow$ no periodic orbits

(M compact)

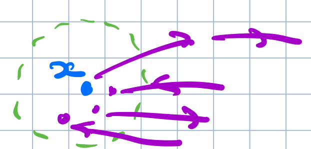
- φ (or φ^t) is topologically transitive if $\exists$ a dense orbit

  $\Longleftrightarrow$ every inv. subset is nowhere dense

- x is recurrent if x comes back to its arbitrarily small nbd infinitely many times:

  $\forall\, U \ni x \;\; \exists\, k_i \to \infty$ s.t. $\varphi^{k_i}(x) \in U$

U



Ex. • x is periodic
• the orbit through x is dense $\Rightarrow$ x is recurrent

• ω-limit set of x:

$$\omega(x) = \{\text{all limits of } \varphi^{k_i}(x),\ k_i \to \infty\}$$

$$= \bigcap_{n=1}^{\infty} \overline{\{\varphi^k(x) \mid k \geq n\}}$$ closure

α-limit set: similarly but $-\infty$

similarly for flows

Ex. • $\varphi^k(x) \xrightarrow[k\to\infty]{} y \Leftrightarrow y = \omega(x)$

• x is periodic $\Leftrightarrow$ $\omega(x) = \alpha(x) = \mathcal{O}(x)$
$=$ the orbit through x

• the orbit through x is dense
$\Leftrightarrow$ $\omega(x) = M = \alpha(x)$ invertible

• x is recurrent $\Leftrightarrow$ $x \in \omega(x) = \alpha(x)$

• For flows

$$\omega(x) = \bigcap_{T \geq 0} \overline{\{\varphi^t(x) \mid t \geq T\}}$$

is connected. Ex

• Many more to follow

Some examples of DS

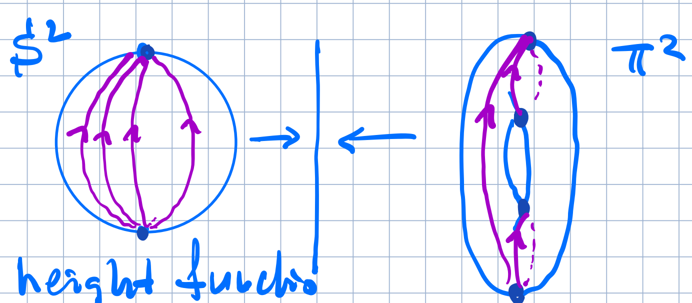
- **Ex1** Gradient-like flows — Boring DS!
  - M is a closed manifold
  - $f: M \xrightarrow{C^\infty} \mathbb{R}$
  - X = gradient-like v.f.:

$$L_X f \geq 0$$

$=$ only at $Crit(f)$

moreover: $\{X=0\} = Crit(f)$

Ex • $X = \nabla f$ for some R. m.



$$\frac{d}{dt} f(\varphi^t(x)) = L_X f(\varphi^t(x)) \geq 0$$

$x \in Crit(f) \iff \longrightarrow = 0$

$\Rightarrow$ • recurrent pts = periodic pts
= fixed pts
= $Crit(f)$

- No dense orbits
- $\forall x \quad \omega(x) \subset \mathrm{Crit}(x)$
  $\quad\quad\quad \alpha(x) \subset \mathrm{Crit}(x)$

If f is Morse:

$\omega(x)$ or $\alpha(x)$ is just one critical pt

Ex - hard

construct f such that
$\exists\ x$ s.t. $\omega(x) \subset \mathrm{Crit}(f)$
is a circle.

Ex2 Rotations of S^1

already much more interesting

$S^1 = \mathbb{R}/\mathbb{Z} = \{|z|=1\} \subset \mathbb{C}$

$\varphi : S^1 \longrightarrow S^1$

$\theta \longmapsto \theta + \alpha \mod 1$, $\alpha \in S^1$ fixed

$e^{2\pi i\theta} \longmapsto e^{2\pi i\theta} e^{2\pi i\alpha}$

Prop

two alternatives

- φ is periodic ($\varphi^q = id$)

$\Leftrightarrow$ $\alpha \in \mathbb{Q}$: $\alpha = \frac{p}{q}$

($\Leftrightarrow$ every pt is q periodic)

- φ is minimal (every orbit is dense)

$\Leftrightarrow$ $\alpha \notin \mathbb{Q}$

Pf $\varphi^k(\theta) = \theta + k\alpha \mod 1$

- $\alpha \in \mathbb{Q}$: $\alpha = \frac{p}{q}$

$\Leftarrow$

$\Rightarrow \varphi^q(\theta) = \theta + q\frac{p}{q} = \theta + p = \theta$

$\Rightarrow$) $\varphi^q = id$: $\varphi^q(\theta) = \theta + q\alpha = \theta \mod 1$

$\Rightarrow q\alpha = p \in \mathbb{Z} \Rightarrow \alpha = \frac{p}{q}$

• $\alpha \notin \mathbb{Q}$ Look at the orbit of $0 = 1$

$\Rightarrow \varphi^k(0) = k\alpha \neq 0$ in S^1

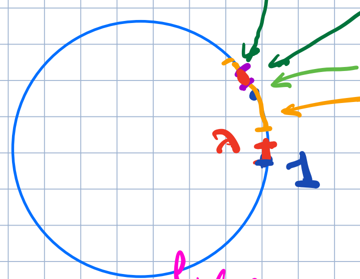
$\Rightarrow \underbrace{\varphi^{\ell}(0)}_{\ell\alpha} \neq \underbrace{\varphi^{m}(0)}_{m\alpha} \ \forall \ell, m$ $\qquad \underbrace{\varphi^{\ell-m}(0) = 0}_{(\ell-m)\alpha = 0 \bmod \mathbb{Z}}$

$\Rightarrow \varphi^k(0)$ has a limit pt

$\Rightarrow \forall \varepsilon > 0 \ \exists \ \ell_0, \ell_1$ s.t.

$0 \neq d(\underbrace{\varphi^{\ell_0}(0)}, \underbrace{\varphi^{\ell_1}(0)}) < \varepsilon$

distance in S^1, rotation invariant



limit pt

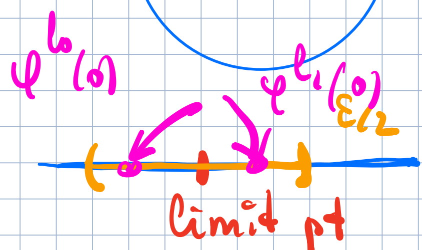
$\varepsilon/2$ - nbd

$\lambda = \varphi^{\ell_1 - \ell_0}(0)$

$= 0 + \ell_1\alpha - (0 + \ell_0\alpha)$

$= (\ell_1 - \ell_0)\alpha = \alpha'$

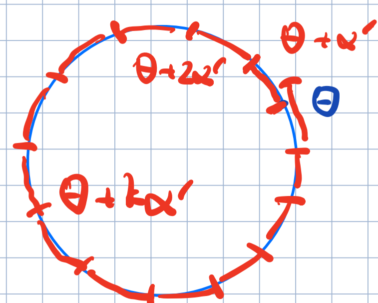
is ε-close to $0 \in S^1$



$\Rightarrow \forall \ \theta \in S^1$

$\varphi^{k(\ell_1 - \ell_0)}(\theta) = \theta + k(\ell_1 - \ell_0)\alpha$

is ε - dense



since ε is arbitrary

$\Rightarrow$ the orbit is dense

Rmk We will come back to this example many times and refine it.

Ex Prove that the decimal expansion of 2^k may begin with any finite sequence of digits:
Given $\lambda_1 \dots \lambda_s = \lambda$ $\exists$ k s.t.

$$2^k = \lambda_1 \dots \lambda_s \dots$$

Ex3 Translations on compact groups

G compact (metrizable) top gp
e.g. a Lie gp

$\varphi(x) = \underbrace{x\cdot\alpha}_{\text{right translation}}$ $\alpha \in G$ fixed

$\Theta(1) = \{\alpha^k \mid k \in \mathbb{Z}\}$ subgroup abelian

use multiplicative notation

$\Rightarrow H = \overline{\Theta(1)}$ is a closed abelian subgroup

$\Rightarrow \Theta(x) = \overbrace{x\Theta(1)}^{x\alpha^k} =$ translation of $\Theta(1)$

$\overline{\Theta(x)} = xH =$ —— · —— H

$\Rightarrow \varphi$ can be minimal only when G is abelian : $G = H = \overline{\Theta(1)}$

$\Rightarrow$ when G is a Lie gp then H is an extension of $\mathbb{Z}_r$ by $\mathbb{T}^n$

need to know a bit of Lie gps

$$1 \to \underbrace{\mathbb{T}^n}_{\text{connected component of id}} \to H \to \mathbb{Z}_r \to 1$$

Ex. $G = S^1$

The only closed subgroups are

- cyclic (roots of unity)
- S^1

$\Rightarrow$ H can only one of the two types:

$\to$ a cyclic subgroup $\Leftrightarrow \alpha \in \mathbb{Q}$

$\to$ $H = S^1 \Leftrightarrow \alpha \notin \mathbb{Q}$

To summarize $\varphi : G \to G,\ x \mapsto x \cdot \alpha$

compact (abelian)

- either all orbits $\mathcal{O}(x) = x\,\mathcal{O}(1)$ are dense
- or none of the orbits are dense:

$\underbrace{\mathcal{O}(x)}_{x\mathcal{O}(1)} \cap U \neq \emptyset \iff \mathcal{O}(1) \cap x^{-1}U \neq \emptyset$

For group translations

minimal $\iff$ top transitive

Lecture 2
01/06 - 2022

Ex4 Translations of $\mathbb{T}^n$

- $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n = \underbrace{S^1 \times \dots \times S^1}_{n}$

$\theta = (\theta_1, \dots, \theta_n)$ mod 1

$\varphi : \mathbb{T}^n \longrightarrow \mathbb{T}^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$

$\theta \longmapsto \theta + \alpha$, use additive notation

Now $\exists$ more possibilities: orbits need not be either dense or periodic

But $\quad \mathcal{O}(x) = \{ \varphi^k(x) \mid k \in \mathbb{Z} \} = \{ x + k\alpha \mid k \in \mathbb{Z} \}$

$= x + \mathcal{O}(0) = x + \{k\alpha\}$

$\Rightarrow \quad \overline{\mathcal{O}(x)} = x + \underbrace{\overline{\mathcal{O}(0)}}$

closed abelian subgroup: an extension of $\mathbb{Z}_r$ by $\mathbb{T}^{m \leq n}$

$\Rightarrow$
- All orbits are periodic $\Leftrightarrow$ $\mathcal{O}$ is periodic $\Big\} \Leftrightarrow \varphi$ is periodic
- All orbits are dense $\Leftrightarrow$ $\mathcal{O}(0)$ is dense $\Big\} \Leftrightarrow \varphi$ is minimal

characterize these two situations

translation (homeo) of a dense set is dense

- 0 is periodic $\Leftrightarrow \alpha \in \mathbb{Q}^n$

$q\alpha = 0 \mod \mathbb{Z}$ for some q

$\Leftrightarrow \underbrace{(q\alpha_1, \dots, q\alpha_n)}_{\varphi^q(0) = 0} = 0 \mod \mathbb{Z}$

$\alpha_i = \frac{p_i}{q_i}$ $\quad i = 1, \dots, n \qquad q = \mathrm{lcm}(q_1, \dots, q_n)$

$\alpha \in \mathbb{Q}^n$

- <u>Prop</u> φ is minimal

$\Leftrightarrow \mathbb{1}, \alpha_1, \dots, \alpha_n$ is linearly ind over $\mathbb{Q}$:

$r_0 \cdot \mathbb{1} + \sum_{i=1}^{n} r_j \alpha_j = 0$, $r_j \in \mathbb{Q}$ or $r_j \in \mathbb{Z}$ $\Rightarrow$ all $r_j = 0$, $j = 0, \dots, n$

<u>Rmk</u> • $\mathbb{R}$ is a v.s. over $\mathbb{Q}$

$\dim_{\mathbb{Q}} \mathbb{R} = \infty$ (continuum)

$\Leftrightarrow \dim_{\mathbb{Q}} \underbrace{\mathrm{span}(\mathbb{1}, \alpha_1, \dots, \alpha_n)}_{\subset \mathbb{R}} = n+1$

• can replace $\mathbb{Q}$ by $\mathbb{Z}$

<u>Rmk</u> A lot can be in between these two cases:

can have

$\underset{\substack{\text{periodic} \\ \text{all } \alpha_i \in \mathbb{Q}}}{1} \leq \dim_{\mathbb{Q}}(\mathbb{1}, \alpha_1, \dots, \alpha_n) \leq \underset{\text{minimal}}{n+1}$

Ex. $n=1$, $\alpha \in \mathbb{R}/\mathbb{Z}$, $\alpha \in \mathbb{R}$

$1, \alpha$ linearly ind over $\mathbb{Q}$

$\Leftrightarrow$ $\alpha \notin \mathbb{Q}$

$\Leftrightarrow$ φ minimal $\Leftrightarrow$ $\mathcal{O}(0)$ is dense

$\Uparrow$ Last lecture

Pf Recall: for translations of compact (abelian) groups:

$$\left\{\begin{matrix}\text{top transitive:}\\ \text{one dense orbit}\end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix}\text{minimal:}\\ \text{all orbits are dense}\end{matrix}\right\}$$

i) lin dependent $\Rightarrow$ not minimal

$\sum \underbrace{r_j}_{\in \mathbb{Z}} \alpha_j = -r_0 \in \mathbb{Z}$: a resonance relation

not all $r_j \neq 0$

$f : \mathbb{T}^n \to S^1 \subset \mathbb{C}$

$$\underbrace{f(\theta) = \exp\left(2\pi i \sum_{j=1}^{n} r_j \theta_j\right)}_{\text{trig. polynomial} \Rightarrow C^0}$$

$\rightarrow$ $f \not\equiv \text{const}$

$\rightarrow$ f is invariant: $f(\theta+\alpha) = f(\theta)$:

$$\begin{aligned} f(\theta+\alpha) &= \exp\left(2\pi i \sum_{j=1}^{n} r_j(\theta_j+\alpha_j)\right) \\ &= \exp\left(2\pi i \sum r_j \theta_j\right) \exp\left(2\pi i \sum r_j \alpha_j\right) \\ &= f(\theta)\, \underbrace{\exp(-2\pi i r_0)}_{1} \quad r_0 \in \mathbb{Z} \\ &= f(\theta) \cdot 1 \end{aligned}$$

$\to$ $f(0)=1$, f is C^0 & $f \not\equiv 1$

$\Rightarrow$ • $\underbrace{X = \{\theta \mid f(\theta)=1\}}_{\supset\, \Theta(0) = \{k\alpha\}}$ proper invariant closed subset

• $\mathbb{T}^n \setminus X \neq \emptyset$, open

$\Rightarrow$ $\Theta(0)$ is not dense

φ is not top transitive

$\Leftrightarrow$ not minimal

Pf $\underbrace{\text{top trans}}$ $\Rightarrow$ $\exists\, k$: $\varphi^k(U) \cap V \neq \emptyset$

$\exists$ a dense orbit : $\{\varphi^j(x)\}$ dense

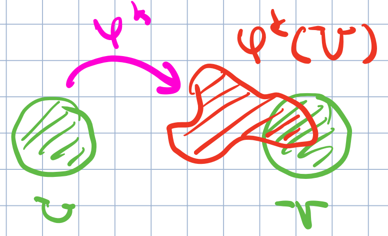
$\Rightarrow$ $\varphi^{j_0}(x) \in U$ & $\varphi^{j_1}(x) \in V$

$\varphi^{j_1 - j_0} \underbrace{\varphi^{j_0}(x)}_{\in U} \in V$

$\Rightarrow$ $\varphi^k(U) \cap V$, $k = j_1 - j_0$

2) lin independent $\Rightarrow$ minimal

Lemma M compact, separable metric space
$\varphi: M \to M$ is top transitive
$\Leftrightarrow$ $\forall\ U \,\&\, V \subset M$ open $\exists\ k$ s.t
$\varphi^k(U) \cap V \neq \emptyset$



Pf: Ex [KH]
(not entirely obvious for $\Leftarrow$)

Cor
φ is top transitive
$\Leftrightarrow$ $\forall\ U, V$ open <u>invariant</u>
$U \cap V \neq \emptyset$

$\Rightarrow$ every C^0 invariant function is const

could have used in 1), clear anyway

To the pf: by contradiction
- U, V open invariant
- Assume $U \cap V = \emptyset$

$f = \chi_U$: $f(x) = \begin{cases} 1 & x \in U \\ 0 & x \notin U \supset V \end{cases}$

$\Rightarrow$ $f \in L^2(\mathbb{T}^n)$, invariant
$f \not\equiv$ const $\Leftarrow$ $f|_V = 0$
$f \not\equiv 1$ a.e.

$$f(\theta) = \sum_k \underbrace{f_k}_{\text{Fourier coeff}} \cdot \exp\Big(2\pi i \underbrace{\sum_j k_j \theta_j}_{\langle k, \theta\rangle}\Big)$$

$$\underbrace{k = (k_1, \dots, k_n)}_{\in \mathbb{Z}^n}$$

$$f(\theta + \alpha) = f(\theta)$$

$$f(\theta+\alpha) = \sum_k f_k \underbrace{\exp\Big(2\pi i \sum k_j(\theta_j + \alpha_j)\Big)}_{\exp(2\pi i \sum k_j\theta_j)\, \exp(2\pi i \sum k_j \alpha_j)}$$

$$= \underbrace{\sum_k \underbrace{f_k \exp\Big(2\pi i \sum k_j \alpha_j\Big)}_{\text{Fourier coeff}} \exp\Big(2\pi i \sum k_j \theta_j\Big)}_{f(\theta+\alpha)}$$

$$\Rightarrow \quad f_k = f_k \exp\Big(2\pi i \sum k_j \alpha_j\Big)$$

at least one $\neq 0 \Leftarrow f \not\equiv \text{const}$

$k \neq 0$

$$\Rightarrow \quad \exp\Big(2\pi i \sum k_j \alpha_j\Big) = 1$$

$$\Rightarrow \quad \sum k_j \alpha_j \in \mathbb{Z}$$

$\Leftrightarrow \quad 1, \alpha_1, \dots, \alpha_n$ lin dependent over $\mathbb{Q}$

$\triangleleft$

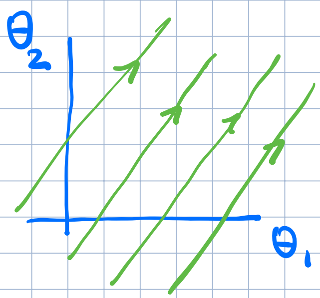
• Linear flows on $\mathbb{T}^n$

$\varphi^t : \mathbb{T}^n \to \mathbb{T}^n$ $\quad\quad$ $\theta = (\theta_1, \dots, \theta_n)$

$\alpha = (\alpha_1, \dots, \alpha_n)$

$\varphi^t(\theta) = \theta + t\alpha$

$t \in \mathbb{R}$



Very similar to translations

Rmk only α_i/α_j matter

Prop • all ratios $\alpha_i/\alpha_j \in \mathbb{Q}$

$\iff$ all orbits are closed

• $\alpha_1, \dots, \alpha_n$ lin ind over $\mathbb{Q}$

$\iff$ all orbits are dense (minimal)

$\iff$ one orbit is dense (top. transitive)

no 1 here: it's easier for the orbits of φ^t to be dense than for $\varphi = \varphi^1$

Pf - Ex

- Digression to number theory:
  Kronecker thm

1D case

$\alpha \notin \mathbb{Q}$ $\forall\ \lambda \in \mathbb{R}$ $\forall\ \varepsilon > 0$

$\exists\ k, m \in \mathbb{Z}$ s.t.

$$|k\alpha + m - \lambda| < \varepsilon$$

This is $\iff$ $\{k\alpha\}$ dense in $S^1 = \mathbb{R}/\mathbb{Z}$

nD case

Thm (Kronecker)

$1, \alpha_1, \dots, \alpha_n$ lin ind over $\mathbb{Q}$

$\iff$ $\forall\ (\lambda_1, \dots, \lambda_n) = \lambda \in \mathbb{R}^n$, $\forall\ \varepsilon > 0$

$\exists\ m = (m_1, \dots, m_n)$ and $k \in \mathbb{Z}$ s.t

$$\underbrace{\| k\alpha + m - \lambda \| < \varepsilon}_{|k\alpha_i + m_i - \lambda_i| < \varepsilon}$$

Pf $\iff$ $\{k\alpha\}$ dense in $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$

Rmk Rich connections

DS $\longleftrightarrow$ number theory

Rmk Translations on compact Lie gps, $\mathbb{T}^n$, are isometries.

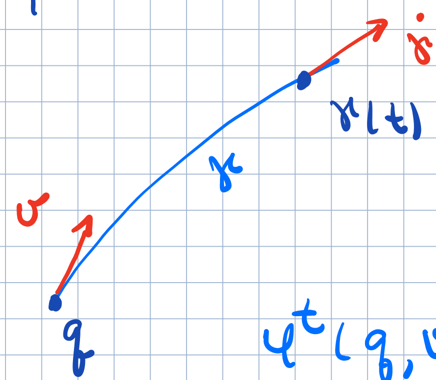
These examples exhaust all the dynamics complexity isometries can have.
Essentially nothing more complicated can happen

Ex 5 Geodesic flows

skipping details for now

- Q a Riemannian manifold, closed
- $M = STQ$ = unit tangent bundle

$\varphi^t : M \longrightarrow M$ the geodesic flow



γ the unit geodesic with

$\gamma(0) = q, \quad \dot\gamma(0) = v$

$$\varphi^t(q,v) = (\gamma(t), \dot\gamma(t))$$

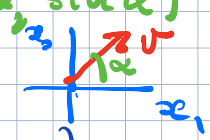
Extremely important.

Ex. $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ as a Riemannian manifold

a flat torus

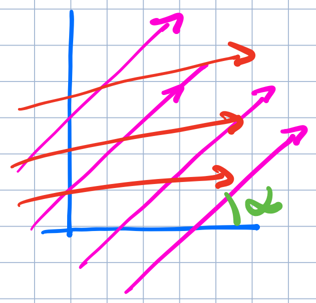
$$ST\,\mathbb{T}^2 = \underset{v\,=\,\alpha}{S^1} \times \underset{(x_1,x_2)\,=\,q}{\mathbb{T}^2}$$

$v = (\cos\alpha, \sin\alpha)$



$$\underbrace{\varphi^t(x,\alpha) = (x_1 + t\cos\alpha,\ x_2 + t\sin\alpha,\ \alpha)}_{\text{parallel transport on } \mathbb{R}^2 \text{ in the direction of } v}$$

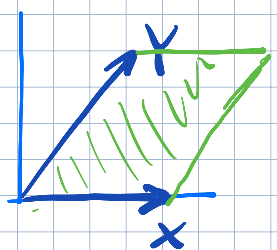
parallel transport on $\mathbb{R}^2$ in the direction of v



Depending on α
the orbits can be all
closed or all dense
in $\alpha = \text{const} = \alpha \times \mathbb{T}^2$

Rmk $\exists$ other flat metrics on $\mathbb{T}^2$:

$$\mathbb{T}^2 = \mathbb{R}^2/\Gamma, \qquad \Gamma = X\cdot\mathbb{Z} + Y\cdot\mathbb{Z}$$
$$X, Y \in \mathbb{R}^2$$



some of them are isometric
and some are not

But their geodesic flows are very similar

More interesting examples:
surfaces of const neg. curvature
Later?