Morse Theory, Math 232

Lecture Notes

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page #s as marked

Presentations (not included in Lecture Notes)

- Erman Cineli: Morse-Novikov Theory
- John Pelais: Lagrangian Floer theory
- Elijah Fender: Equivariant Morse Theory

Morse Theory, Math 232

2021W

Lecture 1
01/05

A few words about the class:

- Not on Canvas – email Zoom links
- Post lecture notes
  Record and post videos
- No exams or hw
  But mention problems in class
- Presentations (optional)
  – the end of the quarter
- No textbook – discuss sources below
- Prerequisites:
  - The manifolds sequence
  - Basic Differential Geom
  - (Co)homology

Textbooks and other sources:

- "Morse Theory" by J. Milnor (1963) ← Rec
- "Lectures on Morse theory, old and new" R. Bott BAMS 7 (1982), 331-358 ← Rec
- Many Diff Top books: Hirsh, Fomenko-Dubrovin-Novikov
  not so bad

Before Floer theory

- "Riemannian Geometry and Geometric Analysis" J. Jost (Chap 7) 2011 ← Recommended
- "Morse Theory and Floer Homology" M. Audin and M. Damian 2014
- Lectures on Morse Homology A. Banyaga and D. Hurtubise 2004

After Floer Theory

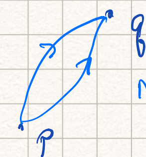
In mid 80s a new way of thinking about Morse Theory – Floer theory – was developed.

It's actually a variant (subset) of Morse theory but it has influenced the theory as a whole.

Our treatment in this class will be modern based on this new perspective but we probably only briefly touch upon Floer theory as such.

Suggested Topics for Presentations

- Morse-Novikov Theory (Ermon)
- Morse-Bott & Equiv Morse Theory
- Morse theory for geodesics (connecting two pts) Milnor's book



Milnor

- Morse theory for closed geodesics and Lusternik-Fet thm (Bott's notes)

Bott

- Convex Hamiltonian systems (Periodic orbits a la Ekeland, Fadel-Rabinowitz, pf of Weinstein conj in the convex case
- h-cobordis thm (Milnor)
- Lefschetz hyperplane thm
- Bott periodicity thm (over $\mathbb{C}$)

} Appl to top & alg geometry

- Hamiltonian circle actions: symplectic geometry, calculations of cohomology a la $\mathbb{C}P^n$, etc

What Morse theory is about

P = a reasonable space
e.g. finite or inf. dim manifold:
a closed manifold, space of closed loops,
or paths with fixed pts

$f: P \longrightarrow \mathbb{R}$ reasonably nice "smooth" func.

e.g. a "generic" smooth function on
a smooth closed manifold or
length or better energy
$x \longmapsto \int |\dot{x}|^2 dt$

Goal: relate critical pts of f
to the topology of P
homology or
homotopy type
Not pt set topology

E.g. • lower bound on $|\mathrm{Crit}(f)|$
in terms of $H_*(P)$

In particular the existence of $\mathrm{Crit}(f)$
How do we know $\exists$ at least one?

E.g. existence of closed geodesics

Conversely: understand the top of P
(e.g. $H_*(P)$) via the str
of f and in particular $\mathrm{Crit}(f)$
+ more info

Note: Many hugely important objects in
math, in physics are critical
pts of some functional f – variational
principles

Part I: Morse Theory for finite-dim manifolds

Setting: $P = M$: $\underbrace{\text{closed}}_{\text{can be relaxed}} = \text{compact}, \ \partial P = \emptyset$

$$f : P \xrightarrow{C^k} \mathbb{R}, \quad k \geq 2 \quad \text{usually } \infty$$

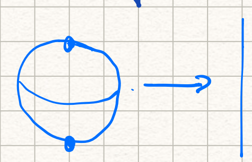
$$\operatorname{Crit}(f) = \{ p \mid \underbrace{df(p) = 0}_{L_X f(p) = 0 \ \forall X \in T_p P} \}$$

the set of critical pts

$$\boxed{\text{Main question: produce a meaningful lower bound for } |\operatorname{Crit}(f)|.}$$

§1 Motivation & Examples:

E.g. $\underbrace{|\operatorname{Crit}(f)| \geq 2}_{\exists \text{ max \& min}}$ — not very interesting



$S^n \subset \mathbb{R}^{n+1}$

Rmk. • But here we can already look at some interaction with top

• what if $|\operatorname{Crit}(f)| = 2$

$\operatorname{Crit}(f) = \{\max, \min\}$

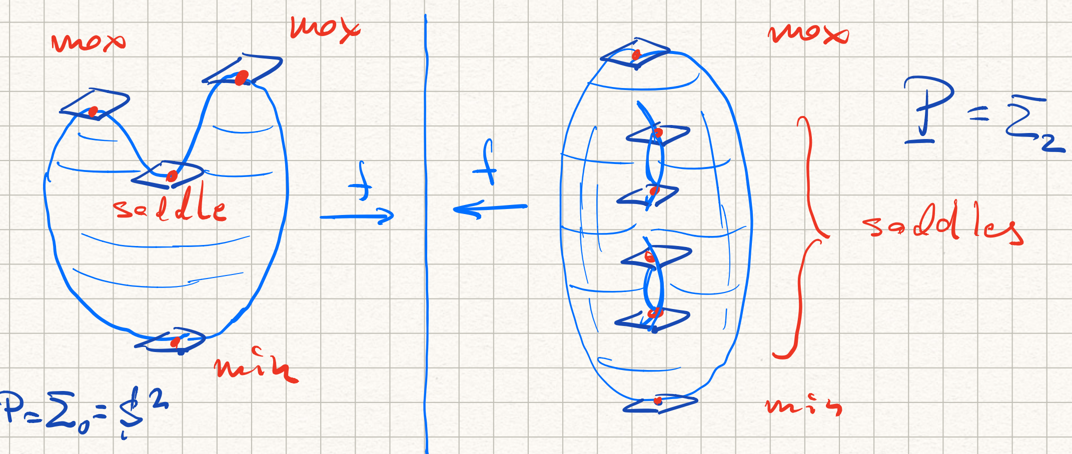
Fact: $P \underset{\text{homeo}}{\simeq} S^n$ but not necessarily diffeo

↑ Return to this later

Ex

$$\begin{array}{ccc} \text{closed surface} \to P & \longrightarrow & \mathbb{R} \\ \cap & \nearrow & \\ \mathbb{R}^3 & & \end{array} \quad \text{height function}$$

$p \in \mathrm{Crit}(f) \iff T_pP$ is horizontal

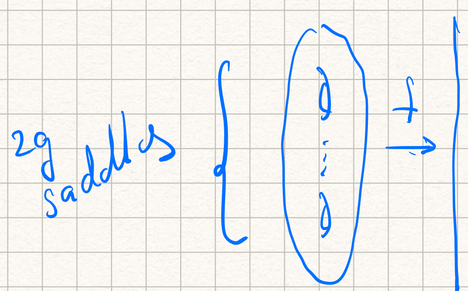


$P = \Sigma_0 = S^2$

$\# = 4 = \underset{\text{min possible}}{2}$ $\qquad\qquad$ $\# = 6$

These and similar examples suggest that

$$\boxed{\left| \mathrm{Crit}\left(P = \underbrace{\Sigma_g}_{\substack{\text{surface}\\ \text{of genus } g}} \xrightarrow{f} \mathbb{R}\right) \right| \geq \underbrace{2 + 2g}_{\dim H_*(\Sigma_g)}}$$



Not true (other than $g=0$)

But almost true

Ex. $\exists\ f: P = \Sigma_g \xrightarrow{C^\infty} \mathbb{R}$

with exactly 3 crit. pts when $g \geq 1$

sphere $\longrightarrow$ 2 —— . —— . $g=0$: S^2

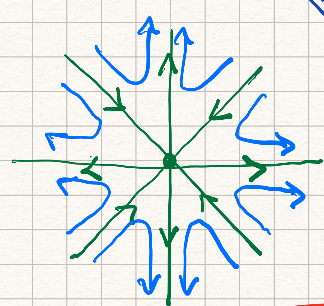
(obvious)

But not fewer?!

Construct such a function!

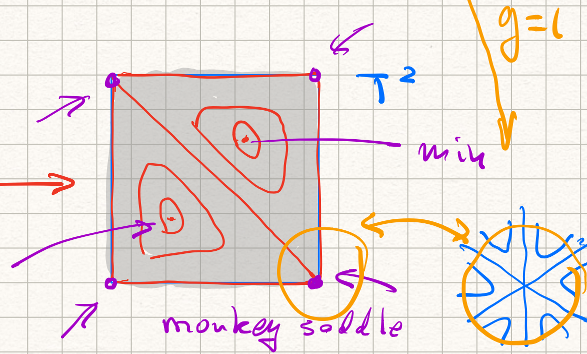
Hint: collapse all $2g$ saddles into one "monkey saddle":

Ex.



smth like $\mathrm{Re}\, z^{2g+1}$

$g=1$



Rmk. This f is not a height function

But true for a broad class of functions = Morse functions

$\Rightarrow$ Key definitions! — p. 8

Rmk • Such a function f cannot

- Ex. be the composition

$$\Sigma_g \underset{\text{embed}}{\hookrightarrow} \mathbb{R}^3 \xrightarrow{z} \mathbb{R}$$

• But it can be

$$\mathbb{T}^2 \underset{\text{imm}}{\looparrowright} \mathbb{R}^3 \xrightarrow{z} \mathbb{R}$$, but not $\Sigma_{g>1}$

Ref (From Elijah):

projecteuclid.org/euclid.ijm/1256050732

= T. Banchoff, F. Takens

"Height functions on surfaces with three critical points", Illinois. J. Math $\underline{19}$ (1975), 325 - 335.

§2 Basic Definitions – Morse functions and the strategy

$f : P \xrightarrow{C^2} \mathbb{R}$
$\quad\;\; p \in \mathrm{Crit}(f)$

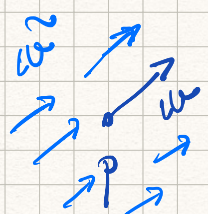
Def. The Hessian of f at p

$$d^2_p f : T_pP \times T_pP \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto (L_{\tilde v} L_{\tilde w} f)(p)$$

directional derivatives

could be v

$\tilde v, \tilde w$ ext of v & w to v.f. near p.



Need to check: independent of $\tilde v, \tilde w$

Prop. a) well-defined ← ind of $\tilde v$ & $\tilde w$
b) symmetric
c) In local coordinates: the ordinary hessian

$$d^2_p f(v, w) = \sum \frac{\partial^2 f}{\partial x_i \partial x_j}(p)\, v_i w_j$$

where $v = \sum v_i \frac{\partial}{\partial x_i}$, $w = \sum w_i \frac{\partial}{\partial x_i}$

$x = (x_1, \dots, x_n)$ ← local coord near p

Rmk. • Condition $p \in \mathrm{Crit}$ is essential:
otherwise $d_p^2 f$ is not well-defined

• Ex. $df(p) \neq 0 \Rightarrow \exists$ a coordinate system $(x_1, \dots, x_n) = x$
near p s.t.

$$f(x) = x_1$$

In fact one can make $f(x) = x_1 + Q(x)$
anything higher order

Pf c) $\Rightarrow$ a) & b)
the dumbest and simplest way to prove the prop

Proving c):

$$L_{\tilde{w}} f = \sum_j \underbrace{\tilde{w}_j}_{\text{functions of } x} \frac{\partial f}{\partial x_j}$$

$$L_v L_{\tilde{w}} f \Big|_p = \sum_{i,j} L_{\tilde{v}_i} \tilde{w}_j \Big|_p \underbrace{\frac{\partial f}{\partial x_j}(p)}_{= 0 \Leftarrow d_p f = 0}$$

$$+ \sum_{i,j} \underbrace{\tilde{v}_i(p)}_{v_i} \underbrace{\tilde{w}_j(p)}_{w_j} \frac{\partial^2 f}{\partial x_i \partial x_j}(p)$$

Using more advanced tools of calculus on manifolds:

Coordinate-free pf of b):

$$L_{\tilde v} L_{\tilde w} f - L_{\tilde w} L_{\tilde v} f = L_{[\tilde v, \tilde w]} f$$
$$= df([\tilde v, \tilde w])$$
$$= 0 \quad @ \; p$$

Again the condition $df_p = 0$ is essential

Coordinate-free pf a):

Already know that $d_p^2 f$ is symmetric

Need to show $\tilde w(p) = 0 \Rightarrow d_p^2 f(\tilde v, \tilde w) = 0$

But $d_p^2 f(\tilde w, \tilde v) = L_{\tilde w}(L_{\tilde v} f)(p) = 0$

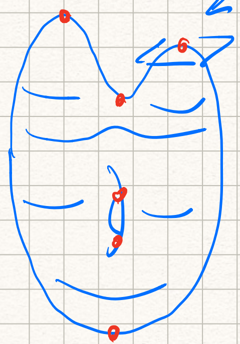
Depends only on $\tilde w(p) = w = 0$

$\triangleleft$

__Ex.__ $P^n \subset \mathbb{R}^{n+1}$

$f =$ projection to some direction

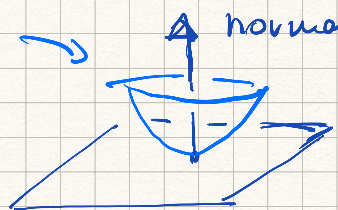
e.g. x_{n+1} - coordinate



$p = 0$ Near p: Locally $P = \text{graph}(f)$

$$f(x) = f(p) + \underbrace{d_p f(x)}_{=0} + \frac{1}{2}\underbrace{d_p^2 f(x)}_{} + \dots$$

the best quadratic approximation of f



P normal direction T_pP $\underbrace{T_pP}_{\mathbb{R}^n}$

The second fundamental form of P

$$\mathrm{II}_p : T_pP \times T_pP \longrightarrow \mathbb{R}$$

$P = \text{graph}(f)$

$$\underbrace{(x_1, \dots, x_n)}_{\cong\, \mathbb{R}^n} \leftarrow \text{local coord on } P$$

Further definitions:

$p \in \mathrm{Crit}(f)$

Def. a) p is non-degenerate if $d_p^2 f$ is non-degenerate

b) The Morse index (or just index) of p is the $\underbrace{\text{index of } d_p^2 f}$

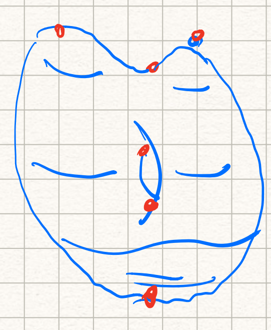
the # of negative squares when $d_p^2 f = x_1^2 + \dots + x_k^2 \underbrace{- x_{k+1}^2 - \dots x_n^2}_{\text{index}}$

$0 \leq \mathrm{ind}_p f \leq \underset{\dim P}{n}$

c) f is Morse if all $\mathrm{Crit}(f)$ are non-degenerate

Notation: $\mathrm{Crit}_k(f) =$ crit pts of index k

Ex



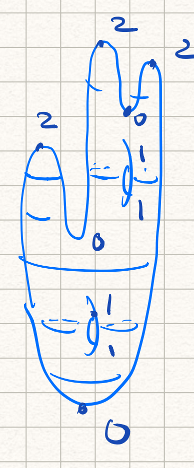
max ind $= 2$

saddles ind $= 1$

min ind $= 0$

$P \subset \mathbb{R}^3$

Or



Ex A Monkey saddle is degenerate!
unless $k = 1$

Goal

Relate the critical pts of a Morse function to the topology of P.

In particular prove:

Morse Inequalities

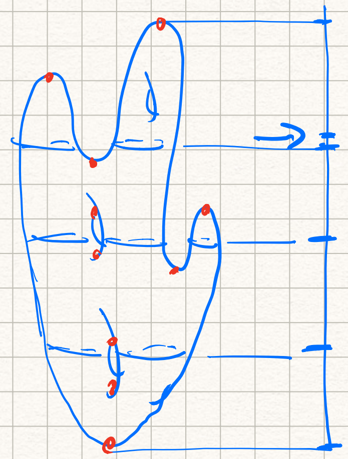
$$\boxed{|\mathrm{Crit}_k(f)| \geq \dim H_k(P)}$$

Strategy:

Do this inductively moving

min f to max f

and looking at how the topology of $\{f \leq c\}$ changes with c



Relevent questions:

- How do we know Morse functions exist on P?
- If so, how common are they?

§3 Regular value intervals

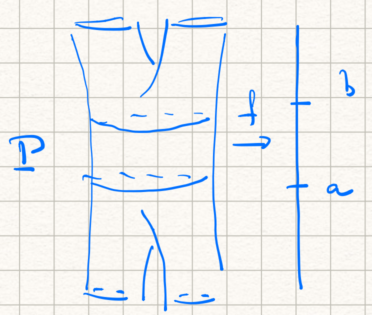
Lecture 3
01/07

<u>Key pt</u>: nothing happens to $\{f \le c\}$ as long as c remains regular

To be more precise:

- $f: P \xrightarrow{C^2} \mathbb{R}$
  P compact or f is proper
- $[a, b] \subset \mathbb{R}$ contains no critical values

<u>Prop</u> $\{f \le a\}$ is diffeo to $\{f \le b\}$



<u>Rmk</u> only matters that $f^{-1}([a, b])$ is compact.

<u>Pf</u>: Use (anti) gradient flow
– key tool in Morse theory

- Fix a Riemannian metric on P:
  $$\langle , \rangle : T_pP \times T_pP \longrightarrow \mathbb{R}$$
  symmetric & pos. def
- ∇f : $\langle \nabla f, \cdot \rangle = df$
  If $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ is orthonormal basis at p:
  $$\text{metric} = \sum dx_i^2 \quad @ p$$
  ∇f is C^1 $\Leftarrow$ f is C^2

$$\boxed{\nabla f(p) = \sum \frac{\partial f}{\partial x_i}(p) \frac{\partial}{\partial x_i}}$$ Same as on $\mathbb{R}^n$

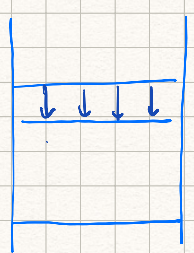
$$L_{\nabla f} f = df(\nabla f) = \langle \nabla f, \nabla f \rangle = \|\nabla f\|^2$$

$$X = -\frac{\nabla f}{\|\nabla f\|^2} \qquad L_X f = -1$$

φ^t = the flow of X, local existence $\Leftarrow \underbrace{\nabla f \; C^1}_{\Uparrow}$ $f \; C^2$

$$\underbrace{\frac{d}{dt} f(\varphi^t(x)) = L_X f(x) = -1}_{\text{f decreases along the flow lines of X with unit speed}}$$

$$\Rightarrow f(\varphi^t(x)) = f(x) - t$$



$\varphi^{b-a} : \{f \le b\} \longrightarrow \{f \le a\}$ diffeo

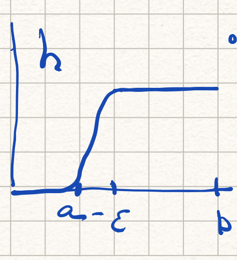
Technical pts:

- compactness $\Rightarrow$ φ^t is defined for all t
- $[a,b]$ regular $\Rightarrow$ $\nabla f \neq 0$ on $f^{-1}([a,b])$ $\lhd$

Rmk. what to change when only $f^{-1}([a,b])$ is compact:

- Ex.

- $f^{-1}([a,b])$ compact $\Rightarrow f^{-1}([a-\varepsilon, b])$ compact
- Replace X by $g.X$:
  $g : P \to \mathbb{R}$ $\quad g = h \circ f$
  cut-off function.



§4. Morse Lemma

Answers the question of what happens at critical pts

But actually more significant than that

Prop (Morse Lemma) $\quad f \in C^3$

$p \in \mathrm{Crit}(f)$ is non-deg $\Rightarrow$

$\exists$ a coord system s.t.

$$f(x) = f(p) + \tfrac{1}{2}(-x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2)$$
$$= f(p) + \tfrac{1}{2} d_p^2 f \quad ; \quad k = \mathrm{ind}_p f$$

In other words, by a local change of coordinates near p one can eliminate higher order terms in the Taylor exp of f

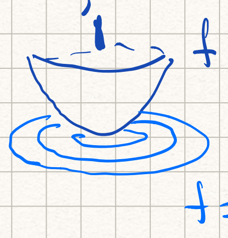
Rmk.: Going from $d_p^2 f$ to $-x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots$ is simply diagonalization of $d_p^2 f$ – linear algebra

Or to rephrase: the morse index is the only local invariant of a non-deg critical pt – two crit pts with the same same index are locally diffeo

Morse lemma gives a local picture of what f looks like near a critical pt of index k.
To be used later

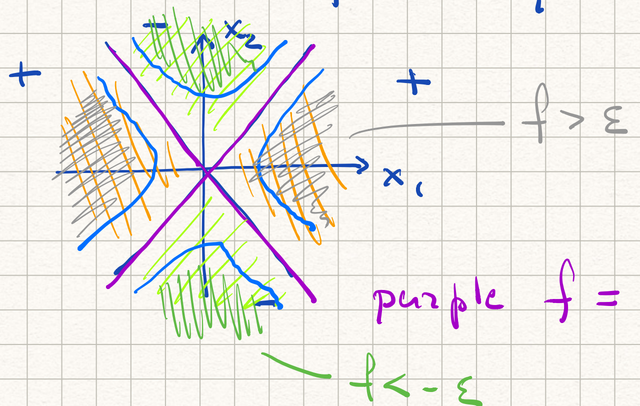
Examples

1) $n=2,\ k=0$: $f(x) = x_1^2 + x_2^2$

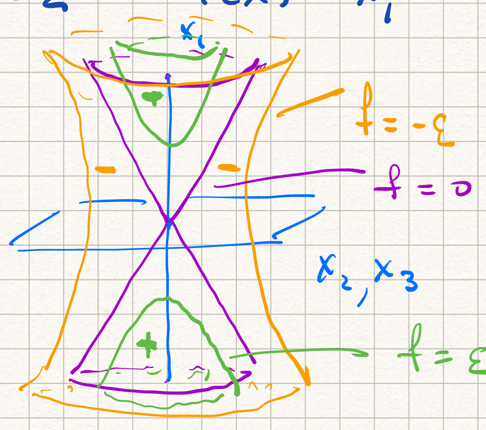


min

2) $n=2 \quad k=1 \qquad f(x) = x_1^2 - x_2^2$ saddle



3) $n=3,\ k=2 \qquad f(x) = x_1^2 - x_2^2 - x_3^2$



Preliminaries - of independent interest

Lemma (Hadamard)

$$f: \mathbb{R}^n \xrightarrow{C^r} \mathbb{R}, \quad f(0)=0$$

$$\Rightarrow \exists\, g_i: \mathbb{R}^n \xrightarrow{C^{r-1}} \mathbb{R} \qquad g_i(0) = \frac{\partial f}{\partial x_i}(0)$$

$$f(x) = \sum g_i(x)\, x_i$$

(near 0)

Rmk. can replace $\mathbb{R}^n$ by a star-shapped domain

Attempts - what does not quite work

$n=1$ $\quad f(x) = x \cdot g(x), \quad g \in C^{r-1}$

1) f is real analytic

$$f(x) = \underbrace{a_0}_{=0} + a_1 x + a_2 x^2 + \dots$$

$$= x\underbrace{(a_1 + a_2 x + \dots)}_{g(x) \text{ converging}} = x\, g(x)$$

But in general the Taylor exp

- need not converge
  (Any power series is the T. exp of some function - Ex)

- if it converges, need not conv to f.

E.g. It can be identically zero

2) Capitalizing on the assumption that $n=1$ set

$$g(x) = \begin{cases} f(x)/x & x \neq 0 \\ f'(0) & x = 0 \end{cases}$$

Need to show that g is C^{r-1}

Not entirely straightforward

Ex. Prove f $C^1 \Rightarrow g$ C^0

f $C^2 \Rightarrow g$ C^1

Rmk. The condition that

f is C^1

is essential to get g C^0 :

and defined at 0

E.g. $f(x) = x^{1/3}$

$$g(x) = \frac{x^{1/3}}{x} = \frac{1}{x^{2/3}} \leftarrow \text{not defined at } 0$$

$$g(0) = \infty$$

Pf

$n=1$

$$f(x) \overset{\text{FTC}}{=} \int_0^1 \frac{d}{dt} f(tx)\,dt \overset{\text{FTC}}{=} f(x)-f(0)$$

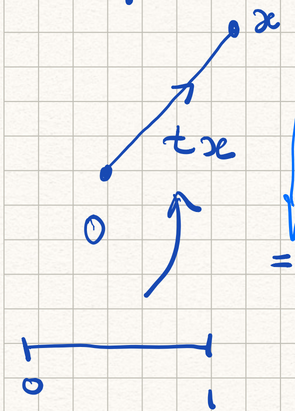
chain rule

$$= \int_0^1 x f'(tx)\,dt$$

$$= x \underbrace{\int_0^1 f'(tx)\,dt}_{g(x) \quad C^{r-1}} = x\,g(x)$$

when $x=0$
$g(0)=f'(0)$

$n \geq 1$



$$f(x) \overset{\text{FTC}}{=} \int_0^1 \frac{d}{dt} f(tx)\,dt$$

$$= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx) \underbrace{\frac{d(x_i t)}{dt}}_{x_i}\,dt$$

$$= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx)\,x_i\,dt$$

$$= \sum_{i=1}^n x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx)\,dt}_{g_i(x) \quad C^{r-1}}$$

$$= \sum_i x_i\,g_i(x)$$

Again at 0:
$g_i(0) = \frac{\partial f}{\partial x_i}(0)$

Cor Assume $f : \mathbb{R}^n \xrightarrow{C^r} \mathbb{R}$, $r \geq 2$,
$\nabla f(0) = 0$

$\Rightarrow$ $h_{ij} : \mathbb{R}^n \xrightarrow{C^{r-2}} \mathbb{R}$ s.t.

- $f(x) = f(0) + \sum_{i,j} h_{ij}(x)\, x_i x_j$
- $h_{ij}(0) = \dfrac{\partial^2 f}{\partial x_i \partial x_j}(0)$

Pf. Apply Hadamard's Lemma twice

Rmk $h_{ij} \overset{?}{=} h_{ji}$ $\leftarrow$ but we don't need this

Pf of the Morse Lemma

The question is local: can work in $\mathbb{R}^n$

$p = 0, \quad f(0) = 0$

$f : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}$ In reality interested in germs

nbd of $0 \subset \mathbb{R}^n$

$$f(x) = \underbrace{Q(x)}_{d_0^2 f(x)} + R(x) \leftarrow \text{remainder}$$

the Hessian

Want a diffeo

$\varphi : (V, 0) \longrightarrow (U, 0)$

nbds of 0

$$\boxed{f \circ \varphi = Q}$$

Homotopy (Moser) method: • important
• ref to sympl. geom class
(e.g. the pf of Darboux thm, Moser's thm)

But the pf of Morse Lemma is a bit harder

Idea: 1) Set $f_t = Q + tR$
$= (1-t)Q + t\cdot f$
$t \in [0,1]$

Looking for a family of diffeo's
φ_t (near 0) $\qquad \varphi_0 = id$

s.t. (*) $f_t \circ \varphi_t = Q$ $\quad$ Seemingly a more difficult question

2) Instead of looking for φ_t look for a generating time-dependent v.f.
v_t : $\qquad t \in [0,1]$

$$\frac{d}{dt}\varphi^t(x) = v_t(\varphi^t(x))$$

Differentiating (*) in t

$$(*) \underset{FTC}{\Leftarrow} \frac{d}{dt} f_t \circ \varphi_t = 0$$

$$\Leftrightarrow \frac{df_t}{dt}(\varphi_t(x)) + L_{v_t} f_t(\varphi_t(x)) = 0$$

$$\Leftrightarrow \quad \boxed{\frac{df_t}{dt} + L_{v_t} f_t = 0} \quad (**)$$

Apply φ_t^{-1} or just $y = \varphi^t(x)$ $\qquad$ ptwise

<u>Want</u>: solve (**) for $v_t = (v_1, \dots, v_n) = \vec{v}_t$

suppress t

Notation $f(x) = Q(x) + R(x)$

$f(x) = \sum x_i x_j \underbrace{g_{ij}}(x)$

$= 0$ at 0

Hessian $\; a_{ij} + R_{ij}(x)$

$$= \underbrace{\sum x_i x_j a_{ij}}_{Q(x)} + \underbrace{\sum x_i x_j R_{ij}(x)}_{R(x)}$$

$$\boxed{\begin{aligned} f_t(x) &= Q(x) + t R(x) \\ &= \sum x_i x_j a_{ij} + t \sum x_i x_j R_{ij} \end{aligned}}$$

one eq
under-determined

(**) $\Leftrightarrow$

probably equal

$$-\sum_{i,j} x_i x_j R_{ij}(x) = \sum_{i,j} x_i v_j \left(2 a_{ij} + t(R_{ij} + R_{ji}) + \dots\right)$$

some more junk

$$-\sum_{i,j} x_i x_j R_{ij}(x) = \sum_{i,j} x_i v_j \left(2 a_{ij} + 2t \hat{R}_{ij}\right)$$

vanishing at 0

$$\Leftarrow \left\{ \underbrace{-\sum_j x_j R_{ij}(x)}_{r_i(x)} = \sum 2(a_{ij} + t\hat{R}_{ij}(x)) v_j \right.$$

system of linear equations at (x,t)

$i = 1, \dots, n$

$$2(\underbrace{a_{ij} + t\hat{R}_{ij}(x)}_{=0 \text{ at } 0}) \leftarrow \text{matric } A_t(x)$$

$$\vec{r} = A_t \vec{v}_t \qquad \underbrace{A_t(0) = 2 \text{ Hessian}}_{\text{non-deg}}$$

non-deg for $t \in [0,1]$
and all x near 0

set $\vec{v}_t(x) = A_t^{-1}(x)\,\vec{r}(x)$

Nuance (important): need to make sure

$$\begin{cases} \bullet\ \varphi^t \text{ is defined for } t \in [0,1] \\ \bullet\ \varphi^t(0) = 0 \end{cases}$$

(Existence & uniqueness is local in t)

$$v_t(0) = 0 \Leftarrow \vec{r}(0) = 0$$

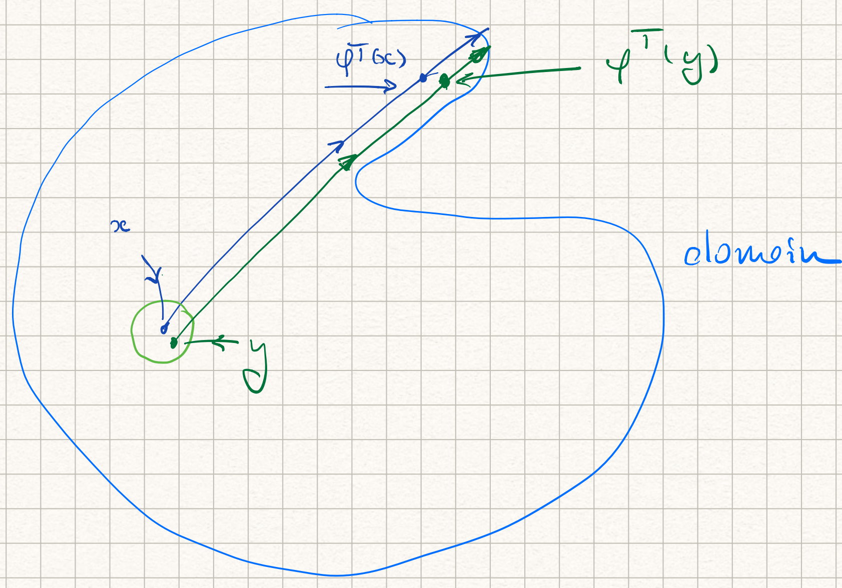
$$r_i(x) = \sum x_j R_{ij}(x)$$
$$= 0 \text{ at } 0$$

Reason: • the existence time is lower semicontinuous

• ∞ at $x = 0$

$\triangleleft$

Pictorial Pf of Lower Semi-continuity



Main pt: $\varphi^t(y)$ exists and is close to $\varphi^t(x)$ as long as $t \in [0, T]$ if y is close to x. $\triangleleft$

Digression - Broader Context - Lecture 3
Normal Forms & Singularities of Functions

01/12

⇒ Consider a class of objects:

(a) • matrices (= linear transf)

(b) • symmetric matr (= quadr forms

(c) • functions near a pt $p = 0 \in \mathbb{R}^n$

⇒ With an equiv relation coming from a gp action = "coord changes"

(a) • $A \leftrightarrow BAB^{-1}$

(b) • $A \leftrightarrow BAB^{T}$

(c) • $f \leftrightarrow f \circ \varphi$ $\quad \varphi$ = diffeo near p, $\varphi(p) = p$

Normal form = a simple form all (or some) objects can be brought to

roughly speaking

E.g (a) • Jordan normal form

(b) • diagonal matrix with ± 1 or 0 on diagonal

__What about (c)?__

Is there a normal form for functions near p?

Not really, but YES under additional conditions

• Fact: $df_p \neq 0$ $\qquad p = 0$

$\iff \exists\ (x_1, \dots, x_n)$ s.t. $f(x) = x_1 + c$ near c

This is Ex on p 9

In other words: all functions with $df_p \neq 0$ are equivalent to each other (up to a const)

• Next step: Focus on functions with

$$df_p = 0$$

Morse Lemma = Normal form, provided that $d_p^2 f$ is non-deg: the only invariant is Morse index

• One can go further and study what happens when $d_p^2 f$ degenerates

E.g. For hol functions of one-variable
Ex the normal form is $z^k + c$

For smooth functions the situation quickly becomes complicated

Subject: "Singularities of smooth functions"

Ex. $f : \mathbb{R} \xrightarrow{C^\infty} \mathbb{R}$ near 0

$$f(0) = \ldots = f^{(k-1)}(0) = 0$$
$$f^{(k)}(0) \neq 0$$

$\Rightarrow \exists$ a change of variable $\varphi : (\mathbb{R}, 0) \circlearrowleft$ near 0 s.t.
$(f \circ \varphi)(y) = \pm y^k$

Hint: Use Hadamard's Lemma

The situation becomes much more involved when $n > 1$

See, e.g., Arnold–Varchenko–Gusein-Zade

Pf.

Hadamard $\Rightarrow f(x) = \pm x^k \cdot g(x)$, $g(0) > 0$

$$= \pm \Big[\underbrace{x\, g(x)^{1/k}}_{y = \varphi(x) \leftarrow \text{diffeo}} \Big]^k \quad \triangleleft$$

$x \mapsto x g(x)$, $g(0) \neq 0$

φ

$\longleftarrow$

§5 Some alg topology: CW-complexes

Generalities

References: Any good alg topology book, e.g. Hatcher

- Attaching a cell — procedure

X = a reasonable top space (metrizable, compact or loc. compact...)

$\psi : S^{n-1} \longrightarrow X$ cont map

Def $Y = X \cup_\psi D^n$; $S^{n-1} = \partial D^n$

closed ball

is obtained from X by attaching an n-cell ($\operatorname{int} D^n$) along ψ

$$Y = X \sqcup D^n / \sim \qquad \underset{S^{n-1}=\partial D^n}{x} \sim \underset{X}{\psi(x)}$$

This is again a reasonable top space

In what follows we are interested in spaces/maps up to homotopy

Ex-Fact (Hatcher): $\boxed{\varphi_0 \sim \varphi_1 \quad \text{(homotopic)} \Rightarrow X \cup_{\varphi_0} D^n \sim X \cup_{\varphi_1} D^n}$:

changing φ within its homotopy class does not effect the homotopy type of $X \cup_\varphi D^n$

Def A CW-complex (or a cell complex) is obtained inductively from

X_0 = finite collection of pts

by attaching cells of increasing dimension:

$$X_0 \subset \underset{\text{graph}}{X_1} \subset \dots \subset X_n = X$$

where X_{k+1} is obtained from X_k by attaching one or several k-cells

Rmk • Here all cell-complexes are finite - finite # of cells

• One is allowed to skip steps:

$X_k = X_{k+1}$

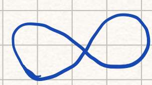
• dim X is max n that occurs

Visualize $\longrightarrow$

Ex.

- $S^n = \quad pt + D^n$
- $\mathbb{R}P^n = \quad pt + D^1 + D^2 + \ldots + D^n$
- $\mathbb{C}P^n = \quad pt + D^2 + D^4 + \ldots + D^{2n}$

obvious notation / not-standard

- T^2:  $+ D^2$

- Σ_g:  $+ D^2$

  $\bigvee_{2g} S^1$

- $\dim X = 1 \underset{\text{def}}{\Longleftrightarrow} X$ is a graph

- more generally a simplicial complex is naturally a CW complex and k-cells = k-simplices

A good ref for this & the next §
is Dubrovin–Fomenko–Novikov, Part III

Prop X = CW complex, $Y = X \cup_\varphi D^{k+1}$

$\Rightarrow Y$ has hom type of a CW complex

Not quite obvious:

In a CW complex have cells of higher dim attached after lower dim:

$$X_0 \subset X_1 \subset \dots \subset X_k \subset X_{k+1} \subset \dots \subset X_n$$
$$\varphi : S^k = \partial D^{k+1} \to X_n, \quad \psi \sim \varphi, \quad \psi : S^k \to X_k$$

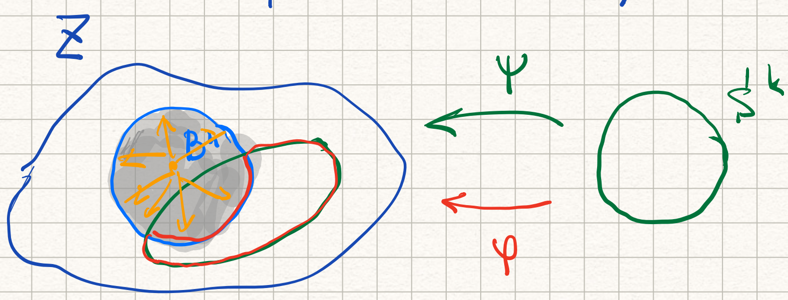
Pf

Need $\underline{\psi \sim \varphi : S^k \longrightarrow X_k}$

Idea: • $Z \supset B^n$
top space, open ball

• $\varphi : S^k \longrightarrow Z$, $\underline{k < n}$

$\Rightarrow \exists\ \psi : S^k \longrightarrow Z \sim \varphi$ s.t.
$\psi(S^k) \cap B = \emptyset$



Then use induction:

$$X = X_{n-1} \cup B^n \cup \dots \cup B^n \leftarrow \text{open balls}$$

Push ψ outside B^n's, etc

Some minor conventions:

- $B^n \subset Z$ replaced by

  $\underset{\text{closed ball}}{F: D^n \longrightarrow Z}$ s.t. $F: B^n \hookrightarrow Z$ homeo on its image, $B^n \subset D^n = \overline{B^n}$

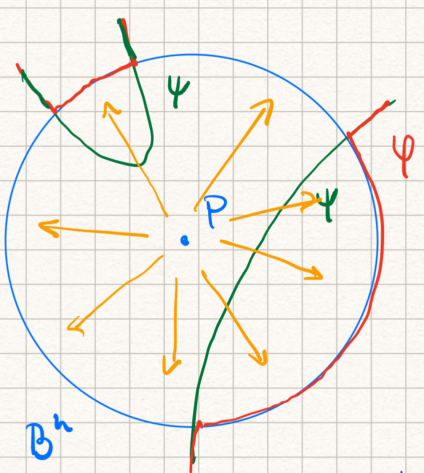
  $F(D^2)$ is a "nice" compactification of B^n

- Need to know that

  $\exists\ p \in B^n$, s.t. $p \notin \psi(S^k)$

  (If so can assume $p = 0 \in B^n$) $\Big\}$ Assume for now

$$R: D^n \setminus p \longrightarrow S^{n-1}$$



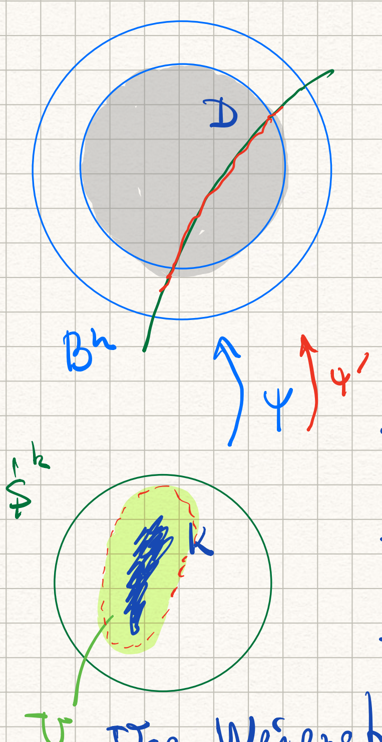
$$\varphi(x) = \begin{cases} \psi(x), & \psi(x) \notin B^n \\ \underbrace{F \circ R \circ \psi(x), \quad \psi(x) \in B^n}_{\text{defined} \Leftarrow p \notin \psi(S^k)} \end{cases}$$

Ex.: Check $\varphi \in C^0$, $\varphi \sim \psi \Leftarrow R$ hom eq.

- Not true in general:
  we can have $\psi: S^k \to D^{n>k}$ onto
  "Peano curve" $\Leftarrow$ cannot be C^1 by Sards
- But $\exists$ $\psi' \sim \psi$ & C^0-close to it
  s.t. $\psi'(S^k) \not\supset B^n$

Idea: approximate ψ by a smooth map or rather a map which is smooth on the part mapped to B^n

Details: $D \subset B^n$ a slightly smaller closed ball



$S^k \supset \underbrace{U}_{\text{small nbh}} \supset \psi^{-1}(D) = K \leftarrow$ compact

$\psi(U) \subset B^n$

$\exists\ \psi': S^k \to Z$ s.t.

$\to \psi' = \psi$ outside B
$\to \psi' \overset{C^0}{\approx} \psi$ inside B $\Rightarrow \psi' \sim \psi$
$\to \psi': U \xrightarrow{C^\infty} B \Rightarrow D \not\subset \psi'(S^k)$

Use Weierstrass to construct ψ'... $\triangleleft$

§6 Some Alg Topology : CW complexes
Homology

Setting

- $\mathbb{F}$ ground field; e.g: $\mathbb{F}_2$, $\mathbb{Q}$ or $\mathbb{Z}$
- X a CW complex
  $e_i^k \leftarrow$ k-dim cells , $i = 1, \dots, r_k$
- C_k = free v.s. (module) over $\mathbb{F}$ generated by e_i^k

Key pt : $\exists\ \partial$

$$0 \leftarrow C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} \dots \leftarrow C_n \leftarrow 0$$

s.t. $H_*(C_*, \partial) = H_*(X; \mathbb{F})$ ← suppress in notation

Assume $\mathbb{Z}$

Construction of ∂

Note • $X_k / X_{k-1} = \bigvee_{r_k} S_j^k$

bouquet of r_k k-spheres labeled by $e_1^k, \dots, e_{r_k}^k$

Want to defined ∂e_i^{k+1}

$$\boxed{\partial e_i^{k+1} = \sum_j m_{ij} e_j^k}$$

need to know what these are

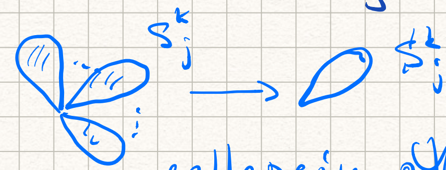
Attaching map for e_i^{k+1}

$$\begin{array}{ccc} \varphi: D^{k+1} & \longrightarrow & X_{k+1} \\ \cup & & \cup \\ S^k = \partial D^{k+1} & \longrightarrow & X_k \end{array} \quad \Rightarrow$$

incidence number of e_i^{k+1} & e_j^k

$$\bar{\varphi}: S^k \longrightarrow X_k \longrightarrow X_k / X_{k-1} = \vee S^k \longrightarrow S^k_j$$

$$\boxed{m_{ij} = \deg(\bar{\varphi})} \in \mathbb{Z}$$



collapsing other spheres

Rmk For other $\mathbb{F}$'s take the image of m_{ij} under $\mathbb{Z} \longrightarrow \mathbb{F}$

Alternatively and more formally:

- $C_k = H_k(X_k/X_{k-1} = \bigvee_{r_k} S^k_i) = \mathbb{F}^{r_k}$

k spheres

- Consider $(\underbrace{X_k/X_{k-2}}_{A}, \underbrace{X_{k-1}/X_{k-2}}_{B})$

Then $A/B \simeq X_k/X_{k-1}$. Long exact seq:

$$\cdots \to H_k(B) \longrightarrow H_k(A) \longrightarrow \underbrace{H_k(A/B)}_{C_k} \xrightarrow{\ \partial\ } \underbrace{H_{k-1}(B)}_{C_{k-1}} \longrightarrow \cdots$$

So we have a connecting map in the long exact seq, and this is:

$$\partial: H_k(X_k/X_{k-1}) \longrightarrow H_{k-1}(X_{k-1}/X_{k-2})$$

- In what follows, it's more convenient to break it down into two steps:

$$\boxed{\partial: H_k(X_k/X_{k-1}) \xrightarrow{\ \delta\ } H_{k-1}(X_{k-1}) \longrightarrow H_{k-1}(X_{k-1}/X_{k-2})} \quad (*)$$

connecting map in LES for (X_k, X_{k-1})

Lemma $\partial^2 = 0$

Not difficult but not entirely trivial

Pf (Suffices over $\mathbb{Z}$) From FDN inspired.

Write $X_k' = X_k / X_{k-1}$. By (*), starting with $k+1$

$$\partial^2 =$$

$$\to \underbrace{\overbrace{H_{k+1}(X_{k+1}')}^{X_{k+1}/X_k} \xrightarrow{\delta} \underbrace{H_k(X_k) \to \overbrace{H_k(X_k')}^{X_k/X_{k-1}} \xrightarrow{\delta} H_{k-1}(X_{k-1})}_{=0} \to H_{k-1}(X_{k-1}')}_{\partial^2} \to$$

$$C_{k+1} \xrightarrow{\partial^2} C_{k-1}$$

Long exact sequence for (X_k, X_{k-1}):

$$\cdots \to \underbrace{H_k(X_k) \to H_k(\underset{X_k/X_{k-1}}{X_k'}) \to H_{k-1}(X_{k-1})}_{=0} \to \cdots$$

$\Rightarrow \partial^2 = 0$ $\triangleleft$

Ex-Problem (see, e.g. DFM)

X is a simplicial complex

$\Downarrow$

X is a CW-complex; simplices = cells

$\Rightarrow$ two chain complexes

$$\underbrace{(C_*^{\Delta}, \partial_\Delta)}_{\text{simpl. chain compl}} \quad \& \quad \underbrace{(C_*^{cw}, \partial_{cw})}_{\text{complex intro above}}$$

Some generators.

Prove that $\partial_\Delta = \partial_{cw}$ $\triangleleft$

Key Hom result: $X =$ CW - complex

Denote: $H_*^{cw}(X) = H_*(C_*, \partial)$

Could possibly depend on the CW-str.:

$$X_0 \subset X_1 \subset \dots \subset X_n = X$$

$$\boxed{\text{Thm} \quad H_*^{cw}(X) = H_*(X)}$$

$H_*(X)$ ← homotopy inv

$\underbrace{H_*}$ whateverflavor of Hom your pref

Pf - induction in dim X:

Assume done for $\dim < n$; $\dim = 0$ clear

what does not work: the five lemma

$$\begin{tikzcd}
\arrow[r] & H_*^{cw}(X_{n-1}) \arrow[r] & H_*^{cw}(X) \arrow[r] & H_*^{cw}\left(\frac{X}{X_{n-1}}\right) \arrow[r] & H_{*-1}^{cw}(X_{n-1}) \\
\arrow[r] & H_*(X_{n-1}) \arrow[r] \arrow[u, phantom, "\parallel"] & H_*(X) \arrow[r] & H_*\left(\frac{X}{X_{n-1}}\right) \arrow[r] \arrow[u, phantom, "\parallel"] & H_{*-1}(X_{n-1}) \arrow[u, phantom, "\parallel"]
\end{tikzcd}$$

No obvious map $H_*(X) \longrightarrow H_*^{cw}(X)$

making the diagram commute

Instead: a bit more subtle argument

still purely formal:

uses only axioms for H_*

Actual pf: $\qquad X = X_n$

- First note $\qquad$ For $k < n-1$

$$H_k(X) = H_k(X_{n-1}) \quad \text{long exact seq}$$
$$\quad\quad \| \longleftarrow \text{induction}$$
$$H_k^{cw}(X) = H_k^{cw}(X_{n-1}) \quad \text{by def}$$

- Need to do $k = n-1, n$ for $X_n = X$

LES for (X, X_{n-1}):

$$0 \to H_n(X) \to \underbrace{H_n\left(\tfrac{X}{X_{n-1}}\right)}_{C_n} \xrightarrow{\delta} \underbrace{H_{n-1}(X_{n-1})}_{Z_{n-1}^{cw}} \to H_{n-1}(X) \xrightarrow{0} \cancel{H_{n-1}\left(\tfrac{X}{X_{n-1}}\right)}$$

$H_n(X) = \ker \delta$

$$H_n^{cw}(X) = \ker \partial : \underbrace{H_n\left(\tfrac{X}{X_{n-1}}\right)}_{C_n} \xrightarrow{\delta} H_{n-1}(X_{n-1}) \to \underbrace{H_{n-1}\left(\tfrac{X_{n-1}}{X_{n-2}}\right)}_{C_{n-1}}$$

$\boxed{\text{mono}}$

LES for (X_{n-1}, X_{n-2})

$$\underbrace{H_{n-1}(X_{n-2})}_{\substack{\| \\ 0}} \xrightarrow{0} \underbrace{H_{n-1}(X_{n-1})} \to \underbrace{H_{n-1}\left(\tfrac{X_{n-1}}{X_{n-2}}\right)}_{C_{n-1}} \to \dots$$

$$\Rightarrow \underbrace{\ker \delta}_{H_n} = \underbrace{\ker \partial}_{H_n^{cw}} \quad \& \quad H_{n-1}(X_{n-1}) = Z_{n-1}^{cw}$$
$$\cap$$
$$C_{n-1}$$

$$\Rightarrow \quad H_{n-1}(X) = \frac{Z_{n-1}^{cw}}{\partial(C_n)} =: H_{n-1}^{cw}(X)$$

$\triangleleft$

<u>Ex.</u> : work out the details I skipped

<u>Ex.</u> Use cw-str to calculate

- $H_*(\Sigma_g) = \begin{cases} \mathbb{F} & 0 \\ \mathbb{F}^{2g} & 1 \\ \mathbb{F} & 2 \end{cases}$

- $H_*(\mathbb{C}P^n) = \underset{0}{\mathbb{F}}, \underset{1}{0}, \underset{2}{\mathbb{F}}, \underset{3}{0} \dots \underset{2n-1}{0}, \underset{2n}{\mathbb{F}}$

- $H_*(\mathbb{R}P^n; \mathbb{F}_2) = \underset{0}{\mathbb{F}_2}, \dots, \underset{n}{\mathbb{F}_2}$

$H_*(\mathbb{R}P^n; \mathbb{Z}) = \ ? \leftarrow$ Harder

§7 Main Thm & Morse Inequalities

Lecture 4 01/14

- $f : P \longrightarrow \mathbb{R}$ <u>morse</u>
  closed manifold
- write $P_a = \{f \le a\}$

<u>Thm</u> $a < c < b$, the only crit value in $[a, b]$

$\Rightarrow$ P_b is obtained from P_a by attaching a k-dim cell ~~for~~ every crit pt x on $\{f = c\}$, where $k = \operatorname{ind}(x)$

<u>Cor</u> P has homotopy type of a CW complex with one cell for each crit pt with dim = ind

Same for P_a or $(\{a \le f \le b\}, \{f = a\})$

In essence:

$\exists$ a complex (C_k, ∂) with
C_k generated by $\mathrm{Crit}_k(f)$ over $\mathbb{F}$
and $H_*(C, \partial) = H_*(P; \mathbb{F})$

Cor (Morse inequalities)

$$\underbrace{|\mathrm{Crit}_k(f)|}_{c_k = \dim C_k} \geq \dim H_k(P; \overset{\text{any field}}{\mathbb{F}}) = b_k$$

or

$$\mathrm{rk}\, H_k(P; \mathbb{Z})$$

Refining Morse Inequalities

set
$$h(t) = \sum \overbrace{\dim H_k(P)}^{b_k} \cdot t^k \quad \text{over } \mathbb{F} \text{ field}$$
$$m(t) = \sum \underbrace{|\mathrm{Crit}_k(f)|}_{c_k} \cdot t^k$$

Prop • $\exists$ a pol $r(t)$ with coeff ≥ 0 such that

$$m(t) = (1+t)\, r(t) + h(t) \qquad (*)$$

• $c_k - c_{k-1} + c_{k-2} - \ldots \pm c_0 \geq b_k - b_{k-1} + \ldots \pm b_0 \quad \forall k$

Cor $\quad \sum (-1)^k c_k = \sum (-1)^k b_k \qquad$ Pf: set $t = -1$ in $(*)$

Pf of Prop

This is a purely algebraic fact

Complex: $\quad C_k \;,\quad c_k = \dim C_k$

$\qquad\qquad H_k \;,\quad b_k = \dim H_k$

Lemma-Ex A complex over a field $\mathbb{F}$ can be decomposed as a direct sum of elementary complexes

Essential

$\mathbb{Z}$ not OK

$$\ldots 0 \to \overset{k}{\mathbb{F}} \to 0 \to \ldots \;:\; H = \mathbb{F}$$

$$0 \to \overset{k}{\mathbb{F}} \xrightarrow{\cong} \overset{k-1}{\mathbb{F}} \to 0 \ldots \;:\; H = 0$$

For elementary complexes (*) & ME obviously hold.

Additivity $\Rightarrow$ General case $\quad \lhd$

Rmk • Can replace $\mathcal{P}$ by $\mathcal{P}_a$ everywhere

• Or even $(\{a \le f \le b\}, \{f = a\})$

Need to take rel. homology

§ 8. Pf of the Main Thm and further refinements: handle bodies

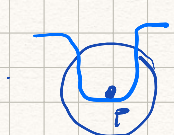
Some preliminary pts:

- can assume $a = c - \varepsilon < c < c + \varepsilon = b$ so that $\{f = a\}$ & $\{f = b\}$ are very close to $\{f = c\}$

- For the sake of simplicity, assume only one critical pt on $\{f = c\}$. Call it p, $k = \mathrm{ind}(p)$. General case — similar.

- Fix a "morse chart" U near p and a Riemmanian metric, Euclidean on U

- $X_0 = -\dfrac{\nabla f}{\|\nabla f\|^2}$ defined on outsid crit pts



← cut-off function near p:

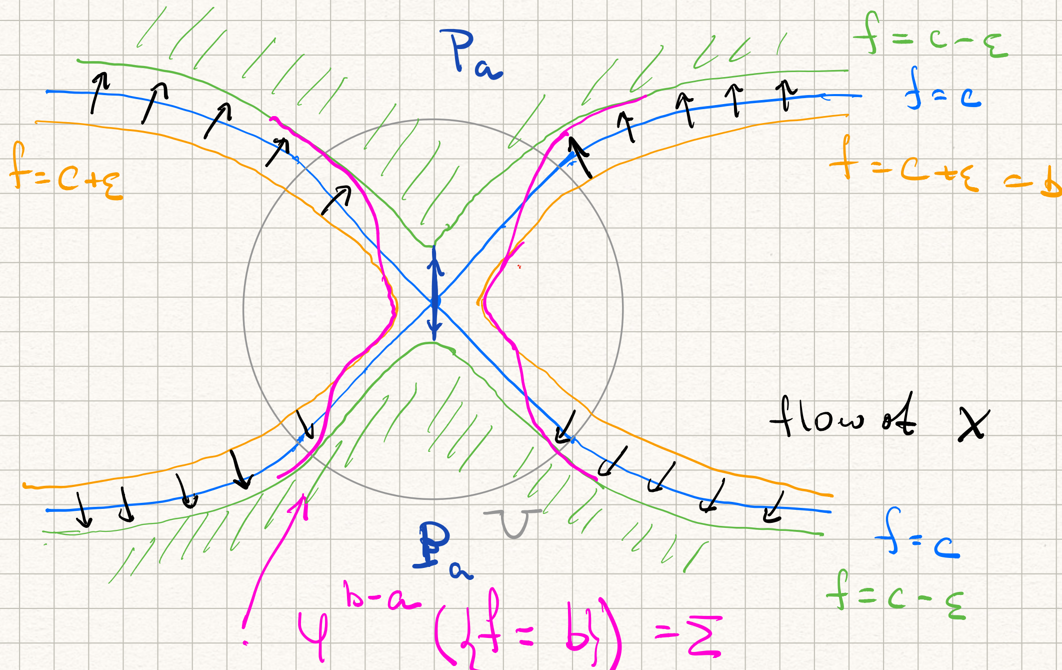
$$h = \begin{cases} 0 & \text{near } p \\ 1 & P \setminus U \end{cases}$$

Replace X_0 by $h \cdot X \cdot \underbrace{(\text{cut-off below } a)}_{\text{function of } f}$

to get rid of other critical ps

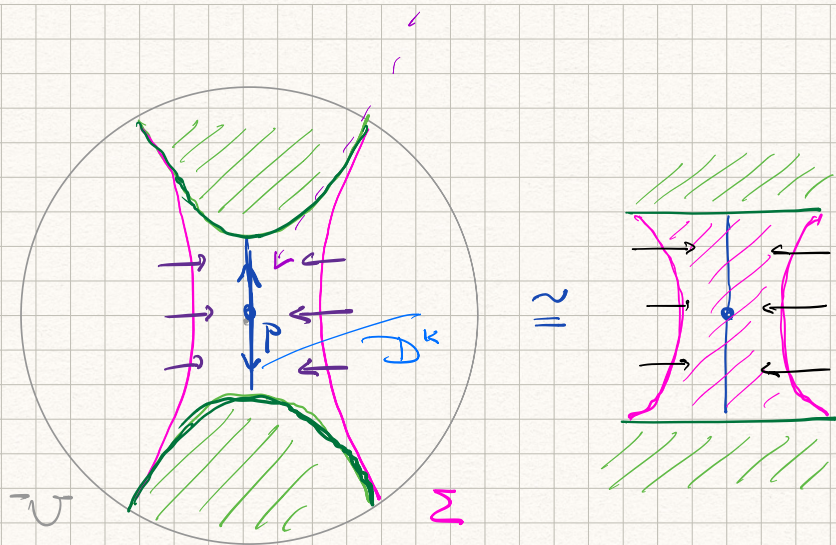
$\Rightarrow$ • the flow of X is defined for all times

• Outside U : "$\varphi^{b-a} : P_b \to P_a$"



$$\varphi^{b-a}(P_b) = P_a \cup \underbrace{\begin{pmatrix}\text{nbd of unstable} \\ \text{manifold of } p\end{pmatrix}}_{D^k}$$

$$\sim P_a \cup D^k$$



$\triangleleft$

In fact, we proved much more:

<u>Attaching handles</u>

= smooth analogue of attaching cells

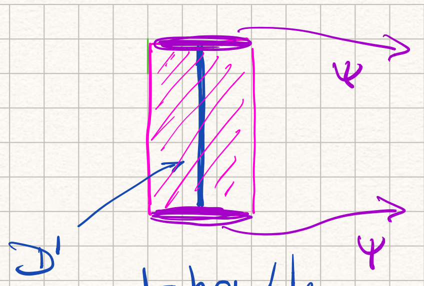
Consider X^n manifold with boundary

$H = D^k \times D^{n-k}$ = k-handle

$\partial(H) = S^{k-1} \times D^{n-k} \cup D^k \times S^{n-k}$

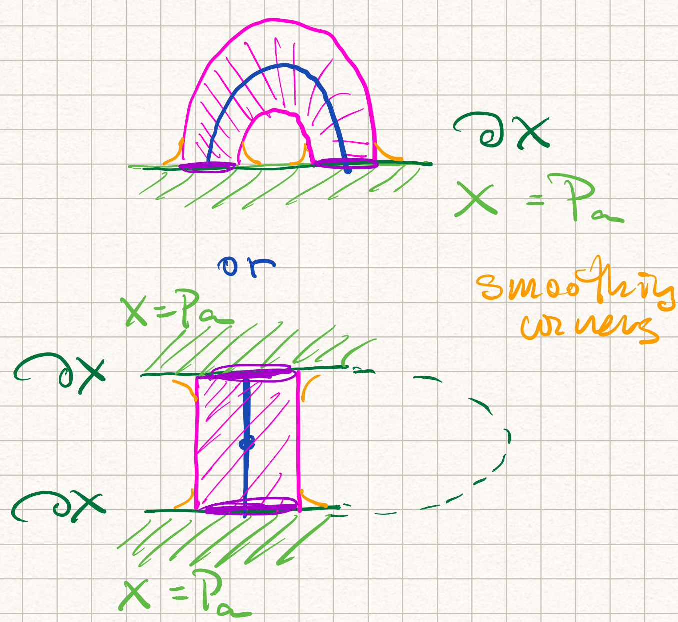
$\psi : S^{k-1} \times D^{n-k} \longrightarrow \partial X$ diffeo

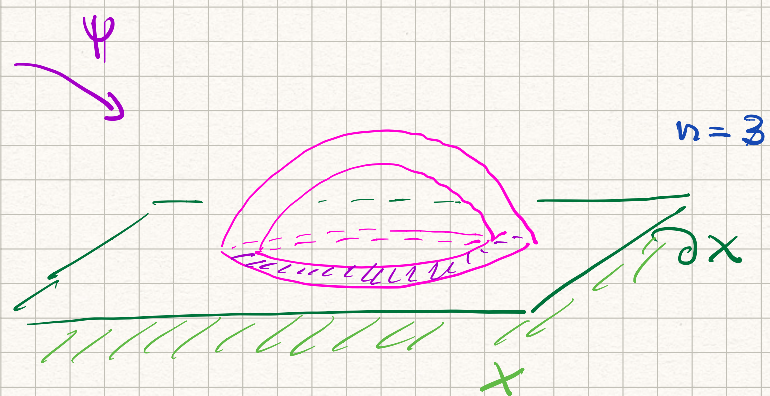
$Y = X \cup_\psi H \leftarrow$ top manifold with ∂ can be made smooth



1-handle

$D^1 \times D^{n-1}$





2-handle

$D^2 \times D^{n-2}$

attaching along

$S^1 \times D^{n-2}$

what we proved is

Thm P_b is obtained from P_a by attaching a k-handle for every crit pt of index k.

Cor A closed smooth manifold has a "handlebody decomposition"

Need to know that Morse functions exist.

Rmk Do not require k-handles to be attached before $k+1$-handles, but this can also be achieved.

Ex. Σ_g is obtained from D^2 by attaching $2g$ 1-handles and one 2-handle

Rmk Why Morse Lemma does not matter

Near a non-deg critical pt: $p = 0$

$$f(x) = \underbrace{Q(x)}_{\substack{\text{non-deg} \\ d_0^2 f}} + \underbrace{R(x)}_{\text{remainder}}$$

Q dominates R

$$\nabla f = \underbrace{\nabla Q}_{\text{completely dominates } \nabla R} + \nabla R$$

$\boxed{\text{smooth and top picture of } f \text{ near } p \text{ is completely determined by } Q.}$

$\Uparrow$ some local dynamics

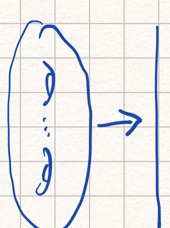
§9 A Glimpse of Applications to Differential Topology

Skipping Details

- The Key Point:
  Use Morse functions ⇝ handle body decomposition
  to understand the str of manifolds
- Strategy: starting with $f: P \xrightarrow[\text{Morse}]{} \mathbb{R}$,
  try to simplify f to get as few $\mathrm{Crit}(f)$ as possible & as simple as possible handle-body decomposition
  Background assumption: P has a Morse function — to prove later
- "Lower-bound" to how simple f can be
  $\Leftarrow$ Morse inequalities

Def f is perfect if $\partial = 0$ in the associated complex: $|\mathrm{Crit}_k(f)| = b_k$
(In reality one should also worry about $\mathbb{F}$ in the background.)

E.x.



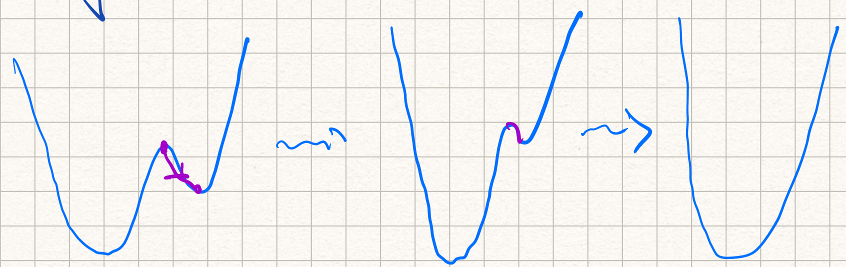
perfect

Rmk In fact perfect Morse function are rare

Ideally we want to start with some f and then modify it to make it perfect, but this is rarely possible

- Two preliminary steps - always work

Ex* → Can modify f to "kill" extra max's & min's to make sure ∃ only one max & min



Ex* → Can move lower index pts below higher index pts

$f(x) < f(y) \iff \mathrm{ind}(x) \leq \mathrm{ind}(y)$

(sliding handles)

Similar to CW-complexes

Beyond these two steps things get tricky and one has to impose extra conditions

Ref. J. Milnor "Lectures on the h-cobordism thm"

Applications:

1) Classification of closed surfaces
("Elementary"; see, e.g. Hirsch)
but tedious

2) The Poincaré conjecture

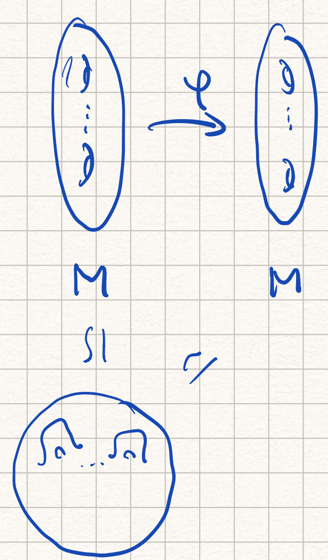
$$P^n \simeq S^n \Rightarrow P \underset{\text{homeo}}{\cong} S^n$$

See Milnor's book or many other sources

$n \geq 5$

3) **Heegaard Splitting**

Thm For every closed orientable 3-manifold P
$\exists\ \varphi : \Sigma_g \circlearrowleft$ diffeo s.t.



$P = M \cup_\varphi M$

$M =$ solid body with g-handles $\partial M = \Sigma_g$

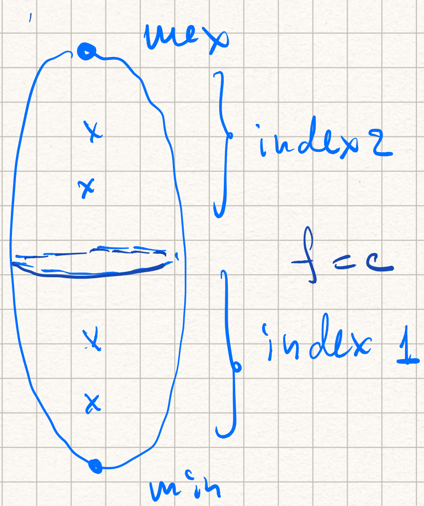
<u>Outline of the pf</u>

- $f: P \longrightarrow \mathbb{R}$ Morse with one max and one min

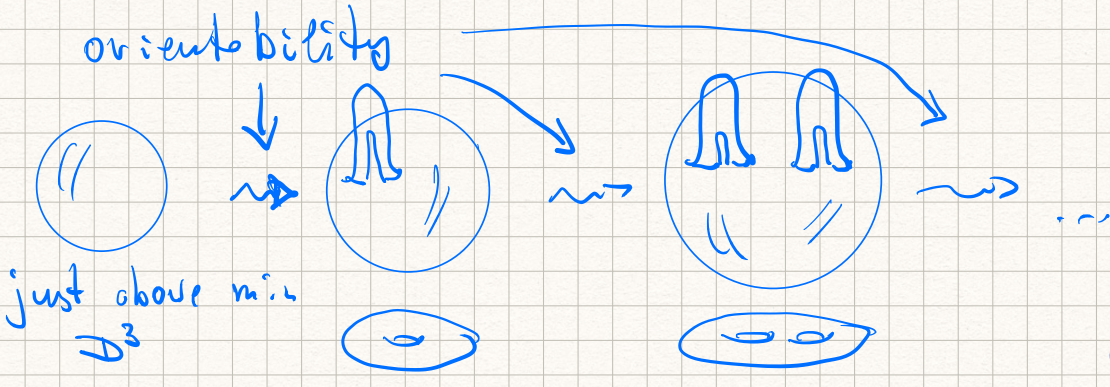
  and index 1 below index 2

- Poincaré Duality $\Rightarrow |Crit_1(f)| = |Crit_2(f)|$

  to be discussed later



Passing throug every pt of index 1
= attaching a 1 handle



"$\Rightarrow$" $\{f \le c\}$ is a solid body with some g handles

similarly for $\{f \ge c\}$ (Replace f by $-f$)

<u>Now</u> $\quad M = \underbrace{\{f \le c\}}_{\cong M} \cup \underbrace{\{f \ge c\}}_{M \cong}$

boundary identification

<u>Rmk.</u> This does <u>not</u> lead to a classification of 3-manifolds

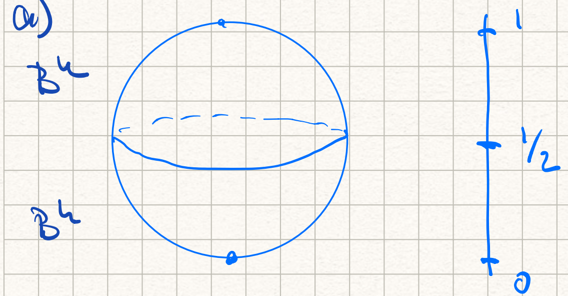
<u>Problem:</u> impossible to tell when $\varphi : \Sigma_g \circlearrowleft$ and $\varphi' : \Sigma_{g'} \circlearrowleft$ give rise to the same M!

4) <u>Thm</u> P admits a Morse function with exactly two critical pts

$$\Rightarrow \quad P \underset{\text{homeo}}{\cong} S^n \quad (\underline{\text{But not diffeo}})$$

<u>Pf - Outline</u>

a)



Morse lemma

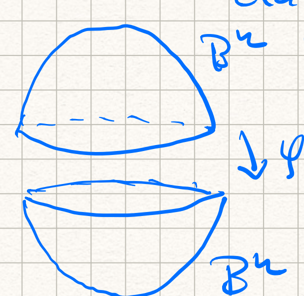
$$\left.\begin{array}{l} f^{-1}([0,\varepsilon]) \cong B^n \\ f^{-1}([1-\varepsilon, 1]) \cong B^n \end{array}\right\} \text{diffeo}$$

$\Rightarrow$ no crit pts

$$\left.\begin{array}{l} f^{-1}([0,1/2]) \cong B^n \\ f^{-1}([1/2, 1]) \cong B^n \end{array}\right\}$$

b) Now we have two copies of B^n a diffeo $\varphi : S^{n-1} \circlearrowleft$ and clutching map

$$P = B^n \cup_\varphi B^n :$$



c) Claim: φ extends to a diffeo $B^n \to B^n$ $\Rightarrow$ $P \underset{\text{diffeo}}{\cong} S^n$ ← not true in general

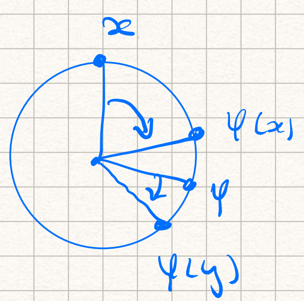
$\Uparrow$ Ex

$\Downarrow$

Claim: φ extends to a homeo $B^n \to B^n$ ← always true: use radial ext:

$\Downarrow$

Thm $\Rightarrow$ $P \underset{\text{homeo}}{\cong} S^n$



$\triangleleft$

§10 Floer Theory Perspective

Lecture 5 01/28

$f : P \longrightarrow \mathbb{R}$ Morse

$\Rightarrow$ CW-complex s.t.

$\mathrm{Crit}_k(f) \xleftrightarrow{1-1} k$-cells

$\Rightarrow$ $\exists$ a complex (C_*, ∂) s.t.

- C_k is generated by $\mathrm{Crit}_k(f)$
- $H_*(C_*, \partial) \cong H_*(P)$

Our goal is to describe ∂ explicitly without using alg topology and CW-complex str.

Def. $(C_*, \partial) = (CM_*(f), \partial) \leftarrow$ new notation

is called the

Morse complex of f

We have

$$\boxed{\partial \underset{\mathrm{Crit}_k(f)}{\underset{\cap}{x}} = \sum_{y \in \mathrm{Crit}_{k-1}(f)} m(x,y)\, y}$$

Want to describe the Morse differential explicitly

Rmk Depends on the metric!

Idea

ind = k

ind = k−1

$\underbrace{\mathrm{ind}(x) - \mathrm{ind}(y) = 1}$

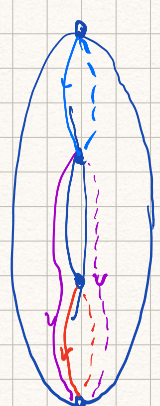
$\Downarrow$ generically

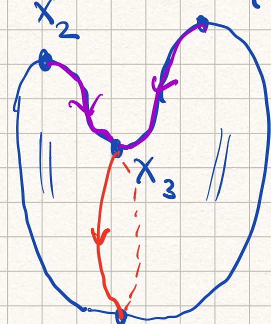
finite number of anti-grad trajectories from x to y

$$\boxed{m(x,y) = \#\text{ of traj } x \rightsquigarrow y}$$

parity over $\mathbb{F}_2 = \mathbb{Z}_2$
or with signs if over $\mathbb{Z}$

Ex Height function on $\mathbb{T}^2$ and on 





$\partial = 0$

$\partial x_1 = x_3 = \partial x_2$

$\partial x_3 = (1-1)x_4 = 0$

In what follows I'll skip most of the pfs, but explain ideas. Usually intuitively clean, but tedious

References:

- Jost
- Audin-Damian
- Banyaga - Hurtubise

§10.1 Construction of the Morse differential ∂: Preliminaries

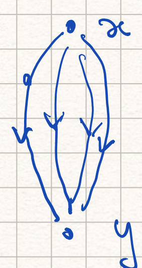
- While $CM_*(f)$ is completely determined by f, ∂ depends on an extra str: a R. metric on M
- Fix a R. m. on M (has to be from a certain open and dense set of R.m.'s)

Consider the antigradient flow of f:

$$\dot{x} = -\nabla f(x) \quad : \quad \varphi_t$$

Set $\mathcal{M}(x, y) = \left\{ z \;\middle|\; \varphi_t(z) \begin{array}{l} \to x \quad t \to -\infty \\ \to y \quad t \to +\infty \end{array} \right\}$

x, y crit



Denote the index of x by $\mu(x)$.

Note: "$\dim \mathcal{M}(x,y) \geq 1$" if $\neq \emptyset$

Thm For a generic metric, $\mathcal{M}(x,y)$ is a smooth manifold of dimension $\mu(x) - \mu(y)$

Explanation

$$W^u(x) = \{ p \mid \varphi^t(p) \to x, t \to -\infty \}$$
$$= \text{unstable manifold}$$

$$W^s(x) = \{ p \mid \varphi^t(p) \to x, t \to +\infty \}$$
$$= \text{stable manifold}$$

Morse Lemma or just non-deg $\Rightarrow$

$$\Rightarrow \begin{cases} W^u(x) \cong D^k \\ W^s(x) \cong D^{n-k} \end{cases}, \quad k = \mu(x) = \operatorname{ind}(x)$$

$$\mathcal{M}(x,y) = W^u(x) \cap W^s(y)$$

generic metric $\Rightarrow W^u(x) \pitchfork W^s(y)$

requires a pf
but not hard

$\Rightarrow$ $\mathcal{M}(x,y)$ is a manifold &

$$\begin{aligned} \dim \mathcal{M}(x,y) &= n - \dim W^u(x) - \dim W^s(y) \\ &= n - \mu(x) - (n - \mu(y)) \\ &= \mu(x) - \mu(y) \end{aligned}$$ $\triangleleft$

<u>Rmk</u> a) $\dim \mathcal{M} < 0 \Rightarrow \mathcal{M} = \emptyset$

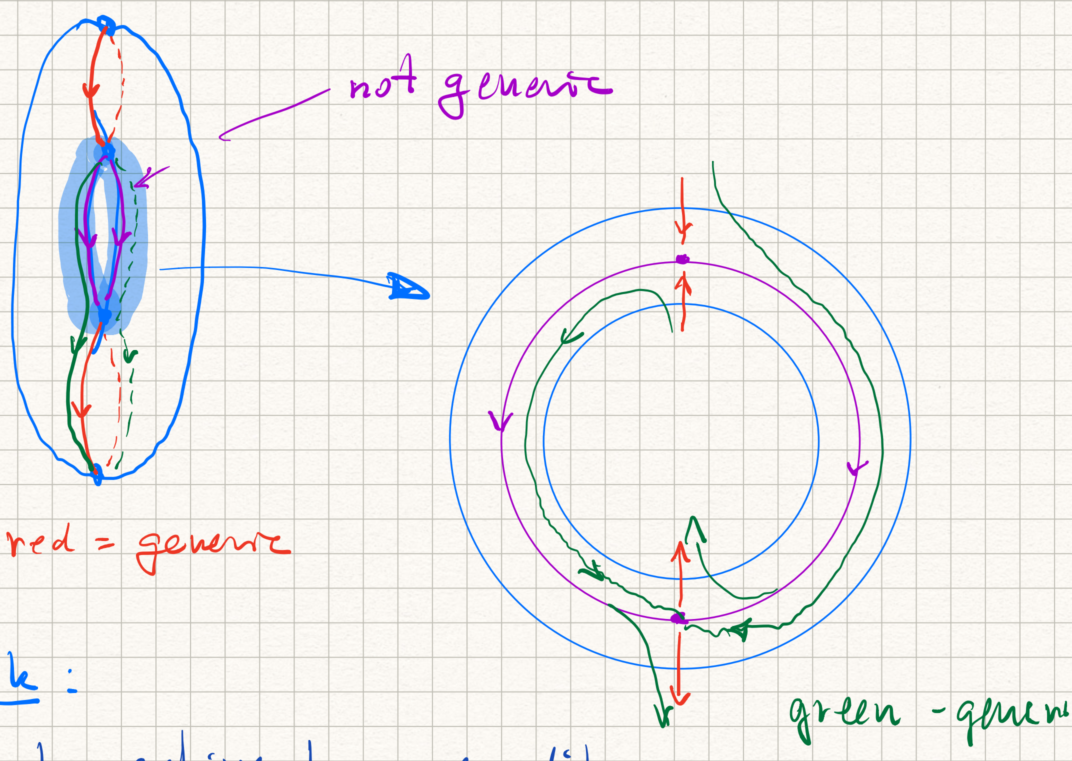
b) $\dim \mathcal{M} \geq 1$: with every pt $\mathcal{M}$ contains a whole trajectory

(Here $x \neq y$)

$\Rightarrow$ c) $\mathrm{ind}(x) \leq \mathrm{ind}(y)$

$\Rightarrow \mathcal{M}(x, y) = \emptyset$

<u>Ex.</u> <u>Height function on $\mathbb{T}^2$</u>



<u>Rmk</u>:

<u>How to achive transversality</u>

Look at $(W^u(x) \cap \{f = c\}) \cap (W^s(y) \cap \{f = c\})$

$f(y) < \underset{\text{reg}}{\underset{\uparrow}{c}} < f(x)$, perturb the metric slightly above c to alter

§10.2 Digression: sliding handles

Recall from §9

Prop: P admits a Morse function s.t.

$$f(x) > f(y) \iff \mu(x) \geq \mu(y) \quad \forall\, x, y \in \mathrm{Crit}(f)$$

Pf-outline: start with some f and modify it by moving critical pts by each other

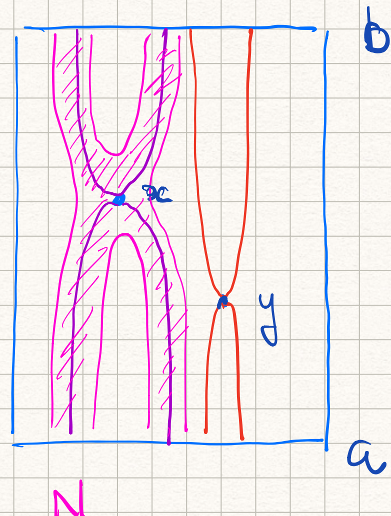
Need this:

x, y the only critical pts with $a < f(y) < f(x) < b$

$\mu(y) > \mu(x)$

$\Rightarrow$ can modify f in $[a, b]$ so that $f(y) \leq f(x)$

Idea



$\Sigma_x = (W^u(x) \cup W^s(x)) \cap f^{-1}([a,b])$

$\Sigma_y = (W^u(y) \cup W^s(y)) \cap f^{-1}([a,b])$

Generically

$$\Sigma_x \cap \Sigma_y = \emptyset$$

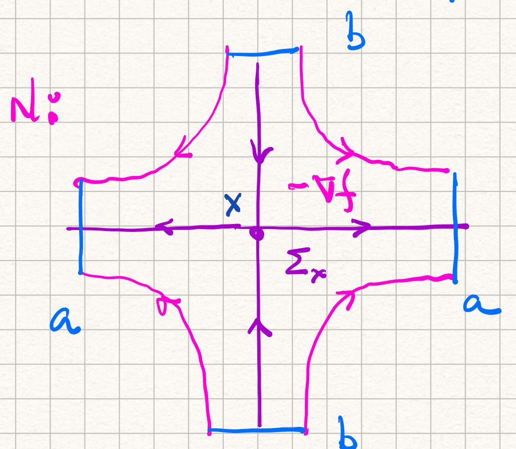
$$\Uparrow$$

$$\mu(y) > \mu(x)$$

modify f in a nbd $N \supset \Sigma_x$
(or $N \supset \Sigma_y$)
keeping it the same near $\partial \Sigma$

$\Rightarrow$ $\boxed{\text{can give } x \text{ any value in } (a,b) \text{ by modifying how } f \text{ changes along integral curves of } \nabla f}$

N is sketched unrealistically.
A better way:



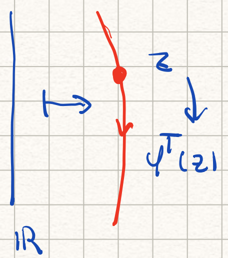
Can change f so that

- remains the same near ∂N
- same $-\nabla f$ up to scaling
- any value $f(x)$ in (a,b)

$\triangleleft$

§10.3 More modern & different perspective

$\mathcal{M}(x,y)$ = the space of parametrized trajectories $t \mapsto \varphi^t(z)$ from x to y



trajectory $\longleftrightarrow$ initial condition

$\varphi^t(z) \longleftrightarrow z = \varphi^0(z)$

Time shift: $t \mapsto \varphi^t(z)$ $\qquad z \mapsto \varphi^T(z)$

$$\begin{array}{c} t \mapsto \varphi^t(z) \\ \Downarrow \\ t \mapsto \varphi^{t+T}(z) \end{array}$$

$\Rightarrow$ free $\mathbb{R}$-action on $\mathcal{M}(x,y)$, $\quad x \neq y$

Space of unparametrized trajectories

$$\hat{\mathcal{M}}(x,y) = \mathcal{M}(x,y)/\mathbb{R}$$

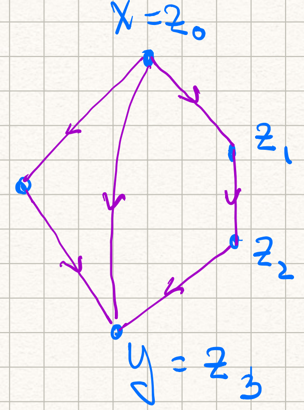
$$\boxed{\text{Cor} \quad \hat{\mathcal{M}}(x,y) \text{ is a smooth manifold of dim } \mu(x) - \mu(y) - 1}$$

E.g. $\mu(x) = \mu(y) + 1 \Rightarrow \hat{\mathcal{M}}$ is discr

Note
- $\mathcal{M}$ & $\hat{\mathcal{M}}$ are usually non-compact
- Geometrically, $\hat{\mathcal{M}}$ can be identified with $\mathcal{M} \cap \{f = c\}$
  $f(y) < c < f(x)$
  $\uparrow$
  regular

For a generic metric:

Thm $\hat{\mathcal{M}}(x,y)$ has a compactification formed by broken trajectories



Such trajectories $x = z_0 \rightsquigarrow z_1 \rightsquigarrow \dots \rightsquigarrow z_k = y$ form a compact manifold with corners.

Rmk $f(x) > f(z_1) > \dots > f(y)$

$\mu(x) > \mu(z_1) > \dots > \mu(y)$

Cor. $\mu(x) = \mu(y) + 1$

$\Rightarrow$ $\hat{\mathcal{M}}$ = compact $\Rightarrow$ finite collection of pts

$\Leftrightarrow$ $\exists$ finite many traj from x to y (for a generic metric)

§10.4 Definition of ∂

Fix a generic metric so that all the thms hold

$$\mu(x) = \mu(y) + 1$$

- Over $\mathbb{Z}_2$, set

$$\mathbb{Z}_2 \ni m(x,y) = |\hat{\mathcal{M}}(x,y)| \bmod 2$$

$$\boxed{\partial x = \sum_{\substack{y \\ \mu(x) = \mu(y)+1}} m(x,y)\, y} \qquad (*)$$

- Over $\mathbb{Z}$ (and hence any ring)

Need to take into account orientations

Fix orientations of $T_x W^u(x)$ $\forall x$

$\Rightarrow$ coorientations of $T_x W^s(x)$

$\Rightarrow \begin{cases} \text{orientations of } W^u(x) \\ \text{coorientation of } W^s(x) \end{cases}$

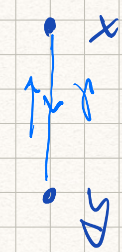
$\Rightarrow$ orientations of

$$\mathcal{M}(x,y) = W^u(x) \cap W^s(y)$$

• when $\mu(x) = \mu(y) + 1$

$\mathcal{M}(x,y)$ = disj union of finite # of trajectories

• Each trajectory γ is also oriented by the flow



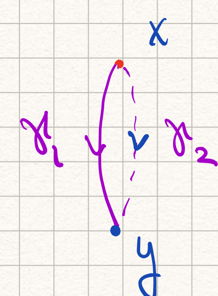
$\Rightarrow$ <u>Two orientations</u> on γ

$$\mathrm{sign}(\gamma) = \begin{cases} +1 & \text{orientations agree} \\ -1 & \text{——.——. disagree} \end{cases}$$

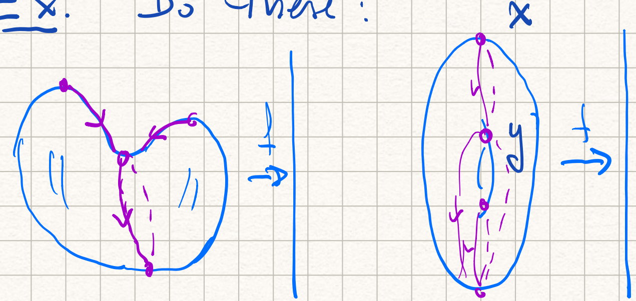
And
$$\underline{m(x,y) = \sum_{x \overset{\gamma}{\rightsquigarrow} y} \mathrm{sign}(\gamma)}$$

$$*: \quad \boxed{\partial x = \sum m(x,y)\, y}$$

$W^s(y) \cap W^u(x) = \gamma_1 \cup \cup \gamma_2$



<u>Ex</u>. Do these:



$W^s(y) \cap W^u(x)$

§10.5 Checking that $(CM(f), \partial)$ is a complex

$\boxed{\textbf{Thm} \quad \partial^2 = 0}$

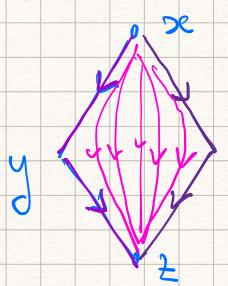
Pf. For the sake of simplicity over $\mathbb{Z}_2$

$$\partial^2 x = \partial \sum_y m(x,y)\, y$$

$\mu(x) = \mu(y)+1$
$\mu(y) = \mu(z)+1$

$$= \sum_y m(x,y) \sum_z m(y,z)\, z$$

$$= \sum_z \Big(\underbrace{\sum_y m(x,y)\, m(y,z)}_{\text{\# of broken trajectories (one break) from } x \text{ to } z \pmod 2} \Big) z \quad \text{mod } 2$$



But $\hat{\mathcal{M}}(x,y)$ one-dim manifold

its compactification: S^1 or I

closed interval

$\Rightarrow$ broken trajectories come in pairs

$\Rightarrow$ # is even

$$\Rightarrow \sum_y m(x,y)\, m(y,z) = 0 \mod 2$$

$\Rightarrow \partial^2 = 0$ $\triangleleft$

Set $HM_*(f) = H_*(CM(f), \partial)$; fixed coefficients

$$\boxed{\textbf{Thm (Morse theory)} \quad HM_*(f) = H_*(P)}$$

Rmk • As a consequence, r.h.s is independent of f ← can be proved directly

• We have already seen some consequences: Morse inequalities, etc

Outline of the pf:

"Classical" Morse theory

Ref
- Audin–Damian
- Banyaga–Hurtubise

Morse function f on M $\rightsquigarrow$ Cellular decomposition of M

$\mathrm{Crit}_k(f) \rightsquigarrow W^u(x)$ ← "cells"

$x \in \mathrm{Crit}_*(f)$

Morse complex $CM_k(f)$ $\longleftrightarrow$ Cellular complex of M: $C(M)$

$\partial_M = \partial_{cw}$

$$\Rightarrow \quad H_*(CM(f), \partial_M) = \underbrace{H_*(C(M), \partial_{cw})}_{H_*(M)} \qquad \lhd$$

§11. Calculations of $H_*(M)$ using Morse homology; Applications

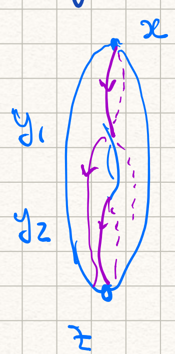
$$H_*(M) = H_*(HM_*(f), \partial_M)$$

very difficult to deal with in general

Works well when $\partial = 0$

Examples (over $\mathbb{Z}$ or $\mathbb{F}$)

1) Σ_g on $\mathbb{T}^2$



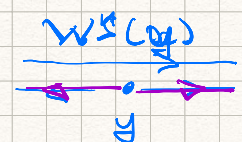
$$\partial x = y_1 + - y_1 + y_2 + - y_2 = 0$$

unstable traj of y_1 & y_2 should come from x

$$\partial y_1 = 0 = \partial y_2$$

$$\Rightarrow \quad H_*(\Sigma_g) = \begin{cases} \mathbb{F} & k = 2 \\ \mathbb{F}^{2g} & k = 1 \\ \mathbb{F} & k = 0 \end{cases}$$

Orientations!



2) $\mathbb{CP}^n$ (over $\mathbb{Z}$ or $\mathbb{F}$)

$$\mathbb{CP}^n = \{ (z_0 : \dots : z_n) \mid \sum |z_j|^2 = 1 \}$$

$$f(z) = \sum \lambda_j |z_j|^2$$

$$\lambda_0 < \lambda_1 < \dots < \lambda_n$$

Ex. a) $\text{Crit}(f) =$ "coordinate axes"

$$= \{ (0, \dots, 0, \underset{j}{1}, 0, \dots, 0) = x_j \}$$

b) In coordinates

$$u = (u_0, \dots, u_{j-1}, 1, u_{j+1}, \dots, u_n)$$

At x_j

$$d^2_{x_j} f = (\lambda_0 - \lambda_j)|u_0|^2 + (\lambda_1 - \lambda_j)|u_1|^2 + \dots \text{ skip } (\lambda_j - \lambda_j)$$
$$\dots + (\lambda_n - \lambda_j)|u_n|^2$$

$\Rightarrow f$ is morse & $\mu(z_j) = 2j \leftarrow z_j$ is a complex #.

& $\partial = 0$

$$\Rightarrow H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{F} & 0 \le k = 2j \le 2n \\ 0 & \text{otherwise} \end{cases}$$

3) $\mathbb{RP}^n$ over $\mathbb{Z}_2$

Similarly $\mathbb{RP}^n = \{ (y_0 : \dots : y_n) \mid \sum |y_j|^2 = 1 \}$

$$f(y) = \sum \lambda_j |y_j|^2$$

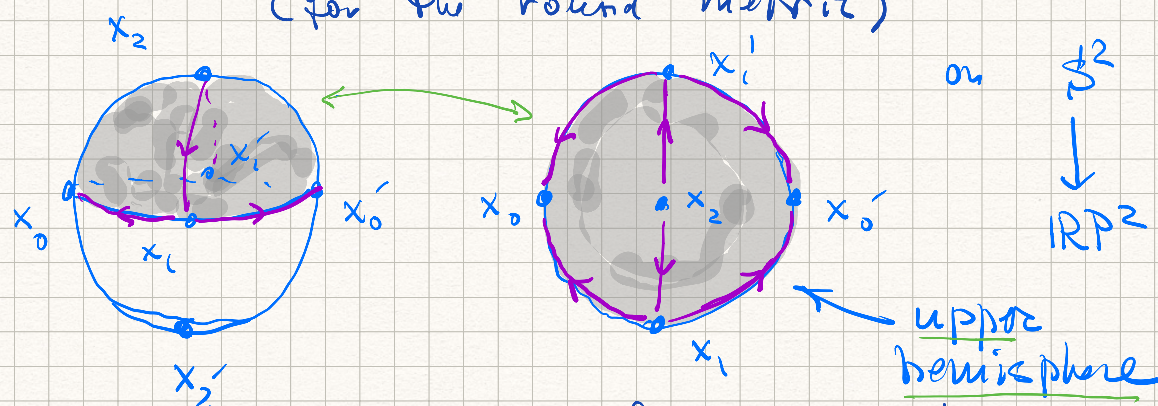
Ex. similarly

a) $x_j = (0, \dots, 0, \underset{j}{1}, 0, \dots, 0) \leftarrow$ Critical pts

b) Hessian: similar — same calculation

$$\Rightarrow \quad \mu(x_j) = j$$

c) $\partial = 0$ over $\mathbb{Z}_2$: exactly two trajectories from x_{j+1} to x_j (for the round metric)



$$\Rightarrow \quad H_k(\mathbb{RP}^n; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

Rmk. Over $\mathbb{Z}$, harder — orientations

Lecture 6 01/21

4) Hamiltonian S^1-actions or torus actions

P a symplectic manifold (closed)

$H: P \longrightarrow \mathbb{R}$ Hamiltonian generating an S^1-action

$$\underbrace{\mathrm{Crit}(H) = \text{Fixed pts} = P^{S^1}}_{\text{assume isolated}} = \{x\}$$

Details: Last year sympl geometry class

- Equivariant Darboux $\Rightarrow$ $d^2_x H$ have even index

$\Rightarrow$ $\partial_M = 0$ Similar to $\mathbb{C}P^n$

$$\dim H_k = |\mathrm{Crit}_k(H)|$$

- Works for a lot of interesting manifolds: Flag manifolds, Grassmannians, symplectic toric manifolds, etc

- Same for $\mathbb{T}^r$-actions with moment map $(H_1, \ldots, H_r)$

Set $H = \sum \lambda_j H_j$ – generalization of $\mathbb{C}P^n$

Two Textbook Applications

i) The Künneth formula for manifolds:

P_0, P_1 two closed manifolds

$$\Rightarrow \boxed{H_*(P_0 \times P_1) = H_*(P_0) \otimes H_*(P_1)}$$

over a field

Pf. • f_0 a morse function on P_0

• f_1 —. —. —. —. P_1

• pick generic metrics $\Rightarrow$ the product metric on $P_0 \times P_1$ is good for us

$$CM_*(f_0+f_1) = CM_*(f_0) \otimes CM_*(f_1)$$

as complexes

Algebraic Künneth ← needs a field

$$\Rightarrow \underbrace{HM_*(f_0+f_1)}_{H_*(P_0 \times P_1)} = \underbrace{HM_*(f_0)}_{H_*(P_0)} \otimes \underbrace{HM_*(f_1)}_{H_*(P_1)}$$

$\triangleleft$

Rmk. Likewise over $\mathbb{Z}$ or a ring but the formula is more involved $\Leftarrow$ algebra

2) Poincaré Duality

P smooth closed manifold, $\dim P = n$

$$\boxed{H^k(P) \cong H_{n-k}(P)}$$

canonical

- over $\mathbb{Z}_2$
- over other fields if orientable

<u>Cor</u>: $b_k = b_{n-k}$

<u>Pf</u>

f morse $\iff$ $-f$ is Morse

∇f $\qquad$ $\nabla(-f) = -\nabla f$

integral curves $\qquad$ same curves but traveled backward

$$\mathrm{Crit}_k(f) = \mathrm{Crit}_{n-k}(-f)$$

$$\Rightarrow \quad CM^*_k(f) = CM_{n-k}(-f)$$

$$\partial^*_M \qquad \partial_M(-f)$$

dual

use the basis

$$H^*(C) := H_*(C^*) = H_*(C)^* \text{ over a field}$$

$$\Rightarrow \quad \underbrace{H_*(CM^*(f), \partial^*_M)}_{H^*(P)} = \underbrace{H_*(CM_{n-*}(-f), \partial_M(-f))}_{H_{n-*}(P)}$$

Morse Theory for $(-f)$!

$\lhd$

§ 12 Existence of Morse functions

Part I: via Transversality thm

P closed manifold, $k \geq 2$ e.g. ∞

Thm Morse functions form an open and dense set in $C^k(P)$

Cor Every manifold admits a Morse function. Moreover, every function can be C^k-approximated by Morse functions.

We'll give two pfs

Pf 1: based on Transversality

Pf 2 (Outline) - direct

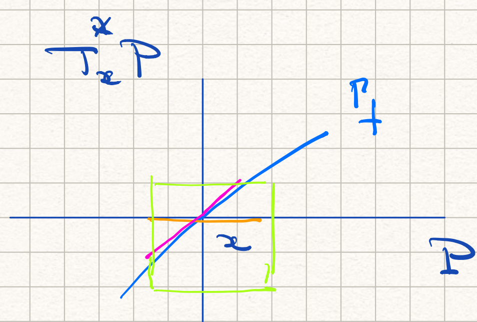
Revisiting the definition of Morse function

$f: P \to \mathbb{R} \rightsquigarrow df$ = section of T^*P

$\Gamma_f = \text{graph}(df)$

$x \in \text{Crit}(f) \iff df(x) = 0$

$\iff x \in \Gamma_f \cap P$ ← zero section

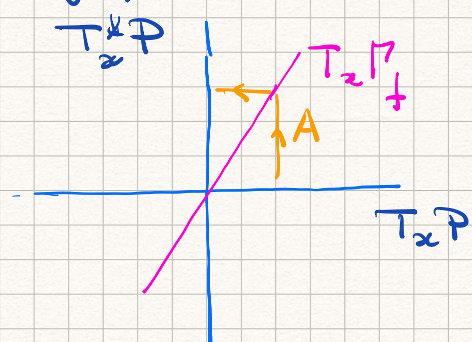


Claim x is non-deg $\iff \Gamma_f \pitchfork P$ at x

$$T_x\Gamma_f + T_xP = \underbrace{T_{(x,0)}T^*P}_{T_xP \oplus T^*_xP}$$

Pf.

- $T_x\Gamma_f = \text{graph}\,(A : T_xP \to T^*_xP)$



A = linearization of $s: y \to df(y)$ at x

- $T_x \Gamma_f \pitchfork P$ at $x \iff \operatorname{Ker} A = 0$

$\iff A$ is onto

$\iff A$ is non-deg

- A is essentially the Hessian

$$d^2_x f(v, w) = \underbrace{A(v)}_{T^*_x P}(w)$$

$$v, w \in T_x P$$

$$\underbrace{A \text{ is non-deg}}_{\iff T_x\Gamma_f \pitchfork P} \iff d^2_x f \text{ is non-deg}$$

$\triangleleft$

Cor f is Morse $\iff \Gamma_f \pitchfork P$

Idea of the pf of $\exists$ Morse functions

- start with some f_0. Need to approx by a Morse function f
- Transversality thm $\Rightarrow df_0$ can be approximated by $\alpha \in \Omega^1(P) =$ sections of T^*P so that $\Gamma_\alpha \pitchfork P$. Difficulty: Need $\alpha = df$

Transversality Theorems: Review (without pfs) & Pf I

<u>Ref.</u> E.g. Arnold–Varchenko–Gusein-Zade or many other textbooks

Based on Sard's Lemma

<u>Definitions</u>

$$X \xrightarrow{F} Y \supset Z$$

$F \pitchfork Z$ if

$$\boxed{DF(T_x X) + T_{F(x)} Z = T_{F(x)} Y}$$

$\forall\ x \in X$ with $F(x) \in Z$

Open condition in C^1-top

<u>Ex</u> $\dim X + \dim Z < \dim Y$, $F \pitchfork Z$

$$\Rightarrow \quad F(X) \cap Z = \emptyset$$

<u>Key pt</u>: Almost all $F \pitchfork Z$ often even extra constraints on F

<u>Ex.</u> $L_0, L_1 \subset \mathbb{R}^3$ two embedded loops

For almost all $v \in \mathbb{R}^3$

$$(L_0 - v) \cap L_1 = \emptyset$$

<u>Hint:</u> • $L_0 \times L_1 \xrightarrow{\Phi} \mathbb{R}^3$

$(x, y) \longmapsto x - y$

• $(L_0 - v) \cap L_1 \neq \emptyset \iff \exists\, x, y$ s.t. $x - v = y$

$\iff x - y = v$

$\iff v \in \Phi(L_0 \times L_1)$

• $\dim(L_0 \times L_1) = 2$, $\dim \mathbb{R}^3 = 3$

Sard's $\Rightarrow$ $\Phi(L_0 \times L_1)$ = zero measure

nowhere dense

closed

$\Rightarrow$ For an open full measure set in $\mathbb{R}^3$

$(L_0 - v) \cap L_1 = \emptyset$

$\triangleleft$

A sequence of settings increasing constraints on F: transversality theorems

Setting 1.

$$X \xrightarrow{F} Y \supset Z$$

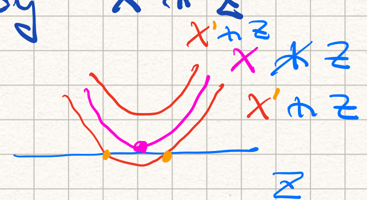
X closed manifold

Z at least proper

Thm 1 $F \pitchfork Z$ for open and dense set in $C^k(X, Y)$

$1 \leq k \leq \infty$

Rmk
- Here and below it's clear that the condition is open in C^k top.
- Pf is based on Sard's lemma
- If $F: X \hookrightarrow Y$ we say $X \pitchfork Z$

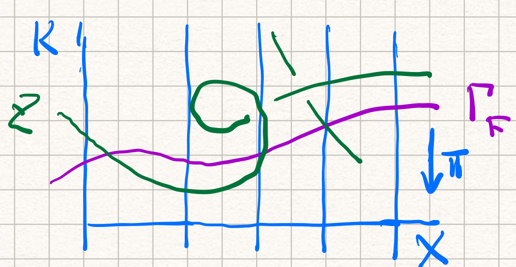


Setting 2

$$Y = X \times K \supset Z, \qquad F: X \to K$$

$$\Gamma_F: X \hookrightarrow Y \qquad x \mapsto (x, F(x))$$

Thm 2 $\Gamma_F \pitchfork Z$ For an open & dense set in $C^k(X, K)$



Much more restrictive class: $\Gamma_F: X \hookrightarrow Y = X \times K$

$$\underbrace{\pi \circ \Gamma_F = id}_{\text{section of } X \times K \to X}$$

Setting 3 : same but

$Z \subset Y$

$$\begin{tikzcd} K \arrow[r] & Y \arrow[d, "\pi"] \\ & X \arrow[u, bend left, "s"] \end{tikzcd} \quad \Bigg\} \quad \text{fiber bundle} \quad \underbrace{\pi \circ s = id}_{\text{section}}$$

Thm 2

$\Uparrow$

Thm 3 $s \pitchfork Z$ For an open and dense set of C^k-sections of π.

Rmk. The class of sections is much smaller than all $C^k(X, Y)$

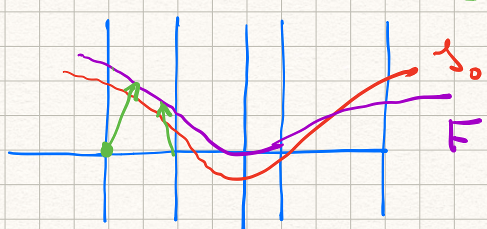
Outline of Thm 1 $\Rightarrow$ Thm 3

Need density

$s_0 : X \to Y$ a given section

$F : X \to Y$, $F \pitchfork Z$ $C^{k \geq 1}$-close to s_0

not a section : $\pi \circ F(x) \neq x$



$\varphi : x \longmapsto \pi \circ F(x)$

$X \longrightarrow X$

Claim (Ex) $\varphi : X \to X$ a diffeo, C^k close to id

Replace F by $s := F \circ \varphi^{-1} \pitchfork Z$

$\approx s_0$ $\triangleleft$

Rmk It is essential that

$$\varphi \overset{C^{k\geq 1}}{\approx} id$$

Not enough $\varphi \overset{C^0}{\approx} id$

Ex $\varphi(x) = x + \varepsilon \sin(\omega x)$

id

Setting 4: Jet Transversality

Short Lecture
01/26
Lecture 7

Next step - incorporating

A very basic minimalist version

$J^1X = T^*X \times \mathbb{R} \longrightarrow X$

$j^1 f = (df, f) : X \to J^1X$ a section

$Z \subset J^1X$ a proper submanifold

$\boxed{\textbf{Thm 4} \quad j^1 f \pitchfork Z \text{ for an open and dense set of } f \in C^k(X),\ k \geq 2}$

Cor Can replace J^1X by T^*X:
$df \pitchfork Z \subset T^*X$ for an open and dense set $f \in C^k(X)$
$k \geq 2$

Pf Replace $Z \subset T^*X$ by $Z \times \mathbb{R} \subset J^1X$
df by $j^1 f$

Rmk much more limited class of maps:

Sections of $T^*X = \Omega^1(X)$

$\alpha \in \Omega^1(X)$:

$\alpha = df \Rightarrow \alpha$ is closed $\Rightarrow \alpha$ is exact

$d\alpha = 0$ a condition on the derivative

Pf of the thm on Morse function:

- set $X = P$, $Z = P \subset T^*P$ zero section
- f Morse $\Leftrightarrow$ $\underbrace{df \pitchfork Z = P \text{ in } T^*P}_{\text{open \& dense condition in } C^{k \geq 2}(P)}$

$\lhd$

More details on transversality approach, by passing Jet Transversality:

Banyaga – Hurtubise

based on the idea in this pf.

On the pf of ordinary transversality

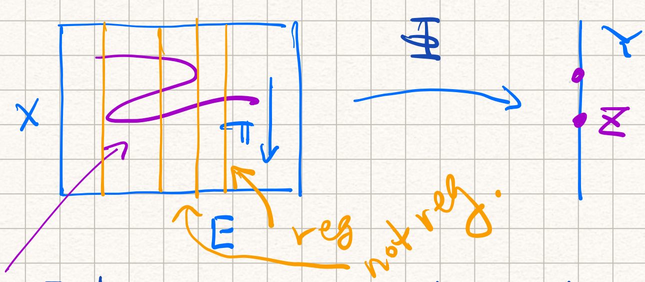
Following Guillemin–Pollock

Assume X & Y are closed for the sake of simplicity

<u>Lemma</u> (Ex) $\Phi : X \times E \to Y \supset Z$, $\Phi_e = \Phi|_{X \times e}$

$$\Phi \pitchfork Z \Rightarrow \Phi_e \pitchfork Z \text{ for almost all } e \in E$$

Hint:



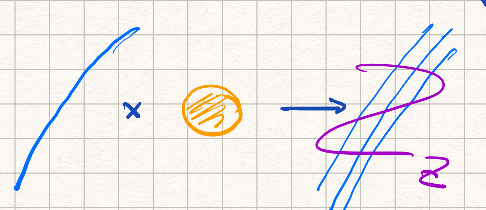
$M = \Phi^{-1}(Z)$ = smooth submanifold of $X \times E$

<u>Claim</u>: $\Phi_e \pitchfork Z \Leftrightarrow e$ is a reg value of $\pi|_M$ ◁

<u>Lemma ⇒ Transv. thm</u>

$Y = \mathbb{R}^n$, $E = \mathbb{R}^n$

$\Phi(x, e) = F(x) + e$



Φ is a submersion (already on the second factor)

$\Rightarrow \Phi \pitchfork Z \quad \forall Z$

$\Rightarrow \Phi_e \pitchfork Z$ for almost all e

Take e close to 0 !

Now when Y is compact

- Take $Y \hookrightarrow \mathbb{R}^n$
- Care only about small $e \in B^n(\varepsilon)$

- Take $F: X \to Y$ and extend to to a submersion

$$\begin{tikzcd} X \times B^n \arrow[rr, "\Phi"] \arrow[dr] & & Y \\ & U \arrow[ur, "\pi"] & \end{tikzcd}$$

smth like $F(x) + e$ is good

- Finish the pf as before

$\triangleleft$

85a

§ 13 Existence of Morse functions
Part II: via Height Functions

Lecture 8
0/28

Recall

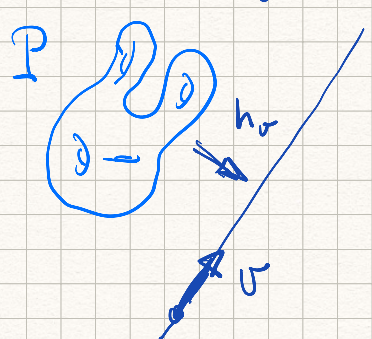
P closed manifold, $k \geq 2$ e.g. ∞

$\boxed{\textbf{Thm 1}\ \text{Morse functions form an open and dense set in } C^k(P)}$

$\exists\ P \hookrightarrow \mathbb{R}^m$, from now on $P \subset \mathbb{R}^m$

Denote by h_v the proj to $v \cdot \mathbb{R}$:

$v \in S^{m-1}$



E.g. The heigh function h

$\boxed{\textbf{Thm 2}\ \text{For an open dense (full measure) subset of } S^{m-1},\ h_v \text{ is Morse}}$

Thm 2 $\Rightarrow$ Thm 1:

$$f : P \longrightarrow \mathbb{R} \qquad \rightsquigarrow \qquad P \xhookrightarrow[(j,f)]{} \mathbb{R}^{m+1} \xrightarrow{\;h\;} \mathbb{R}$$

$$j : P \hookrightarrow \mathbb{R}^m$$

$h \Leftarrow$ height, with f the composite $P \to \mathbb{R}^{m+1} \to \mathbb{R}$.

Take $v \in S^m$ close to $e_{m+1} \subset \mathbb{R}^{m+1}$ so that $h_v : P \longrightarrow \mathbb{R}$ is Morse

$$\Rightarrow \qquad h_v \overset{C^\infty}{\approx} h$$

$$f = h : P \longrightarrow \mathbb{R}$$

$$h_v : P \xrightarrow{\;SS\;} \mathbb{R}$$

$\triangleleft$

Rmk. Milnor's "Morse theory":

A related but different argument

$$M \xrightarrow[\mathrm{dist}_p^2]{} \mathbb{R}, \qquad M \subset \mathbb{R}^m$$

$\mathrm{dist}_p^2(x) = \| x - p \|^2$

is Morse for almost all p.

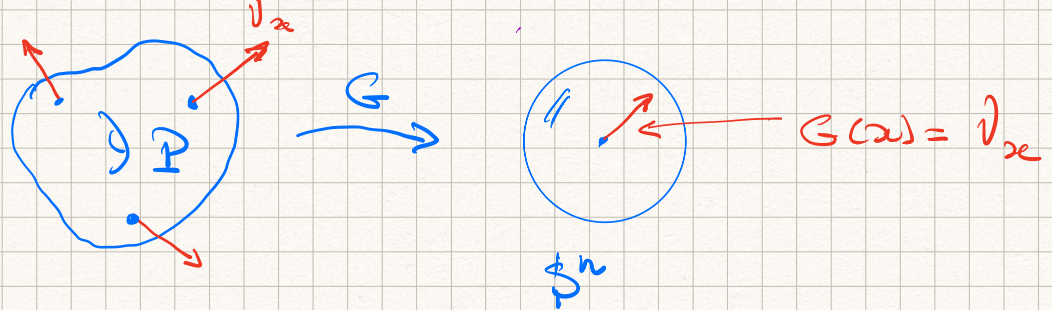
<u>Pf of Thm 2</u>

$*$ <u>Particular case P = hypersurface:</u>

$$P^n \subset \mathbb{R}^{n+1}$$

$G : P \longrightarrow S^n$ the Gauss map

$x \longmapsto \nu_x \leftarrow$ unit outer normal



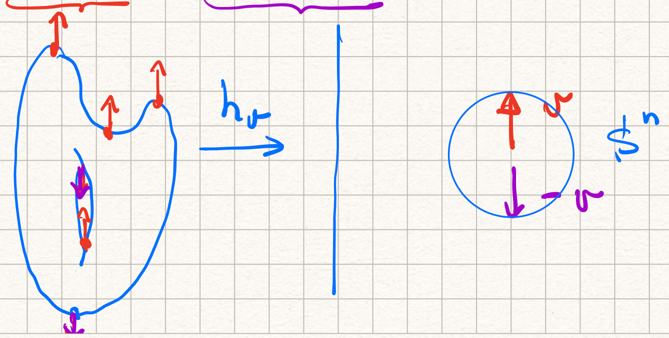
<u>Prop</u> $\underline{h_v \text{ is Morse} \Leftrightarrow \pm v \text{ are reg values of } G}$

$\Downarrow \leftarrow$ Sard's

Thm 2 when P is hypersurface

Pick: $v \in S^n$, $v = e_{n+1} \leftarrow$ convenient

$$\underbrace{G^{-1}(v)} \cup \underbrace{G^{-1}(-v)} = \mathrm{Crit}(h_v)$$



Pf of the proposition

$G(p) = v = e_{n+1}$

Suffices to show:

p is a reg pt of G $\Longleftrightarrow$ $d^2 h_v$ is non-deg at p

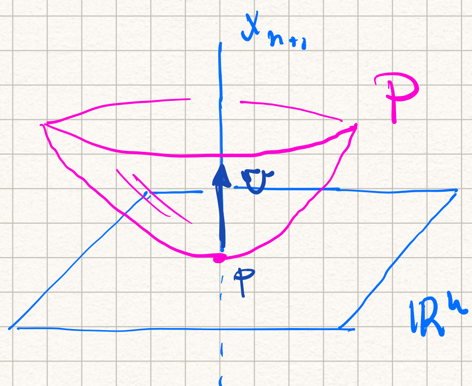
$\Updownarrow$ def

$DG : T_p P \to T_v S^n$ is onto

From now on

$v = e_{n+1}$, $h_v = h$

Now we can assume locally



$P = \text{graph}(g : \mathbb{R}^n \to \mathbb{R})$

$g = h$

$T_p P = \mathbb{R}^n$

$p = 0$

The second fundamental form

$$(\mathrm{II}_p : T_p P \otimes T_p P \longrightarrow \mathbb{R}) \overset{\text{def}}{:=} d_p^2 g$$

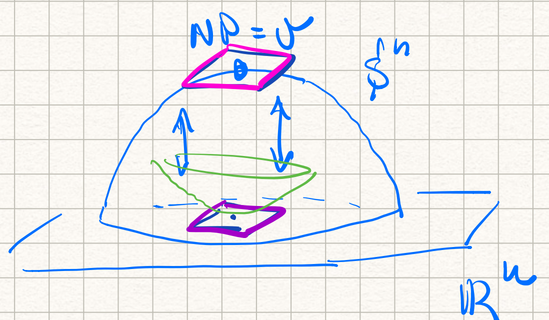
$$\mathrm{II}_p = d_p^2 h$$

<u>Goal</u> : relate II_p to DG_p

<u>Preliminary steps:</u>

- First of all
  Can assume g is a quadratic form
  $$g(x) = \mathrm{II}_p(x) + \underbrace{\dots}_{\text{do not affect } DG}$$
  $\Rightarrow$ Can drop from the calculation
  $$\underbrace{g = \frac{1}{2}\langle Ax, x\rangle}_{\mathrm{II}_p(x)}, \quad A = A^T$$
  Not really using this...

- Can recycle $\mathbb{R}^n$ as a local chart near $v = NP \in S^n$
  using orthogonal proj $\mathbb{R}^{n+1} \to \mathbb{R}$
  along x_{n+1}



Now : $DG_p : \underbrace{T_p P}_{\mathbb{R}^n} \longrightarrow \underbrace{T_v S^n}_{\mathbb{R}^n}$

The horizontal $\mathbb{R}^n$

• <u>Key Claim</u> $DG_p = -A$

$\Rightarrow$ $\underbrace{\left(p \text{ is a reg point of } G \Leftrightarrow \mathrm{II}_p \text{ is non-deg}\right)}_{\text{exactly what we need}}$

Pf of the claim

$\rightarrow$ $\nu_x = \dfrac{(-\nabla g(x), 1)}{\sqrt{1+|\nabla g(x)|^2}}$ $\leftarrow$ vector calculus 23B

$\rightarrow$ <u>In the $\mathbb{R}^k$-chart</u>

$\nu_x = -\dfrac{\nabla g(x)}{\sqrt{1+|\nabla g(x)|^2}}$; $g(x) = \frac{1}{2}\langle Ax, x\rangle + \ldots$

$= -Ax + \ldots$

$\Rightarrow$ at p: $DG = -A$ $\triangleleft$

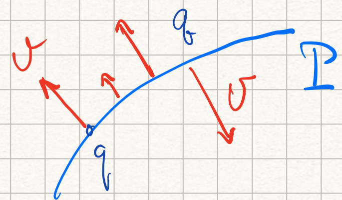
This finishes the pf in the case where P is a hypersurface

General case: $P^n \subset \mathbb{R}^{n+k+1=m}$

The Gauss map and Normal bundle

N_P = normal bundle to $P \subset \mathbb{R}^n \times \mathbb{R}^n$

$= \{(q, v) \in P \times \mathbb{R}^n \mid v \in T_q P, \text{ normal to } P \text{ at } q\}$



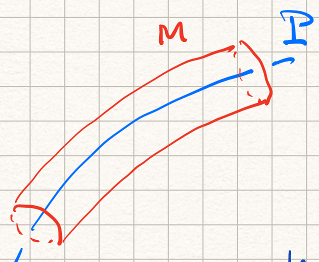
$\dim N_P = m = n+k+1$

$N_P \xrightarrow{\pi} P$ vector bundle

$SN_P = M$ = the unit normal bundle

$= \{ (q, v) \mid \|v\| = 1 \} \subset N_P \xrightarrow{\pi} P$

should think of M as the boundary of infinitesimally small nbd of P



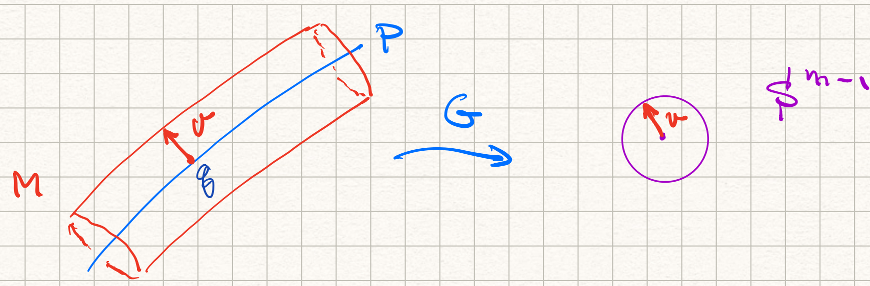
Rmk: P hypersurface

$\Rightarrow M$ = two copies of P: inward & outward

$$\dim(M = SN_P) = m-1$$

Gauss map: $G: M = SN_P \longrightarrow S^{m-1}$

$$(q, v) \xmapsto{G} v$$



Prop $\pm v$ are reg values of $G \iff h_v$ is Morse.

$\Downarrow$ $\leftarrow$ Sards

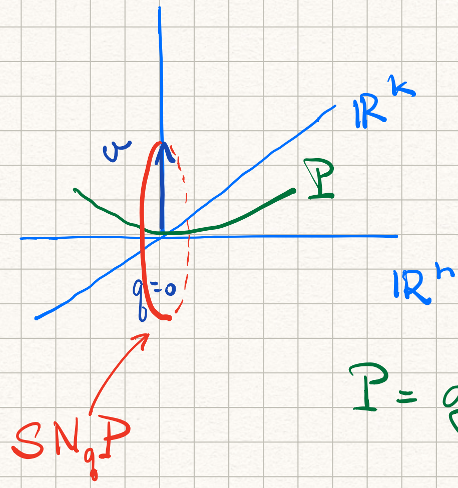
Thm 2

$$\underbrace{\pi\left(G^{-1}(v) \cup G^{-1}(-v)\right)}_{\text{base pts}} = Crit(h_v)$$

Pf (outline)

As before can assume $v = e_m \leftarrow$ vertical vec.

$q = 0$.



$T_qP = \mathbb{R}^k$

Locally

$P = \text{graph}(g : \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R})$

Write $g = (F, h)$ $\leftarrow$ the height function

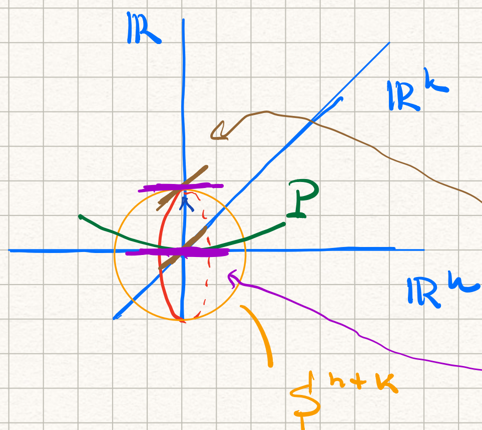
By construction $g(q) = 0$, $Dg(q) = 0$

<u>Note</u>: as before care only about the quadratic part of g

What is DG ?

$$T_v S^{n+k} = \mathbb{R}^n \times \mathbb{R}^k = T_{(q,v)} M$$

$$\underbrace{T_q P}_{\text{purple}} \qquad \underbrace{T_{(q,v)} SN_q P}_{\text{brown}}$$



$$G|_{SN_qP} = id$$

$$DG_q = \begin{array}{c|cc} & \mathbb{R}^k & \mathbb{R}^n \\ \hline \mathbb{R}^k & I & DF \\ \mathbb{R}^n & 0 & -A \end{array}$$

$DF \leftarrow$ does not matter

$h(x) = \frac{1}{2}\langle Ax, x\rangle + \dots$

same calculation as for hypersurfaces

$\Rightarrow DG_q$ is onto $\Leftrightarrow$ A is non-deg

$\Leftrightarrow$ $d_q^2 h$ is non-deg

An application (added)

$M \subset \mathbb{R}^n$ $\qquad\qquad h_v(x) = \langle x, v \rangle$: proj to $v\mathbb{R}$

$\downarrow f \qquad\qquad\qquad\qquad\qquad\qquad v \in \mathbb{R}^n$

$\mathbb{R}$

What we have shown is that:

$$\boxed{\text{For almost all } v \in \mathbb{R}^n \quad f + h_v \text{ is Morse on } M}$$

<u>Cor</u> For a dense set of p

$x \mapsto \|p - x\|^2$ is Morse

<u>Pf.</u> $f(x) = \| x \|^2$

$$f(x) + h_v(x) = \langle x, x\rangle + \langle x, v\rangle$$
$$= \langle x + \tfrac{1}{2}v, x + \tfrac{1}{2}v\rangle - \tfrac{1}{4}\|v\|^2$$

$\Rightarrow \langle x + \frac{1}{2}v, x + \frac{1}{2}v\rangle$ is Morse

Can replace 0 by any pt. $\triangleleft$

Next Lecture - Short

§14. Digression: Cup Product in Morse homology

Ref: Jost

skipping details
quite technical

Recall

* P any reasonable space

  $\Rightarrow$ $H^*(P)$ is a graded unital alg

  Product: cup product "$\cup$"

  E.g. • P a manifold, in $H^*_{dR}(P) = H^*(P; \mathbb{R})$

  $$[\alpha] \cup [\beta] = [\alpha \wedge \beta]$$

* Assume now P^n is a closed manifold

  Intersection product

  $$a \cdot b := PD^{-1}(PD(a) \cup PD(b))$$

  $\rightarrow$ $|a \cdot b| = |a| + |b| - n$

  $\rightarrow$ unit $= [M]$

$$\boxed{H_{*_k}(P) \otimes H_{*_\ell}(P) \longrightarrow H_{*_{\ell+k-n}}(P)}$$

Conceptually $a = [A]$, $b = [B]$ $\rightsquigarrow$ $a \cdot b = [A \cap B]$

Key pt: the intersection product has a simple interpretation in Morse theory.

Even two

setting : $f: P \longrightarrow \mathbb{R}$ morse

$\Rightarrow$ $CM_*(f)$ morse complex

$HM_*(f) = H_*(P)$ morse hom

Interpretation I: $HM_*(f)$ is a module over $H_*(P) \overset{PD}{\cong} H^*(P)$

$$\boxed{H_*(P) \otimes HM_*(f) \longrightarrow HM_*(f)}$$

Define an "action" of $H_*(P)$ on $HM_*(f)$

$b \in H_*(P)$, $b = [B]$ a cycle

$a \in HM_*(f)$, $a = [A]$, $\underbrace{A = \sum \beta_i x_i}_{\text{morse cycle}} \in CM_*(f)$

$B \circ A := \sum \beta_i \underbrace{(B \cdot x_i)}$ need to def this

Think of B as a generic, immerse submanifold: of codim ℓ Discuss

$$B \cdot x = \sum_z \langle B \cdot x, z \rangle z$$

$\langle B \cdot x, z \rangle =$ " $|\mathcal{M}(x,z) \cap B|$ " # of constraints

$$\boxed{\mu(z) = \mu(x) - \operatorname{codim} B} \quad (*)$$

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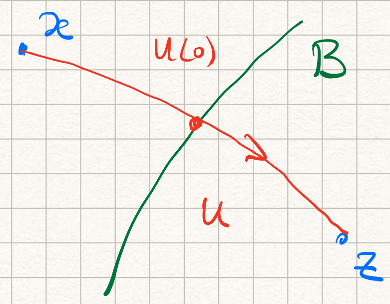


$$\langle B\cdot x, z\rangle = \Big|\underbrace{\{x \overset{u}{\rightsquigarrow} z \mid u(0)\in B\}}_{\text{a finite set when } (*) \text{ holds \& } B \text{ is generic}}\Big|$$

Then

$$\partial_M (B\cdot w) = B\cdot \partial_M w$$

a cycle

$\Rightarrow$ $B\cdot$ descends to $HM_*(f)$

The resulting map depends only on $[B] = b$

Geometrically and very informally

$$\text{"}CM_* \longrightarrow C_*\text{"}$$

$$x \longmapsto W^u(x)$$

$$B\cdot x = \sum \underbrace{(\qquad\qquad)}_{|\mathcal{M}(x,z)\cap B|} W^u(z)$$

$$|(W^u(x) \pitchfork W^s(z)) \pitchfork B|$$

Interpretation II: Product on $HM_*(f) \cong H_*(P)$

$=$ intersection product

Product on $CM_*(f)$

Need to define $\;$ $f_1 \& f_2$ Take two functions for transversality

$$CM_*(f_1) \otimes CM_*(f_2) \longrightarrow CM_*(f_1 + f_2)$$

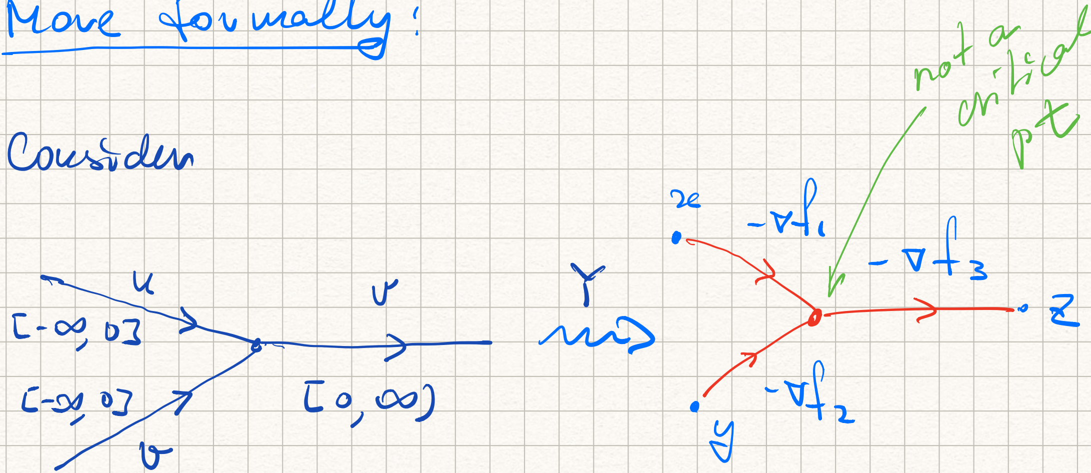
$$x \cdot y = \sum \langle x \cdot y, z \rangle z$$ can take f_3 here

$$\mu(z) = \mu(x) + \mu(y) - n$$

Idea: Replace B in the previous constr by $W^u(y)$ ← informally the chain y gives rise to.

More formally:

Consider



$$\mu(z) = \mu(x) + \mu(y) - n$$

For a generic metric $\exists$ finitely many such Y shapped traj and their # is $\langle x \cdot y, z \rangle$ (with signs)

As defined it's not associative, but
$$\partial_M (x \cdot y) = \partial_M x \cdot y \pm x \cdot \partial_M y$$
(I think)
and on the level of homology we have

$$\underbrace{\begin{array}{ccccc} HM_*(f_1) & \otimes & HM_*(f_2) & \longrightarrow & HM_*(f_3) \\ \| & & \| & & \| \\ H_*(P) & \otimes & H_*(P) & \longrightarrow & H_*(P) \end{array}}_{\text{intersection product}}$$

<u>Rmk</u> Other homological features of P also have similar descriptions via <u>graph flows</u>

<u>Rmk</u>: Want:
$$H_*(P_a) \otimes H_*(P_b) \longrightarrow H_*(P_{a+b})$$
$$\Rightarrow \quad f_3 \leq f_1 + f_2$$

Part II Lusternik-Schnirelmann Theory

Lecture 9: 02/02 (short)

§15 Introduction

<u>Setting</u>: $f: \underset{\text{closed}}{\Sigma} \xrightarrow{C^\infty} \mathbb{R}$, $P_a = \{f \leq \overset{\text{reg}}{a}\}$

<u>Q</u>: Get a lower bound on $|\mathrm{Crit}(f)|$ without non-degeneracy

Underlying idea (both Morse & LS):

top change from P_a to P_b $\Rightarrow$ crit pts between a & b

But Morse & LS process this change differently

<u>Rmk.</u> • LS is more versatile and better adapted to higher dim generalizations

• In $\dim < \infty$, Morse is deeper

Comparison – Examples

P	Morse	LS
S^n	2	2
$\Sigma_{g\geq 1}$	$2g+2$	3
T^n	2^n	$n+1$
$\mathbb{C}P^n$ or $\mathbb{R}P^n$	$n+1$	$n+1$

different

don't know this yet

__Rmk__ • In general LS is very far from sharp

__But__: any P^n (closed) admits $f: P \xrightarrow{C^\infty} \mathbb{R}$ with $|\mathrm{Crit}(f)| \leq n+1$ (Takens)

• Sharpness of Morse Theory $\Longleftrightarrow$ Diff topology; Poincaré conj, h-cobordism etc

__Next Goal__: Proving LS lower bounds

start with a very traditional approach in topology __Eg.__ Fomenko-Dubrovin-Novikov

§16 Topological Preliminaries: LS Category & Cup-length

X = a reasonably good space

Def: LS category of X:

$$\mathrm{cat}(X) = \min\{k \mid X = \bigcup_{i=1}^{k} A_i\}$$ closed or open

where A_i's are contractible to pt in X

$$\begin{tikzcd} A_i \arrow[r, hook] \arrow[dr] & X \\ & pt \arrow[u] \end{tikzcd}$$

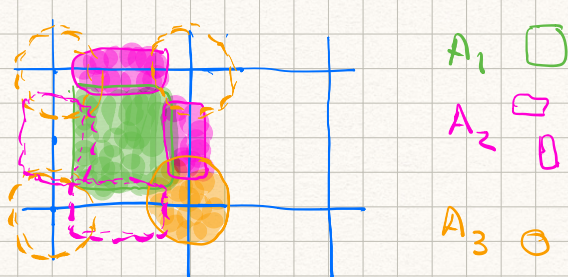
need not be injective

Rmk A_i's need not be connected

Ex.

- $\mathrm{cat}(S^n) = 2$
- $\mathrm{cat}(\mathbb{RP}^n \text{ or } \mathbb{CP}^n) \overset{(=)}{\leq} n+1$ see later
- $\mathrm{cat}(\Sigma_{g\geq 1}) = 3$

T^2



$\Sigma_g \setminus D^2 =$ 

Rmk cat(X) is in general very difficult to determine. Upper bounds ⇑ expl. constructions

more generally

X = n-dim CW complex
$\Rightarrow$ $\text{cat}(X) \leq n+1$

Pf: same argument: proceed inductively

Also note: homotopy invariant

Def The cup-length of X (over $\mathbb{F}$) connected

$$cl(X) = \max\{k \mid \exists\, \alpha_1, \dots, \alpha_k \in H^{*>0}(X)\}$$
$$\alpha_1 \cup \dots \cup \alpha_k \neq 0$$

<u>Clearly</u>: $0 \leq cl(X) \leq \underbrace{\dim X}_{\text{when } X \text{ is a manifold}}$

<u>Rmk</u>:
- depends on the ground field
- $\exists$ more general notions...

<u>Ex.</u>
- $cl(S^n) = 1$
- $cl(\underbrace{\mathbb{RP}^n}_{\text{for } \mathbb{Z}_2} \text{ or } \underbrace{\mathbb{CP}^n}_{\text{any field}}) = n$
- $cl(\mathbb{T}^n) = n \qquad \alpha_i = [dx_i]$
- $X = P^{2n}$ closed symplectic
  $\Rightarrow cl(P) \geq n = \frac{1}{2}\dim P, \quad \alpha_i = [\omega]$
- $cl(\Sigma_{g\geq 1}) = 2$
- More generally P closed $\Leftarrow$ PD
  $cl(P) \geq 2$ over $\mathbb{Z}_2$
  over anything when orientable
  when $H^i(P) \neq 0$
  $0 < i < n$

Rmk Usually $cl(X)$ is not so hard to calculate – pretty primitive inv.

Prop

$$\boxed{cat(X) \geq cl(X) + 1}$$

Rmk The gap "$>$" can be huge but examples are not "easy" to construct

not among manifolds you think about at this level

Con-Ex. $cat(\mathbb{RP}^n \text{ or } \mathbb{CP}^n) = n+1$

Pf

claim $A \subset X$ contractible to a pt

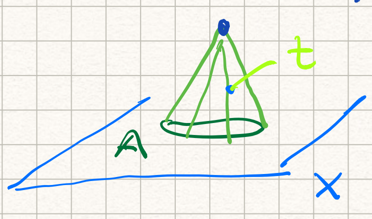
$\Rightarrow H^*(X,A) \xrightarrow[\text{onto}]{i^*} H^*(X)$ $*>0$

Rmk. In de Rham $H^*(X,A)$ comes from forms vanishing on A

Pf of the claim

$C_A(X)$ = cone over $A \subset X$

$= X \cup A \times [0,1] / \sim$ $(a,0) \sim a$, $A \times 1 \sim 1$ pt



$H^{*>0}(X,A) = H^{*>0}(C_A(X))$ essentially by definition

$f_t : A \longrightarrow X$ contraction to a pt

$f_0 : A \hookrightarrow X$

$f_1(A) = pt$

Define $F : C_A(X) \longrightarrow X$

$$\begin{cases} F(x) = x & \text{for } x \in X \\ F(a,t) = f_t(a) \end{cases}$$

$$X \underset{F}{\overset{i}{\rightleftarrows}} C_A(X)$$

$F i = id \implies \underbrace{i^* F^* = id \text{ in } H^*(X)}$

$\Downarrow$

i^* is onto

$\lhd$

<u>Pf of Prop</u>: Assume the contrary:

$cl(X) \geq cat(X)$

no need to do it by contradiction

$X = A_1 \cup \ldots \cup A_k$ $\leftarrow$ contr in X

$$\begin{array}{ccc} \alpha_1 \cup \ldots\ldots \cup & \alpha_k & \neq 0 \text{ in } H^{*>0}(X) \\ \uparrow & \uparrow & \\ \tilde{\alpha}_1 & \tilde{\alpha}_k & \text{in } H^{*>0}(X, A_i) \end{array}$$

Recall $H^*(X,A) \otimes H^*(X,B) \longrightarrow H^*(X, A\cup B)$

E.g. think diff forms vanishing on A & B

$$\underbrace{0 \neq \alpha_1 \cup \dots \cup \alpha_k}_{\cap \atop H^*(X)} \leftarrow \underbrace{\tilde\alpha_1 \cup \dots \cup \tilde\alpha_k}_{\cap \atop H^*(X, \underbrace{A_1 \cup \dots \cup A_k}_{X})} = 0$$

$$\Uparrow$$

$$H^*(X) \leftarrow H^*(X, A_1 \cup \dots \cup A_k) = 0$$

$\rightarrow\, \leftarrow$ $\triangleleft$

Ref Handbook of Alg. Top
Edited by I. M. James

Discussion

LS cat is a peculiar notion
"not quite happy with itself"

some things to keep in mind:

- Not monotone: $X \subset Y \not\Rightarrow \mathrm{cat}(X) \le \mathrm{cat}(Y)$
- $\exists$ a notion of $\underline{\mathrm{cat}_X(A)}$: covering A, but contractible in X

Then $A \subset B \subset X \Rightarrow \mathrm{cat}_X(A) \le \mathrm{cat}_X(B)$

- But $A = $ retract of $X \Rightarrow \mathrm{cat}(A) \le \mathrm{cat}(X)$

$A \subset X$, $\mathrm{cat}(A) \neq \mathrm{cat}(X)$

- $\mathrm{cat}_X(A)$ is continuous in A
- $F \hookrightarrow E \to B$ $\quad \mathrm{cat}(E) \le \mathrm{cat}_E(F) \cdot \mathrm{cat}(B)$
- homotopy invariant (not quite obvious)
- $\mathrm{cat}(X \cup Y) \le \mathrm{cat}(X) + \mathrm{cat}(Y)$

§17 Lower bound via LS cat

$f : \underset{\text{closed}}{P} \xrightarrow{C^2} \mathbb{R}$

This is how they do it in topology textbooks

Thm (LS) Assume that Crit(f) are isolated

$$\boxed{\big|\underbrace{\text{crit values of } f}_{cv(f)}\big| \geq \text{cat}(P)}$$

Cor $|\text{Crit}(f)| \geq \text{cat}(P) \geq \text{cl}(P) + 1$

Rmk can be strict

Cor

P	$\lvert\text{Crit}(f)\rvert \geq$
$\mathbb{C}P^n$ & $\mathbb{R}P^n$	$n+1$
$\Sigma_{g\geq 1}$	3
$\mathbb{T}^n$	$n+1$

completing the set of inequalities from p. 101

Pf (outline)

- $f: P \to \mathbb{R}$ with exactly m critical values $c_1 < \ldots < c_m$ & isolated $\mathrm{Crit}(f)$

$\Rightarrow$ cover of P by m contr. to pt sets

- Do inductively by moving upward

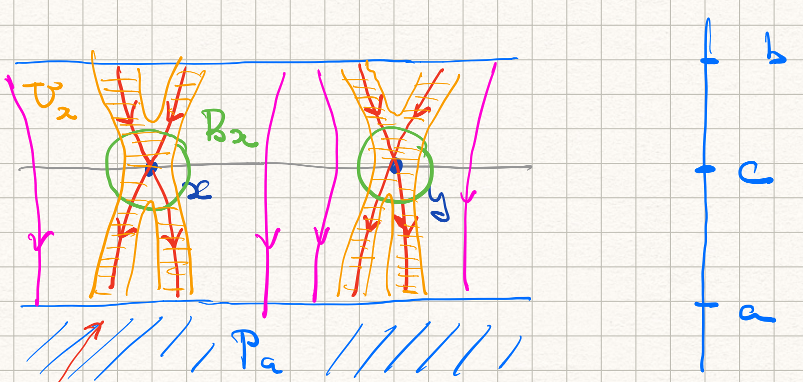
$b \leftarrow$ reg

$c_k \to c \leftarrow$ crit

$a \leftarrow$ reg

Assume the cover for P_a is constructed ($(k-1)$ sets)

$\Rightarrow$ a cover of P_b by k sets

- 

$\Sigma_x := \big(\underbrace{W^s(x) \cup W^u(x)}_{\text{stable unstable sets}}\big) \cap \{a \le f \le b\}$

$U_x := \begin{cases} \text{"flow invariant (in } a < f < b) \\ \text{nbd of } \Sigma_x \end{cases}$

Observations:

→ • U_x is contractible by
upward/downward flow
in a small nbd B_x of x

• B_x is contr to x

$\Rightarrow A_k = \coprod_x U_x$ is contr to a pt
in P_b

→ • $P_b \setminus (\coprod_x U_x)$ is homotopy
equiv to P_a (in fact homeo)

$\Rightarrow$ a cover of $P_b \setminus (\coprod_x U_x)$
induction
by $k-1$ sets $A_1, \dots, A_{k-1}$

Together $A_1, \dots, A_{k-1}, A_k$ the required
cover of P_b

$\triangleleft$

Q But how few crit pts can a function on P have?

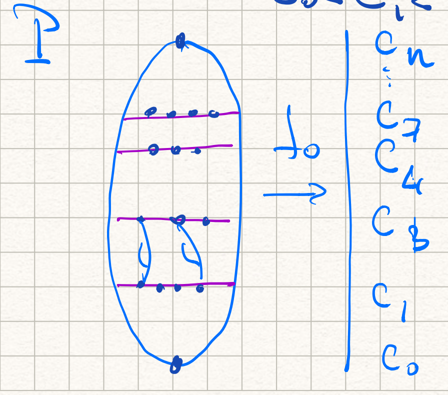
Thm (Takens, Inventiones, 1968)
dim $P = n$
$\Rightarrow \exists\, f: P \xrightarrow{C^\infty} \mathbb{R}$ with $|\mathrm{Crit}(f)| \le n+1$

Cor $\mathrm{cat}(P) \le \dim P + 1$
↑ • also know because P = CW of dim $\le n$
or • selecting disjoint subsets in a cover

Outline of the pf

- Start with $f_0 : P \to \mathbb{R}$ a Morse function with one max & one min
- Sliding handles $\Rightarrow$ can have all critical pts of index k on one level $f = c_k$ and

$c_0 < c_1 < \dots < c_n$ ← can skip some values



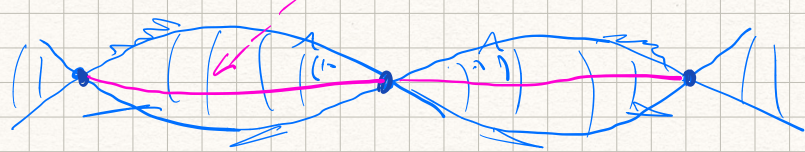
• Ex — show that

$\{f = c_k\}$ is connected

<u>Rmk</u> Need to treat the case $n = 2$ separately

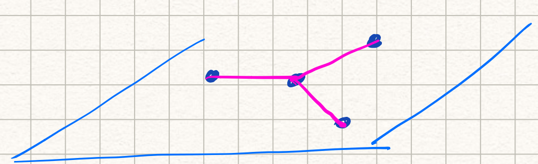
• Pick a $\underbrace{\text{tree}}$ $\Gamma_k \subset \{f = c_k\}$

a union of smooth arcs intersecting only at their ends
homeo to a tree



$f = c_k$

containing all critical pts on $f = c_k$



$f = c_k$

- Contract all Γ_k to pts

$\underline{P} \underset{\text{diffeo}}{\cong} \underline{P} / \amalg \Gamma_k$ more precisely

$\exists$ a smooth map $\underline{P} \xrightarrow{G} \underline{P}$

s.t. $\underline{P} \setminus \amalg \Gamma_k \longrightarrow \underline{P} \setminus \underbrace{\{x_0, \dots, x_m\}}$ X can skip some i's

is a diffeo & $\Gamma_i = G^{-1}(x_i)$

- Now $f_1 = f_0 \circ G^{-1} : \underline{P} \xrightarrow{C^0} \mathbb{R}$

is smooth outside X and only C^0 at the pts of X

- Modify f_1 near each x_i to make it smooth and have only one critical pts (Takens, Thm 2.7 – elementary but non-obvious) $\triangleleft$

§ 18. The min/max principle

And this is how they do it in dynamics/calculus of variations

- $f: P \longrightarrow \mathbb{R}$ anti-grad flow
  closed manifold $\longleftarrow$ this condition can be significantly relaxed

- $\mathcal{F}$ = a class of compact subsets of P closed under $\varphi^{t\geq 0}$

<u>Ex.</u>
- Fix $\alpha \in \pi_k(P)$

$$\mathcal{F} = \{ S^k \xrightarrow{u} P \mid [u] = \alpha \}$$

- Fix $\alpha \in H_*(P)$

$$\mathcal{F} = \{ \text{cycles representing } \alpha \}$$

<u>E.g.</u>
- images of singular cycles (over $\mathbb{Z}_2$)
- maps $\sigma: M \longrightarrow P$ s.t. $\sigma_*([M]) = \alpha$

Set

$$c_{\mathcal{F}}(f) = \inf_{A \in \mathcal{F}} \underbrace{\sup}_{\max} f|A$$

Thm (Minimax Principle, I) $\boxed{c_{\mathcal{F}}(f) \text{ is a critical value}}$

Rmk
- Versatile & important
- conditions can be relaxed
- a lot of applications

Ex.
- $\mathcal{F}$ = collection of all pts in P

$$c_{\mathcal{F}}(f) = \inf_{A \in P} f(A) = \min f$$

- $\mathcal{F} = \{P\}$ ← just one set P itself

$$c_{\mathcal{F}}(f) = \max_{P} f = \max f$$

Pf of the min/max principle ← nearly obvious

- $c := \inf_{A \in \mathcal{F}} \overset{\max}{\sup_{A}} f|_A$ ← A is compact

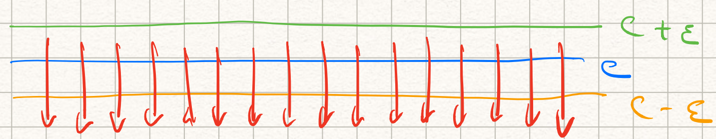
$\mathcal{F} = \{A\}$; A is compact
$\mathcal{F}$ is closed under $\varphi^{t \geq 0}$

Assume c is not a critical value
— a pf by contradiction

- $\exists$ $\varepsilon > 0$ & $\tau > 0$ s.t.
$f(x) \leq c + \varepsilon \Rightarrow f(\varphi^{\tau}(x)) \leq c - \varepsilon$

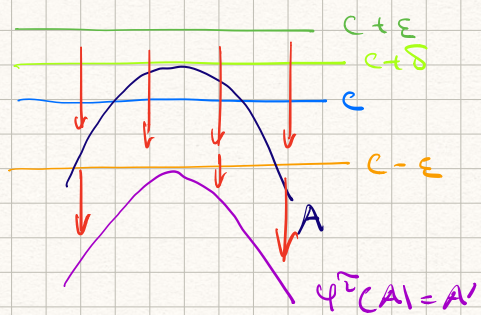
$\Uparrow$

Compactness & c is not a critical value



φ^t = the flow of $-\nabla f$

- As said assume c is not a critical value
take A so that $\max_A f = c + \delta < c + \varepsilon$



$\underbrace{\max_{\varphi^{\tau}(A)}}_{A' \in \mathcal{F}} f = c - \varepsilon < c$ → ←

because $\mathcal{F}$ is $\varphi^{t \geq 0}$ invariant

◁

- Our next goal is to illustrate how Min/max works by several simple applications of the Idea.

- We'll keep on coming back to it over & over

§19 Connecting with LS Theory

Lecture 11
02/09
Review minimax

- Critical Value Selectors

This is the first application of of the Min/max Principle

Goal: reprove

$$\boxed{|\mathrm{Crit}(f)| \geq \mathrm{cl}(P)+1}$$

To be more precise we'll prove

$$\boxed{\mathrm{Crit}(f) \text{ isolated} \Rightarrow \mathrm{cv}(f) \geq \mathrm{cl}(P)+1}$$

By a much more flexible construction

- As before $f : P \xrightarrow{C^\infty} \mathbb{R}$
  - P closed
- Fix $\mathbb{F}$ ← the background ring (or field)
  $H_*(P) = H^{n-*}(P)$ when a field

Critical value selectors

- $\alpha \in H_k(P) = H^{n-k}(P)$
- Stick to some construction of $H_*(P)$ as the homology of $C_*(P)$

$$\mathcal{F}_\alpha = \{ A \mid [A] = \alpha \}$$

Rmk
- A is not necessarily a subset of P but rather a combination of maps to P ($A : M \longrightarrow P$ or simpl. chain) and then we mean the image of this map
- Or it can be literally a subset:
  - pseudo-cycles [McDuff–Salamon]

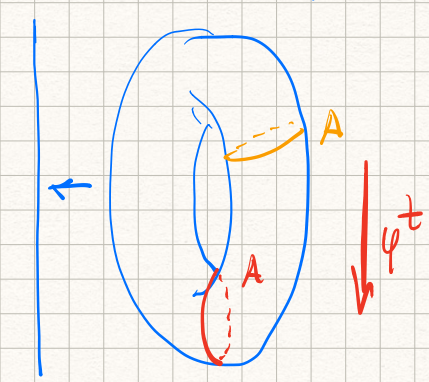
Def 1 Critical value selectors

$c_\alpha(f) := c_{\mathcal{F}_\alpha}(f) :=$ min/max for $\mathcal{F}_\alpha = \{ A \mid [A] = \alpha \}$

$$\boxed{c_\alpha(f) = \inf_{[A]=\alpha} \max_A f}$$

Ex.

$P = \mathbb{T}^2$



α = homology class of these loops

$c_\alpha(f)$ is attained by pushing down A by $\varphi^{t\to\infty}$ till it "hangs on" a critical pt

Def 2 • Assume f is Morse

a finite collection

$$A = \sum m_i x_i \quad , \quad [A] = \alpha \text{ in } CM_*(f)$$

$$c_\alpha(f) = \inf_A \max_{m_i \neq 0} f(x_i)$$

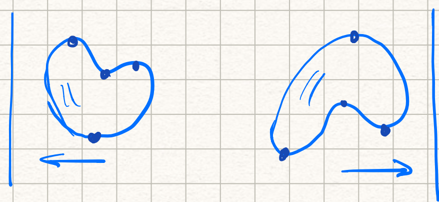
min

• If f is not Morse, approximate it by Morse $\tilde{f} \xrightarrow{C^\infty} f$ and set

$$c_\alpha(f) = \lim_{\tilde{f}\to f} c_\alpha(\tilde{f})$$

Examples • $c_{[M]}(f) = \max f$

• $c_{[pt]}(f) = \min f$



Def. 3

$$c_\alpha(f) = \inf\{\underset{\nwarrow \text{ regular}}{a} \mid \alpha \in \operatorname{im}\left(H_*(P_a) \longrightarrow H_*(P)\right)\}$$

Pf of equiv : $\text{Def } 2 \overset{\text{by def}}{\Longleftrightarrow} \text{Def } 3 \overset{\text{by def when } f \text{ is Morse}}{\Longleftrightarrow} \text{Def } 1$

Properties

• Criticality ← Min/max : $c_\alpha(f)$ = crit value of f

• Monotonicity : $f \leq g \Rightarrow c_\alpha(f) \leq c_\alpha(g)$ By def

• C^0 continuity : $|c_\alpha(f) - c_\alpha(g)| \leq \underbrace{\|f-g\|_{C^0}}_{\max|f-g|}$

Hint : use $f - \varepsilon \leq g \leq f + \varepsilon$ and monotonicity

Rmk $\quad c_\alpha(f) = f(x), \quad x \in \operatorname{Crit}(f)$

Cannot make x cont in f

- <u>Sub-additivity</u> $\quad \alpha, \beta \in H_*(P)$

(1) $\boxed{c_{\alpha \cap \beta}(f+g) \leq c_\alpha(f) + c_\beta(g)}$

In particular $\Downarrow$ Take $g=0$

(2) $\boxed{c_{\alpha \cap \beta}(f) \leqslant c_\alpha(f)}$

Informal explanation: for (1)

cycles

$\alpha = [A]$, $\beta = [B]$, $\alpha \cap \beta = [A \pitchfork B]$

$$c_{\alpha \cap \beta}(f+g) = \inf_{[C]=\alpha \cap \beta} \max(f+g)|_C$$

to be $C = A \cap B$

$$\leq \max(f+g)|_{A \cap B}$$

$$\leq \max f|_A + \max g|_B$$

Now take inf over all A & B

- Can be turned into a pf once "cycles" and $A \cap B$ are defined

(2) is even easier:

$$\max f|_{A \cap B} \leq \max f|_A$$

Pf of (1) using Y-graph flows

Can assume that f & g are Morse

$\alpha \in HM_*(f) = H_*(P)$, $A \in CM_*(f)$

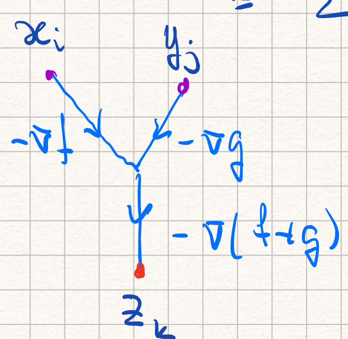
$\beta \in HM_*(g) = H_*(P)$, $B \in CM_*(g)$

$$A = \sum a_i x_i, \quad x_i \in \mathrm{Crit}(f) \qquad B = \sum b_i y_i, \quad y_i \in \mathrm{Crit}(g)$$

$$\alpha \cap \beta = [A \cdot B]$$

$$= \sum a_i b_j \, x_i \cdot y_j$$

$$= \underbrace{\sum a_i b_j \, \#\left\{ \begin{smallmatrix} x_i & & y_j \\ & \searrow \swarrow & \\ & \downarrow & \\ & z_k & \end{smallmatrix} \right\} z_k}$$



Pick A & B so that (exists)

$$c_\alpha(f) = \max f(x_i)$$

$$c_\beta(g) = \max f(y_j)$$

one of these C

$$(f+g)(z_k) \leq f(x_i) + g(y_j)$$

Then
$$c_{\alpha \cap \beta}(f+g) = \inf_{C = \sum c_k w_k} \max (f+g)$$

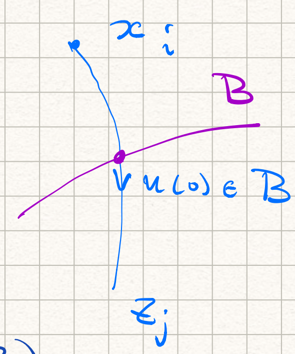
$$\leq \max f(z_k)$$

$$\leq \max f(x_i) + \max g(y_j) = c_\alpha(f) + c_\beta(g) \quad \triangleleft$$

Ex. Give a direct pf of (2) using the $H_*(P)$-mod-str on $HM_*(f)$

$$HM_*(f) \otimes H_*(P) \longrightarrow HM_*(f)$$

$\alpha = [A]$ $\qquad\qquad$ $\beta = [B]$



$$A = \sum a_i x_i$$

$$B \cdot A = \sum a_i (B \cdot x_i)$$

$$= \sum a_i \, \#\{ \ \ \} \, z_j$$

f decreasing along u $\Rightarrow$ (2)

$\lhd$

<u>Cor</u> $\quad \min f \leq c_\alpha(f) \leq \max f$

A more subtle result

Thm ("LS inequality")

> Assume • $\mathrm{Crit}(f)$ isolated
> • $|\beta| < n$: $\beta \neq [\mathbb{I}]$
>
> $\Rightarrow$ $c_{\alpha \cap \beta}(f) < c_\alpha(f)$
>
> strict

I don't know a very simple pf

Explanation:

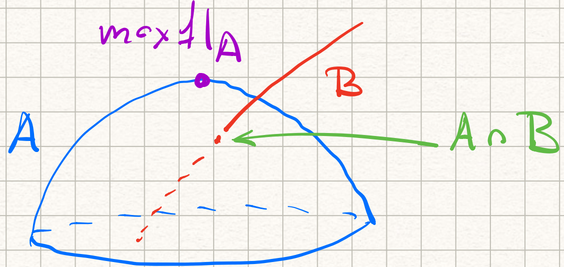
Assume $c = \inf_{[A]} \max f|_A$ attained on some A (Not essential)

$\Rightarrow$ $c = \max f|_A$ and $\max f|A$ is attained at one pt on A $\Leftarrow$ $\mathrm{Crit}(f)$ isolated

$|\deg \beta| < n \Rightarrow \operatorname{codim} B \geq 1$

$\Rightarrow$ B does generically not pass through max of $f|A$

$\Rightarrow$ $\max f|_{A\cap B} < \max f|_A$ $\triangleleft$



Not hard to turn into a a real pf

Now we are ready to re-prove

Prop (LS) $f: P \xrightarrow{C^2} \mathbb{R}$

$\mathrm{Crit}(f)$ isolated

$$\Rightarrow \boxed{|cv(f)| \geq cl(P)+1}$$

Pf. $k = cl(P)$, use intersection product $\cap$

$$\underbrace{\beta_1 \cap \dots \cap \beta_k}_{\substack{\text{necessarily } [pt] \\ \text{by PD}}} \neq 0 \quad \text{in } H_*(P)$$

$|\beta_i| < n$

not $\sim$ units

$$\Rightarrow \underbrace{[M]}_{\alpha_0}, \underbrace{[M]\cap\beta_1}_{\alpha_1 = \alpha_0 \cap \beta_1}, \underbrace{[M]\cap\beta_1\cap\beta_2}_{\alpha_2 = \alpha_1 \cap \beta_2}, \dots, \underbrace{[M]\cap\beta_1\cap\dots\cap\beta_k}_{\alpha_k = \alpha_{k-1}\cap\beta_k}$$

Prop

$$\Rightarrow \underbrace{\underbrace{c_{\alpha_0}(f)}_{\max f} > c_{\alpha_1}(f) > \dots > \underbrace{c_k(f)}_{\min f}}_{k+1 \text{ distinct values}}$$

◁

Refinement:

Q. What happens when we have an equality of two crit value selectors?

$$c_{\alpha\cap\beta}(f) \overset{\leq}{=} c_\alpha(f)$$

$<$ — if isolated $|\beta| < n$

Let K = crit set on the level

$f = c$, $c = c_{\alpha\cap\beta}(f) = c_\alpha(f)$

$\alpha, \beta \in H_*(P)$

Prop (Ex) The restriction of $PD(\beta) \in H^*(P)$ is $\neq 0$ in $H^*(K)$

Ex. $\beta = [P] \Rightarrow PD(\beta) = 1 \in H^0(P)$

$$H^0(P) \longrightarrow H^0(K)$$

$$1 \overset{\text{onto}}{\longmapsto} 1 \qquad \neq 0 \text{ always}$$

and $\alpha\cap\beta = \alpha$

$c_{\alpha\cap\beta}(f) = c_\alpha(f)$

$\triangleleft$

Cor $|\beta| < n$ ($\Leftrightarrow |PD(\beta)| > 0$)

$\Rightarrow$ card(K) = continuum (cannot be countable)

Cor $|cv(f)| \leq cl(P)$

$\Rightarrow$ continuum of crit pts

<u>Rmk</u> Need to be careful about the def of $H^*(K)$, for K could be a very bad set.

§20 Another Application: the Courant-Fischer Thm

Lecture 12 02/11

Setting

- $Q(x) = \langle Ax, x\rangle : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, $A^T = A$, a quadratic form
- $f = Q|_{S^n} : S^n \longrightarrow \mathbb{R}$.

  Also know as the Rayleigh-Ritz quotient:

$$f(x) = \frac{\langle Ax, x\rangle}{\langle x, x\rangle} : \mathbb{R}^{n+1} \setminus 0 \longrightarrow \mathbb{R}$$

  Here we think of f as $S^n \longrightarrow \mathbb{R}$
- $\lambda_0 \leq \ldots \leq \lambda_n$ the eigenvalues of Q (or A)

Thm (Courant - Fischer)

$$\lambda_k = \inf_{\{L_k\}} \max f|_{L_k},$$

E.g.
$\lambda_0 = \min f$
$\lambda_n = \max f$

where $L_k = S^n \cap \{k+1 \text{ dim lin space}\} \cong S^k$
$\uparrow Gr(n+1, k+1)$

Rmk A similar statement for Hermitian forms on $\mathbb{C}^{n+1}$ (Same pf)

$A = A^*$

E.g. $k=1$
$\{L_0\} \cong \mathbb{RP}^n$

Pf.

- $\text{crit}(f) \longleftrightarrow$ unit eigenvectors

  $cv(f) \longleftrightarrow$ eigenvalues

Pf Lagrange multipliers

Crit of Q on $\{g=c\}$ = sol of $\nabla Q = \lambda \nabla g$

$\nabla Q = 2Ax \qquad (A^T = A)$

$\nabla g = 2x \qquad g = \|x\|^2$

$\nabla Q = \lambda \nabla g$; $\underbrace{Ax = \lambda x}$, $\|x\|=1$

x is an eigenvalue

Then $f(x) = Q(x) = \langle Ax, x\rangle = \lambda$

Rmk unit eigenvectors ($=\text{Crit}(f)$) come in pairs $\pm x$ with the same crit value λ

- $\mathcal{F} = \{L_k\} \longleftrightarrow Gr(n+1, k+1)$

To check : closed under φ^t

HW (requires a pf)

∇Q is linear on $\mathbb{R}^{n+1}$

→ ∇f = proj of $\nabla Q|_{S^n}$ to S^n

→ Need to check that φ^t sends equators to equators min/max $\Rightarrow$ $\inf_{\mathcal{F}} \max f|_{L_k} = \mu_k$

are critical values

- Routine: $\mu_k \le \mu_{k+1}$

next page

Get $\mu_0 \le \dots \le \mu_n$: $n+1$

$\lambda_0 \le \dots \le \lambda_n$:

Need to check that every c.v. is obtained in this way

μ_i's are among λ_j's

clear when distinct — for a generic Q

then pass to a limit

$\Rightarrow$ $\mu_i = \lambda_j$

$\triangleleft$

<u>**Rmk.**</u> Rather than working on S^n could work on $\mathbb{RP}^n$: $f(-x) = f(x)$

$\to$ Then L_k projects to $\hat{L}_k \subset \mathbb{RP}^n$

linear proj subspace $\cong \mathbb{RP}^k$

$$[\hat{L}_k] = [\mathbb{RP}^k] = \alpha_k \in H_k(\mathbb{RP}^n)$$

$\mathcal{F}_k \subset \mathcal{F}_{\alpha_k}$ $\leftarrow$ as defined previously

all cycles in α_k

$$\lambda_k' = \inf_{\hat{L}_k \in \mathcal{F}_k} \max f|_{L_k} \ge \inf_{A \in \mathcal{F}_{\alpha_k}} \max f|_A = \lambda_k$$

$$\begin{matrix} \lambda_0' \le \dots \le \lambda_n' \\ \shortparallel \quad\quad\quad\quad \shortparallel \\ \lambda_0 \le \dots \le \lambda_n \end{matrix} \Bigg\}$$ both sequences formed by crit values (generically distinct)

Fixing the gap: all eigenvalues occur.

By continuity, assume λ_i's are distinct

Assume that one of the eigenvalues does not occur

$$\begin{array}{ccc} \lambda_i & < & \lambda_{i+1} \\ \shortparallel & & \shortparallel \\ \mu_i & = & \mu_{i+1} \end{array}$$

Two ways to reason:

1) As in refinement of LS:
   Then $Crit(Q)$ are not isolated

2) By passing to $\mathbb{RP}^n$
   Then $\{Q < \lambda_{i+1}\}$ is contractible to $\mathbb{RP}^i$ and
   $\{Q < \lambda_{i+1}\} \supset \mathbb{RP}^{i+1}$
   impossible $\rightarrow\leftarrow$

§21 History Digression: Lusternik-Schnirelmann Thm

The origins of LS theory is

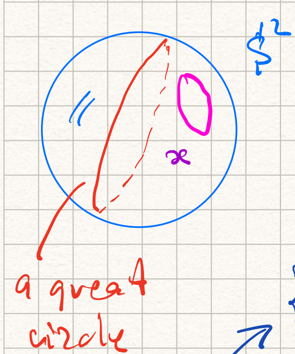
Thm (LS, 1929)
Any metric on S^2 has ≥ 3 simple closed geodesics

embedded (no self-intersections)

- very difficult
- Complete detailed pf: Ballmann 1978

Idea of the pf $\longleftrightarrow$ LS theory

- Λ = smooth embedded loops in S^2 (unoriented, unparametrized)



$S^1 \hookrightarrow S^2$

For instance
$\{\text{great circles}\} \subset \Lambda$
$\{S^2 \cap V\}$ ← 2-dim lin. subspace
$Gr(3,2) \cong \mathbb{RP}^2$

$\mathbb{RP}^2 \xhookrightarrow{\sim} \Lambda$ ← Homotopy equivalent (absolutely non-obvious)

- $L : \Lambda \longrightarrow \mathbb{R}$ the length functional

By def $\mathrm{Crit}(L)$ = simple closed geodesics

- Punch line : $\mathrm{cl}(\Lambda) = \mathrm{cl}(\mathbb{RP}^2) = 2$

If we could do LS on Λ
we would have

$$|\mathrm{Crit}(L)| \geq \mathrm{cl}(\Lambda) + 1 = 2 + 1 = 3$$

One can do an analogue of the LS theory on Λ (that was the starting pt for them), but this is very difficult.

$\triangleleft$

§22 Getting rid of compactness: the Palais-Smale condition

Goal: Find a good replacement for the condition that P is compact. Too restrictive.

Background assumption:

P = finite dimensional (but possibly not compact) (preferable)

on

Hilbert or Banach manifold (never compact)

$f: P \longrightarrow \mathbb{R}$ sufficiently smooth

Def f satisfies the Palais-Smale condition (PS) if

- every seq $\{x_i\} \subset P$ s.t.

$$\boxed{|f(x_i)| < C < \infty \quad \text{and} \quad df(x_i) \longrightarrow 0 \quad (\Leftrightarrow \nabla f(x_i) \longrightarrow 0)}$$

contains a convergent subseq

Rmk $df(x_i) \longrightarrow 0 \Leftrightarrow \nabla f(x_i) \longrightarrow 0$ when $\exists$ a R.m.: $\dim P < \infty$ or P Hilbert but not when P is Banach.

Rmk PS condition does not involve the flow of $-\nabla f$. One can have PS satisfied without the flow

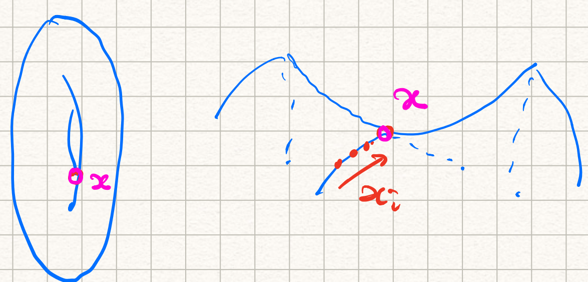
Important: PS condition easily breaks down

Don't need anything exotic

Construction: • Start with $f: \underset{\text{closed}}{P} \longrightarrow \mathbb{R}$

• $\mathring{P} = P \setminus \{\text{a crit pt}\}$

$\Rightarrow$ $f: \mathring{P} \longrightarrow \mathbb{R}$ does not satisfy the PS but everything else is fine (φ_t is defined for all t)

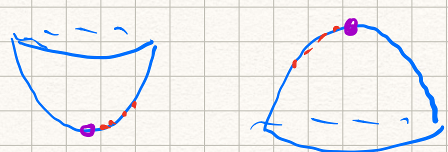


$|f(x_i)|$ bounded

$\nabla f(x_i) \to 0$

but x_i does not conv

$x_i \to x \notin \mathring{P}$



Upshot: Change of topology from P_a to P_b

Without PS $\not\Rightarrow$ existence of critical pts

Basic example – application – illustration

Assumptions:

- f bounded from below
- f satisfies PS
- The flow for anti-grad-like vector field is defined for all $t \geq 0$

$\Rightarrow$ f has a critical pt

actually min is attained

X s.t. $L_X f \leq 0$

$L_X f(x) = 0 \Leftrightarrow x \in \mathrm{Crit}(f)$

E.g $X = -\nabla f$.

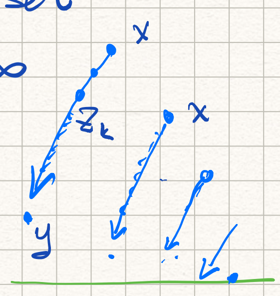
For many purposes as good as $-\nabla f$ but more flexible & does not need metric

Pf • Start with some x and set

$$z_k = \varphi^k(x_0) \qquad k \to \infty$$

PS + bounded from below

$\Rightarrow$ $z_k \to y \leftarrow$ crit pt



• To get min: take x_i s.t. $f(x_i) \to \inf f$. Apply this process and get y_i

$df(y_i) = 0$ & $f(y_i) \to \inf f$

$\overset{PS}{\Rightarrow}$ $y_i \to y$ & $f(y) = \inf f$ $\lhd$

Back to Minimax

Setting

with some care
can replace by
anti-grad-like

- P as above
- $f: P \longrightarrow \mathbb{R}$ satisfies PS
- the anti-grad flow φ^t of f is defined for all times $t \geq 0$

(*)

- $\mathcal{F}$ is closed under φ^t
- f on $\mathcal{F}$ is bounded from below:
$$\inf_{A \in \mathcal{F}} \max f|_A > -\infty$$

(**)

Thm (Minimax, II) $\quad c = \inf_{A \in \mathcal{F}} \max f|_A$ is a crit value

Ex. Take $\mathcal{F}$ = pts of P in the previous example.

The pf is very similar to the example

Pf

- Bounded from below $\Rightarrow$ $\underset{\mathbb{R}}{\underset{\Uparrow}{c}} > -\infty$

- Will prove: $\forall\, \varepsilon > 0\ \exists\, x$ s.t.

$c - \varepsilon < f(x) < c + \varepsilon$ & $|\nabla f(x)| < \varepsilon$

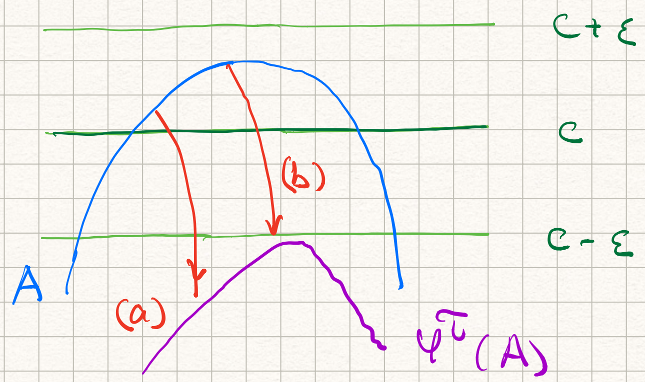
- Arguing by contradiction, assume that:

$$\|\nabla f(x)\| > \varepsilon \quad \text{when } c - \varepsilon \le f(x) < c + \varepsilon$$

for some $\varepsilon > 0$

- $\exists$ $A \in \mathcal{F}$ s.t.

$$\max f|_A < c + \varepsilon$$



- Claim $\underline{\tau = \frac{2}{\varepsilon}} \Rightarrow \max f|_{\varphi^\tau(A)} < c - \varepsilon$

Indeed, take $x \in A$

(a) if $f(\varphi^t(x)) < c - \varepsilon$ for some $t \in [0, \tau)$ we are done for $f(\varphi^t(x)) \searrow$

(b) if $c-\varepsilon < f(\varphi^t(x)) < c+\varepsilon$

for $\underline{t \in [0, \varepsilon]}$, we have

$$|\nabla f(\varphi^t(x))| > \varepsilon$$

$$\Rightarrow \quad f(\varphi^t(x)) \underbrace{< c+\varepsilon - \varepsilon^2 t}$$

$$\frac{d}{dt} f(\varphi^t(x)) \leq -\langle \nabla f, \nabla f \rangle$$

$$\leq -\varepsilon^2$$

$$\Rightarrow \quad f(\varphi^\tau(x)) < c+\varepsilon - 2\varepsilon$$

$$< c-\varepsilon$$

$$\Rightarrow \quad \max f|_{\varphi^\tau(A)} < c-\varepsilon$$

$\rightarrow\leftarrow$ $\triangleleft$

Part III: Geodesics and all that

Toward ∞-dim applications

Lecture 13
02/16

§23 Preliminaries

Review of relevant notions from Differential Geometry – minimalist

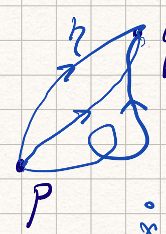
M^n = closed manifold with a fixed Riemannian metric

"Def 1" A geodesic $\gamma: [a,b] \longrightarrow M$ is a curve locally minimizing length: if t_0 & t_1 are close to each other

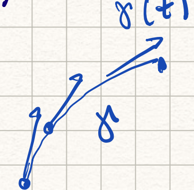
$$\ell(\gamma|_{[t_0,t_1]}) = \rho(\gamma(t_0), \gamma(t_1))$$

Here

$$\ell(\gamma) = \int_a^b \|\dot\gamma(t)\|\, dt$$

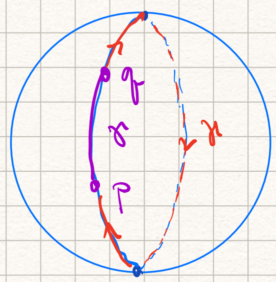


$$\rho(p,q) = \inf_{\substack{\eta \\ \text{connects } p \,\&\, q}} \ell(\eta)$$



Rmk
- $\ell(\gamma)$ is independent of parametrization or orientation
- ρ is a metric & M is complete

Ex.



A geodesic $p \overset{\gamma}{\rightsquigarrow} q$ need not minimize dist between p & q.

Rmk • If compact (complete as a R.M.)
$\Rightarrow$ any p & q can be connected by a geodesic (Hopf–Rinow Thm)

Ex. • $\exists$ a minizing geodesic
use Arzela–Ascoli, also will be clear later

In what follows we will be interested in two types of geodesics: connecting two fixed pts p & q and closed geodesics
$\gamma : S^1 \xrightarrow{C^\infty} M$ $\qquad S^1 = \mathbb{R}/\mathbb{Z} = [0,1]/\sim$
$\qquad \dot\gamma(0) = \dot\gamma(1)$

Set sufficiently smooth
$$\Lambda = \Lambda(p,q) = \{ \gamma : [0,1] \to M \mid \gamma(0) = p,\ \gamma(1) = q \}$$
$$\Lambda \qquad = \{ \gamma : S^1 \longrightarrow M \}$$

Def 2 A geodesic is a "critical pt" of
$$\ell : \Lambda \longrightarrow \mathbb{R}$$
Get geodesics connecting p & q, etc

Rmk Not quite satisfactory:

- ℓ is not smooth: $\| \ \|$ not smooth

  $x \mapsto y = |x|$ not smooth

  But smooth at immersions:

  $$\mathring{\gamma}(t) \neq 0 \quad \forall t$$

- ℓ is independent of parametrization

  $\Rightarrow$ huge sets of critical pts

  along with γ contains $\gamma \circ \varphi$, $\varphi : [0,1] \circlearrowleft$

Def 3 A geodesic is a critical pt of the energy functional:

$$E : \Lambda \longrightarrow M$$

Depends on parametrization $\longrightarrow$

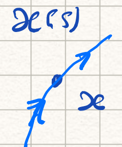
$$E(\gamma) = \frac{1}{2} \int_0^1 \| \mathring{\gamma}(t) \|^2 \, dt$$

ignore

An issue to deal with: what is a crit pt

Recall $f : P \longrightarrow \mathbb{R}$ $\quad x$ is a crit. pt if

$$df_x = 0 \iff \underbrace{(L_v f)(x)}_{df_x(v)} = 0 \quad \forall v$$



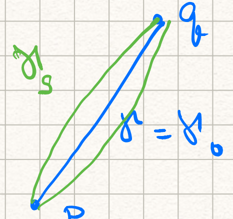
$$\iff \overbrace{\underbrace{\frac{d}{ds} f(x(s)) \Big|_{s=0}}_{=: (L_v f)(x)}}^{df_x(v)} = 0 \quad \forall v$$

$x(0) = x$

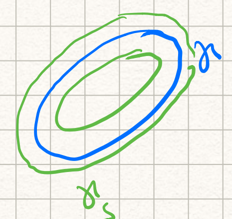
$\frac{d}{ds} x(0) = v$

Def $\gamma \in \mathrm{Crit}(E)$ if $\forall$ any variation γ_s of γ we have $\gamma_0 = \gamma$

$$\boxed{\frac{d}{ds} E(\gamma_s) \Big|_{s=0} = 0}$$



connecting p & q



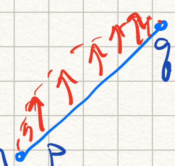
closed loops

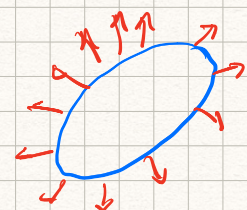
$$\frac{d\gamma_s}{ds}\Big|_{s=0}$$

Should think

$T_\gamma \Lambda$ = v.f. along γ

(vanishing at p & q)





$$v = \frac{d}{ds} \gamma(s)$$

$$(L_v E)(\gamma) = dE_\gamma(v) := \frac{d}{ds} E(\gamma(s))\Big|_{s=0}$$

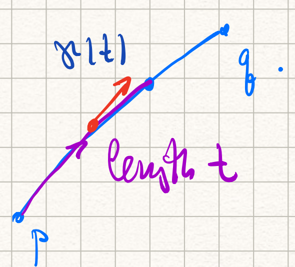
Fact: • geodesics in the sense of Def 3 also satisfy Def's 1 & 2: locally length minimizing and critical pts of L (unless $\gamma \equiv pt$)

• and $\dot{\gamma}(t) \neq 0$ for any t unless $\gamma \equiv pt$

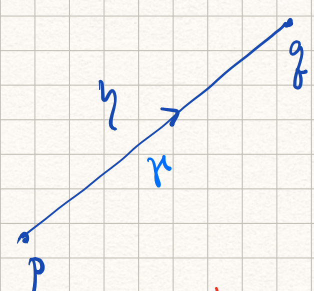
Fact: As a critical pt γ comes with a natural parametrization:
(E pics up a parametrization)
γ is parametrized proportionally to arc length to get from p to q or close up in time 1.

Arclength parametrization:

$$\ell(\eta|_{[0,t]}) = t \Rightarrow \|\dot\eta\| = 1$$



Likewise for closed geodesics



$\ell = \rho(p,q) = L(\eta)$
η parametrized by arc length: $[0,\ell]$

$$\gamma(t) = \eta(\ell \cdot t)$$

$$\Rightarrow \boxed{\|\dot\gamma(t)\| = \ell}$$

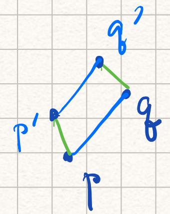
$$\Rightarrow \boxed{E(\gamma) = \int_0^1 \|\dot\gamma(t)\|^2 dt = \ell^2}$$

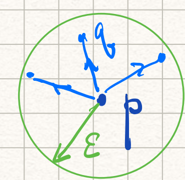
For any γ parametrized by $[0,1]$ $\sim$ arclength

Fact: • $\exists\ \varepsilon > 0$ s.t. $\forall\ p, q \in M$

(injectivity radius) | with $\rho(p,q) < \varepsilon$
$\exists$ unique <u>minimal</u> geodesic $\gamma_{p,q}$
connecting p to q : $\underline{\ell(\gamma_{p,q}) = \rho(p,q)}$

• Moreover, $\gamma_{p,q}$ depends smoothly on $(p,q) \in M \times M$





Rmk. γ is a geodesic (in the sense of Def 3)
$\iff \nabla_{\dot\gamma} \dot\gamma = 0 \leftarrow$ acceleration
This is how geodesics are usually defined in R.G.

Rmk In the next two sections we follow Bott's "Lectures on M.T."

§24 Closed Geodesics, I: Cartan's Thm

M^n closed Riemannian manifold

Recall: free homotopy classes $S^1 \xrightarrow{C^\infty} M$

$= [S^1, M]$

$=$ conjugacy classes in $\pi_1(M)$

$\Rightarrow \quad \pi_1(M) \neq 1 \iff \exists$ non-contractible loops

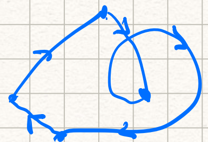
Thm (Cartan) $\pi_1(M) \neq 1$
for any non-trivial free homotopy class $[\alpha] \in [S^1, M]$ $\exists$ a closed geodesic in the class $[\alpha]$.

Idea: take the shortest loop in α as this geodesic or minimize E over all loops in α

Minor adjustment to the setting:

$\Lambda =$ continuous piece-wise smooth loops

Note: E, ℓ etc are still defined ...



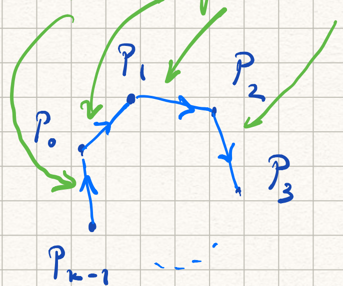
Recall: $[S^1, \Lambda] \longleftrightarrow \pi_0(\Lambda)$ = connected components in Λ

Λ_α = the connected component $\ni [\alpha] \neq 1$

Pf

- Def Geodesic polygons = broken geodesics

short geodesic segments $\quad \varepsilon <$ inj. rad (with short edges)



collection of pts

$\vec{p} = (p_0, \dots, p_{k-1})$ s.t. inj. radius

$\rho(p_i, p_{i+1}) \leq \varepsilon$

$(p_k = p_0)$

$\vec{p}$ gives rise to a well-defined element in Λ, "broken geodesic" (terminology) ρ

- Every loop α can be approximated by broken geodesic (with large k) in the same free homotopy class.

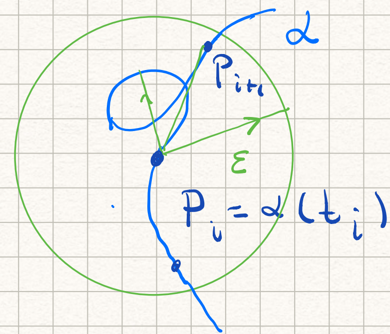
$\to$ Partition $[0,1] = S^1$ into

$$0 = t_0 < t_1 < \dots < t_{k-1} < t_k = 1 = 0$$

$$\begin{array}{cccc} \alpha(t_0) & \alpha(t_1) & \alpha(t_{k-1}) & \alpha(t_k) \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ p_0 & p_1 & p_{k-1} & p_0 = p_k \end{array}$$

so that $\quad \rho(p_i, \alpha(t)) < \varepsilon \quad \forall t \in [t_i, t_{i+1}]$

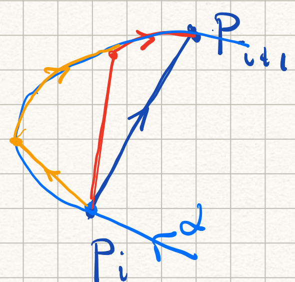
$p_i = \alpha(t_i)$

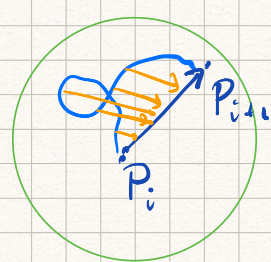


$\Rightarrow$ broken geodesic $\vec{P} = (P_0, \ldots, P_{k-1}) = \xi$
C^0-approximating α

parametrized by $[0,1]$ ~ arc length

<u>Homotopy from α to ξ:</u>





Not clear how to do this explicitly

<u>Cor</u>: Every Λ_α contains a broken <u>geodesic</u>

<u>Next</u>: $\quad P_k \subset \underbrace{M \times \ldots \times M}_{k}$ s.t.

$$\mathcal{E}(\vec{P}) = \rho(P_0, P_1)^2 + \ldots + \rho(P_{k-1}, P_0)^2 \leq \varepsilon^2$$

Important:

(1) $\vec{p} \in P_k \Rightarrow \rho(p_i, p_{i+1}) < \varepsilon$

$\Rightarrow$ get a broken geodesic ξ $\forall \vec{p} \in P_k$

$\Rightarrow P_k \subset \Lambda$

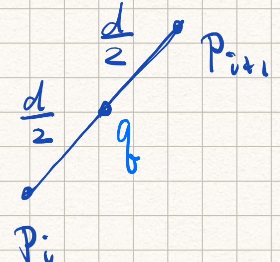
(2) ξ is not necessarily short:

any α can be approximated by an element in P_k ← large

Pf. Approximated α for some k & δ (in place of ε)

$$\rho(p_i, p_{i+1}) < \delta$$

and start subdividing



$$\rho(p_i, p_{i+1}) = d < \delta$$

$$\rho(p_i, q) = \rho(q, p_{i+1}) = \frac{d}{2}$$

$\rho(p_i, p_{i+1})^2 = d^2$ gets replaced by

$$\rho(p_i, q)^2 + \rho(q, p_{i+1})^2 = \left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2 = \frac{d^2}{2}$$

Can make the total sum $\Sigma \rho(p_i, p_{i+1})^2$ arbitrarily small

Lecture 14
02/28

Instructive: relation between $\mathcal{E}(\vec{p})$ & $E(\xi)$

$$\mathcal{E}(\vec{p}) = \sum \rho(p_i, p_{i+1})^2$$

- depends on partition and parametrization of ξ
- can be really small

E.g. $\ell(\xi) = \ell$, $\rho(p_i, p_{i+1}) = \frac{\ell}{k}$

$$\Rightarrow \mathcal{E}(\vec{p}) = k \cdot \left(\frac{\ell}{k}\right)^2 = \underline{\frac{1}{k} \ell(\xi)^2} \approx \frac{1}{k} E(\xi)$$

if arclength

Overall depends on where p_i's are:

$$\boxed{E(\xi) = \sum_{i=0}^{k-1} \frac{\rho(p_i, p_{i+1})^2}{t_{i+1} - t_i}}$$

$\leftarrow$ More generally

when parametrized by $[0,1]$, arclength on each $[t_{i-1}, t_i]$

Rmk • P_k approximates $\{E<a\}$ in Λ very well (homotopy eq) when k is large

Upshot $P_{k,\alpha} = \Lambda_\alpha \cap P_k \neq \emptyset$ (k large)

smooth manifold with boundary : $\mathcal{E} = \varepsilon$ ← generic

$\Rightarrow$ $P_{k,\alpha}$ = union of some connected components of P_k

Idea: Replace E by $\mathcal{E}$ on P_k

– soon see $\underbrace{\mathrm{Crit}(\mathcal{E}) = \mathrm{Crit}(E)}_{\text{closed geodesics}}$

But not crucial.

Then: minimize $\mathcal{E}$ on $P_{k,\alpha}$

Rmk • $\mathcal{E}|_{\partial P_k} = \max \mathcal{E}|_{P_k} = \varepsilon^2$

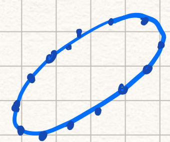
• $\mathcal{E} \geq 0$

• $\min \mathcal{E}|_{P_{k,\alpha \neq 1}} > 0$ ← don't need this

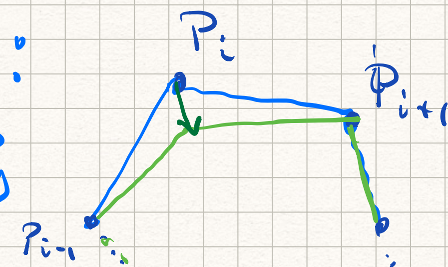
Lemma $\gamma \in \mathrm{Crit}(\mathcal{E}) \Rightarrow \gamma$ is a closed geodesic
$(\Leftarrow)$

$\Downarrow$

Carton's Thm



Idea: changes $\mathcal{E}(\vec{p})$



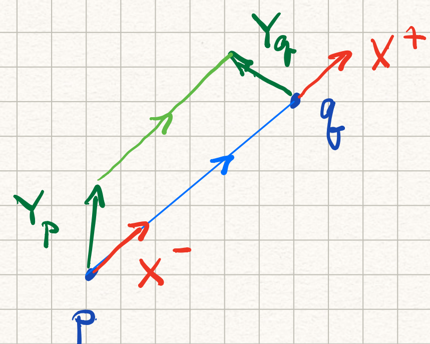
Pf of the lemma

Step 1 (p,q) near diagonal in $M \times M$:

$$\rho(p,q) \leqslant \varepsilon < \text{inj. radius}$$

How does $\rho(p,q)$ changes when we move p & q?

$X^{\pm}$ = unit tangents



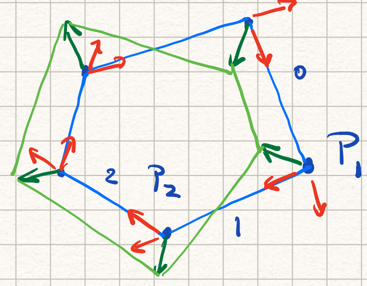
$$\rho^2 = \rho(p,q)$$

$$T_{(p,q)} M \times M = T_p M \times T_q M$$
$$\qquad\qquad\qquad\quad\ \ \, \Downarrow \qquad\ \ \Downarrow$$
$$\qquad\qquad\qquad\quad\ \ Y_p \qquad\ Y_q$$

$$L_{(Y_p, Y_q)} \rho^2 = 2(\langle X^+, Y_q \rangle - \langle X^-, Y_p \rangle) \cdot \rho \quad (*)$$

Put the pf of $(*)$ aside for now and finish the pf of the lemma

Step 2



At every p_i we have X_i^+ & X_{i-1}^- and Y_i

Summing up $(*)$ for each segment we get

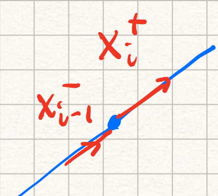
$$L_{(Y_0,\dots,Y_{k-1})} \mathcal{E}(\vec{p})$$

$$= 2\sum_{i=0}^{k-1} \left(\langle Y_{i+1}, X_{i+1}^+\rangle - \langle Y_i, X_i^-\rangle\right)\rho_i$$

$$= 2\sum_{i=1}^{k} \langle Y_i, X_i^+\rho_i - X_{i-1}^-\rho_{i-1}\rangle$$

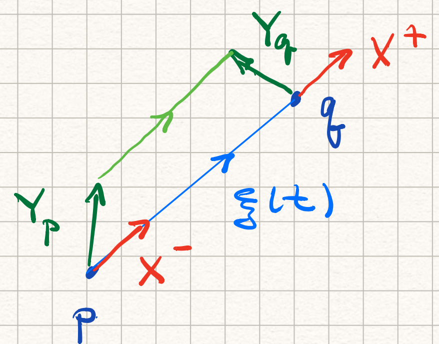
$\Rightarrow \quad \vec{p} \in \mathrm{Crit}(\mathcal{E}) \iff X_i^+\rho_i = X_i^-\rho_{i-1} \quad \forall i$

$\iff$ no corner at at p_i and parametrizations of the edges match.



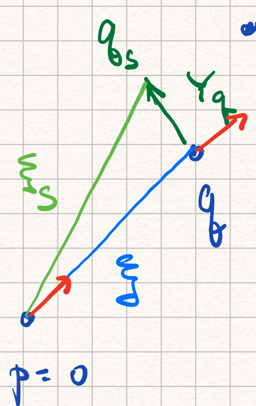
Remains to prove $(*)$

Pf of (*) in $\mathbb{R}^n$



Similar in general but need a bit more D.G.
See Milnor's book p. 71.

Can assume one of the vectors Y_p or $Y_q = 0$ (by additivity). Say $Y_p = 0$



- $p = 0 \Rightarrow \xi(t) = qt$

$$X^+ = X^- = q/\|q\|$$

$$q_s = q + sY_q$$

$$\rho^2(s) = \|q_s\|^2 = \|q + sY_q\|^2$$

$$\left.\frac{d}{ds}\rho^2(s)\right|_{s=0} = \left.\frac{d}{ds}\|q + sY_q\|^2\right|_{s=0}$$

$$= 2\langle q, Y_q\rangle = 2\langle \underbrace{\frac{q}{\|q\|}}_{X^+}, Y_q\rangle \underbrace{\|q\|}_{\rho}$$

$$= 2\langle X^+, Y_q\rangle \cdot \rho$$

$\triangleleft$

Rmk What happens if $[\alpha]=1$
The argument goes through but gives a <u>point geodesic</u>

Rmk Same argument shows the existence of a minimizing geodesic between any two points.

Rmk Could have worked with E on P_m, just need to be a bit more careful near ∂P_m

§25 Closed Geodesics, II : Lusternik - Fet Thm

But what if $\pi_1(M) = 1$?

Thm (Lusternik - Fet)

$\pi_1(M) = 1 \implies \exists$ a non-const closed geodesic

The argument builds on the machinery we developed in the prev. section

Note: Now Λ is connected

Pf

1) First need a bit on top of Λ.

Evaluation map

$$\left.\begin{aligned} e : \Lambda &\longrightarrow M \\ \gamma &\longmapsto \gamma(0) \end{aligned}\right\} \text{ Serre fibration}$$

$$\begin{tikzcd} \Omega \arrow[r, "j"] & \Lambda \arrow[d, shift left, "e"] \\ & M \arrow[u, shift left, "s"] \end{tikzcd}$$

marks loops

$\Omega = e^{-1}(p)$

$= \text{loops throug } p$

$= \{\gamma \mid \gamma(0) = p\}$

a section $p \longmapsto$ constant loop $\gamma(t) \equiv p$

$e s = id : M \hookrightarrow \Lambda$

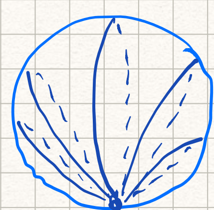
Fibration $\Rightarrow$ long exact seq in homotopy gps

$$\begin{tikzcd}
\pi_{i+1}(\Omega) \arrow[r, "0"] & \pi_i(\Omega) \arrow[r, "j_*"] & \pi_i(\Lambda) \arrow[r, "e_*"] & \pi_i(M) \arrow[l, bend left, "s_*"] \arrow[r, "0"] & \pi_{i-1}(\Omega) \arrow[r, "j_*"] & \cdots
\end{tikzcd}$$

$es = id \Rightarrow e_* s_* = id$

$\Rightarrow \pi_i(\Lambda) = \pi_i(\Omega) \oplus \pi_i(M)$

Also $\pi_i(\Omega) = \pi_{i+1}(M)$



$\longleftrightarrow$ loop in Ω

$A \in \pi_2(M)$ $\qquad$ $\in \pi_1(\Omega)$

$i \geq 1$

$$\Rightarrow \boxed{\pi_i(\Lambda) = \pi_{i+1}(M) \oplus \pi_i(M)}$$

$\Rightarrow \pi_i(\Lambda) \neq 0$ for some $\underline{i \geq 1}$

Take min i so that $H_i(M; \mathbb{Z}) \neq 0$

$$\begin{array}{c} H_i(M;\mathbb{Z}) \neq 0 \\ \parallel \\ \pi_i(M) \neq 0 \end{array}$$

$\boxed{\begin{array}{l} \underline{\text{Moreover}}\ \exists\, i \geq 1 \text{ such that} \\ \quad \pi_i(\Lambda) \neq 0\ \&\ \pi_i(M) = 0 \end{array}}$ $(*)$

2) Need state exactly how P_k approximates Λ:

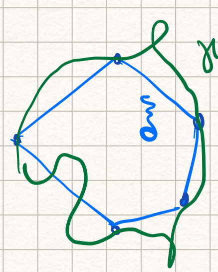
<u>Lemma</u> For any i_0 $\exists$ k_0 such that

$$\pi_i(P_k) = \pi_i(\Lambda) \quad \text{for} \quad \begin{array}{l} i \leq i_0 \\ k \geq k_0 \end{array}$$

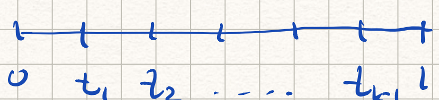
<u>Pf</u>

Here we have
$$\begin{tikzcd} P_k \arrow[r, hook] & \Lambda \\ & S^i \arrow[u, "A", "\sim"'] \arrow[ul, dashed, "\text{need}"] \end{tikzcd}$$

- The pt $A(\xi)$, $\xi \in S^i$ is a loop, Need to contract it continuously (in ξ) to a loop in P_k

- 

We have constructed such a contraction for an individual γ Depending <u>only</u> on a sufficiently fine partition



$$t_j = \frac{j}{k}$$

The value k is determined by

$$\max_{t \in S} \|\dot{\gamma}(t)\| = \|\dot{\gamma}\|_{C^0}$$

Now it suffices to take the same k for all $A(s)$:

$$\max_{s \in \xi^i} \left\| \frac{d}{dt} A(s) \right\|_{C^0} \rightsquigarrow k$$

Gap

<u>Rmk</u> • In this construction and the original one, we did not specify how to parametrize ξ.

• Can parametrize $\sim$ arclength

$$\Lambda \underset{\sim}{\Longleftrightarrow} \{\text{loops parametrized} \sim \text{arclength}\}$$

• Deform γ to ξ with whatever parametrization, reparametrize

3) Punchline

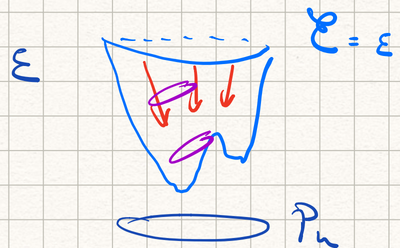
- Take $i \geq 1$ so that $\pi_i(\Lambda) \neq 0$ and $\pi_i(M) = 0$ By (*) in 1)

Take k so large that

$$\pi_i(P_k) \neq 0 \leftarrow \text{exists by 2)}$$

- $\mathcal{F} = \{A : S^i \to P_k \hookrightarrow \Lambda\}$, $[A] = \alpha \neq 0$ in $\pi_i(P_k)$

Closed under the positive anti-grad flow for $\mathcal{E}$ on P_n



$\mathcal{E} > 0$

P_n has boundary but it does not matter!
$\mathcal{E}|\partial P_n = \max$

- Apply the Minimax Principle for $\mathcal{F}$

Get a critical value attained on some γ:

$$c = \mathcal{E}(\gamma) = \inf_{A \in \mathcal{F}} \max \mathcal{E}\big|_A \geq 0$$

- Need to know that $c > 0$

$\Rightarrow$ γ is non-trivial.

But if not, A gets contracted into

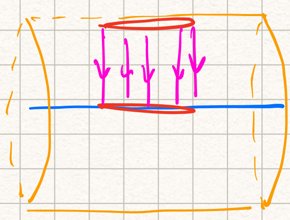
const loops $\longrightarrow$ $M \subset P_n \subset \Lambda$ by the anti-grad flow

$\qquad\qquad\qquad\quad \overset{\shortparallel}{\mathcal{E}=0}$

$\Rightarrow A$ is represented in M

But $[A] \neq 0$ in $\pi_i(P_n)$

and $\pi_i(M) = 0$



$\longrightarrow\longleftarrow$

$\triangleleft$

§26 Infinite-dim approach

Lecture 15
02/23

Very briefly

Recall

Sobolev space (not cohomology) of H^1-functions on S^1: $H^1(S^1, \mathbb{R})$

Abs. continuous functions f such that f' is L^2

Hilbert space with the inner product

$$\langle f, g \rangle_{H^1} = \langle f, g \rangle_{L^2} + \langle f', g' \rangle_{L^2}$$

Rmk Can also define as

could be $\bar{f}\bar{g}$

$$\langle f, g \rangle'_{H^1} = f(0)g(0) + \langle f', g' \rangle_{L^2}$$

Different product but "equivalent"

Likewise: f can take values in $\mathbb{R}^n$

Def Λ = H^1-maps $\gamma : S^1 \longrightarrow M$

closed Riemannian manifold

γ is abs. continuous

$\dot{\gamma}$ is L^2:

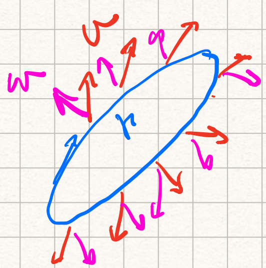
$$\int_{S^1} \|\dot{\gamma}(t)\|^2 dt < \infty$$

Fact Λ is a Hilbert manifold (modeled on $H^1(S^1 \to \mathbb{R}^n)$)

Tangent space:

$$T_\gamma \Lambda = H^1\text{- v.f. along } \gamma$$

$$= \{ v \mid v \text{ abs cont } \& \ \nabla_{\dot\gamma} v \text{ is } L^2 \}$$



Hilbert space

$$\langle v, w \rangle_{H^1} = \langle v, w \rangle_{L^2} + \langle \nabla_{\dot\gamma} v, \nabla_{\dot\gamma} w \rangle_{L^2}$$

Ref Klingenberg "Lectures on closed geodesics"
Chap 1 & 2 (ex caution)

Important

$E : \Lambda \longrightarrow \mathbb{R}$ is defined & smooth

$$E(\gamma) = \frac{1}{2} \int_0^1 \| \dot\gamma \|^2 \, dt$$

Elements of Λ have just enough smoothness

<u>Key pt</u>: one can do basic LS for E on Λ

<u>Steps</u> (without pfs)

- Anti-grad flow for $\nabla_{H^1} E$ is defined on Λ for $t \geq 0$
- E satisfies the PS condition

$\Downarrow$

$\Rightarrow$ min E is attained on every connected component
$\Rightarrow$ Corotan's theorem

and

$\Rightarrow$ minimax holds
$\Rightarrow$ Lusternik-Fet thm

<u>Important rmk</u>

- E would still be defined if we required more smoothness for elements of Λ $H^{k>1}$-Sobolev space
- But PS condition would then break down

Underlying Principle:

In all problems of this type require as little smoothness as necessary to have the function defined.

E.g. For the action functional one would have to work with $H^{1/2}$.

Important underlying features of E that make things work

- E is bounded from below
- At every $\gamma \in \mathrm{Crit}(E)$ = closed geodesic $d^2_\gamma E$ has
  - → finite # of neg squares (finite index)
  - → finite nullity

$V \subset T_\gamma E$ s.t. $d^2_\gamma E|_V \leq 0 \Rightarrow \dim V < \infty$

Not always true: e.g. fails for the action functional on $H^{1/2}$

⇒ Huge difficulties

Rmk (Bumpy Metrics)

- In general, E or $\mathcal{E}$ are <u>not</u> Morse

- A closed geodesic γ is non-deg if its non-deg as a periodic orbit of the geodesic flow on its level

$\iff$ $d^2_\gamma E$ is non-deg in the Morse-Bott sense

- For a C^∞-generic metric all closed geodesics are non-deg ($\iff$ E or $\mathcal{E}$ is Morse)
Such metrics are called Bumpy Metrics

Abraham (1970) – without complete pf
Anosov (1983) – complete pf

Application: $\operatorname{rk} H_k(\Lambda) \xrightarrow[k\to\infty]{} \infty$ usually satisfied

$\Rightarrow$ a <u>bumpy</u> metric has ∞ many prime geodesics

can be eliminated
(Gromoll-Meyer c73?)

Not true for S^n. Not known if every metric on S^3 has ∞ many prime geodesics.

§ 26 A quick glance at other calculus of variations questions

Recap: what we have learned

M a R.m.: $\langle\,,\,\rangle$ = R.m.

$L : TM \longrightarrow \mathbb{R}$ ← a Lagrangian

$$(q, v) = \frac{1}{2}\langle v, v\rangle_q$$

Fix $T > 0$

$$E(\gamma) = \int_0^T L(\gamma, \dot{\gamma})\, dt$$

$E : \Lambda \longrightarrow \mathbb{R}$, $\Lambda = \{ S = \mathbb{R}/T\mathbb{Z} \longrightarrow M \}$

Crit(E) = closed geodesics parametrized by $[0, T]$ ~ arc length

solutions of:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \leftarrow \text{acceleration}$$

Governs the motion of a unit mass particle confined to M

Cartan & Lusternik - Fet

$\Rightarrow$ Existence

Other types of Lagrangians

(1) $\quad L(q,v) = \frac{1}{2}\langle v, v\rangle_q - V(q)$

Potential energy $\quad V: M \longrightarrow \mathbb{R} \quad$ or $\quad V: M \times S^1 \longrightarrow \mathbb{R}$, $\quad S^1 = \mathbb{R}/T\mathbb{Z}$

$$\begin{cases} \text{Same } \Lambda \\ \mathcal{L}(\gamma) = \int_0^T L(\gamma, \dot{\gamma})\,dt \end{cases}$$

$\mathrm{Crit}(\mathcal{L}) =$

$$\gamma: S^1 \longrightarrow M$$
$$\nabla_{\dot{\gamma}}\dot{\gamma} = -\nabla V \leftarrow \text{Force}$$

Governs the motion of a unit mass particle on M, ext. force $F = -\nabla V$.

$$\frac{1}{2}\langle v, v\rangle + \langle v; g^{-1}(\alpha)\rangle - V$$

(2) $\quad L(q,v) = \cancel{\frac{1}{2}\langle v-\alpha, v-\alpha\rangle_q - \nabla V}$

V the same : potential energy

$\alpha \in \Omega^1(M)$: magnetic potential

$\mathrm{Crit}(\mathcal{L}) = \quad \gamma: S^1 \longrightarrow \mathbb{R}$

$$\nabla_{\dot{\gamma}}\dot{\gamma} = -\nabla V + \underbrace{F(\gamma, \dot{\gamma})}_{\text{Lorentz Force}}$$

$F(q,v) := g^{-1} i_v d\alpha$

Rmk: only $\omega = d\alpha$ matters

$$\begin{array}{rcccccc} F: & TM & \xrightarrow{i_v} & T^*M & \xrightarrow{g^{-1}} & TM \\ & (v,q) & \longmapsto & i_v d\alpha & \longmapsto & F \end{array}$$

$d\alpha =$ magnetic field

<u>Ex</u> a) B = magnetic field in $\mathbb{R}^3$

$$\omega = i_B \underbrace{(dx \wedge dy \wedge dz)}_{dVol} \leftarrow \text{closed}$$

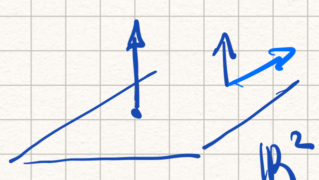
$\Uparrow$ Maxwell

$\omega = d\alpha$

$$F(q,v) = B \times v \quad \text{at } q$$

b) $\omega = B\, dx \wedge dy$ in $\mathbb{R}^2$, $B : \mathbb{R}^2 \to \mathbb{R}$
$= d\alpha$

$$F(q,v) = B \cdot Jv \quad \text{at } q$$



Magnetic field $\perp \mathbb{R}^2$

$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Governs the motion of a unit mass, unit charge particle on M, ext. force $F = -\nabla V$, and magnetic field $d\alpha$.

<u>Ex.</u> Determine the flow on $\mathbb{R}^2$ when B = const.

3) More generally:

$$L : TM \longrightarrow \mathbb{R} \qquad \text{s.t.}$$

or $$L : TM \times \underbrace{S^1}_{\mathbb{R}/T\mathbb{Z}} \longrightarrow \mathbb{R}$$

E.g.
$L(q,v) \geq A\|v\| - B$

a) fiberwise convex
b) quadratic or superlinear growth as $v \longrightarrow \infty$

exact conditions vary (Tonelli Lagr)

Rmk. When L is time-dependent and T-periodic in time,
$\gamma : S^1 = \mathbb{R}/T\mathbb{Z} \longrightarrow M$ some T.
Can set $T=1$, but don't want to yet.

Goal: Existence of Crit $(\mathcal{L})$

$$\mathcal{L}(\gamma) = \int_0^T L(\gamma, \dot{\gamma})\,dt$$

Solutions $\gamma : S^1 \longrightarrow M$ of the Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial v}(\gamma, \dot{\gamma}) = \frac{\partial L}{\partial q}(\gamma, \dot{\gamma})$$

Thm (Benci, 86) ⟵ sample result

- $\pi_1(M) \neq 1 \Rightarrow$ sol in every $\neq$ trivial free homotopy class
- $\pi_1(M) = 1 \Rightarrow$ inf many contractible T-periodic sol's γ_k

Moreover: $\underbrace{\mathcal{L}(\gamma_k) \longrightarrow \infty}$

this is what makes it more interesting
In some sense γ_k's are distinct
(loose)

Rmk: → no longer have the notion of trivial (const) vs non-trivial (non-const) sol

→ For geodesic flow get reparametrization of the same iterated geodesic:

$\gamma_1, \gamma_2, \gamma_3, \ldots$

$\gamma_2(t) = \gamma_1(2t), \ \gamma_3(t) = \gamma_1(3t), \ldots$

On the pf

→ Fix a background R.M.

→ The H^1-anti-grad-flow for $\mathcal{L}$ is defined for all $t \geq 0$

→ P.S. condition is satisfied

- $\pi_1(M) \neq 1$: Exactly as for geodesics in § 25: minimize $\mathcal{L}$ on $\Lambda_{\alpha \neq 1}$

- $\pi_1(M) = 1$. To get just one solution could again just minimize

To get ∞ many, find a seq of classes

$$\beta_k \in H_{m_k}(\Lambda; \mathbb{R}) \qquad m_k \to \infty$$

$\overset{PS}{\Rightarrow}$ minimax orbits γ_k

Moreover β_k's are not supported in the lower energy regions of Λ:

energy for some R.M.

$$\beta_k \notin \operatorname{im}\left(H_*\{E < e_k\} \to H_*(\Lambda)\right)$$

for some $e_k \to \infty$

$\Rightarrow \mathcal{L}(\gamma_k) \to \infty$ $\triangleleft$

Rmk

A quick digression: A fixed energy problem

- So far we looked for periodic orbits of a fixed period
  But one can look for periodic orbits with "fixed energy" - dual problem (and much harder)

- Assumption:
  $L =$ independent of time

set $H(q,v) = \frac{\partial L}{\partial v} \cdot v - L$

Then E-L $\Rightarrow \frac{d}{dt}\left(H(\gamma, \dot{\gamma})\right) = 0$

$\Rightarrow$ Trajectories on the same level of H (compact) $\leftarrow$ Ex

Q! Find per solutions of E-L eq of unspecified period on a fixed level $E = c$ or some specified levels

Much harder

Further ref + names:
Rabinowitz, Benci, Figalli, Mañé
Abbondandolo, ...

§28 Geodesics connecting two points
— Morse Theory

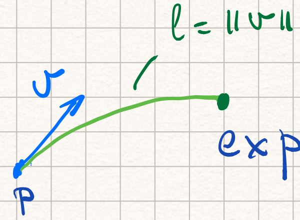
Lecture 16
02/25

- Very briefly
- Skipping most of R.G.
- Following Milnor — a great intro to R.G.

• A Brief Review: exp map & conj. points

$$\exp : T_pM \longrightarrow M$$
$$v \longmapsto \gamma_v(1)$$

γ_v = geodesic starting at p with $\dot\gamma_v(0) = v$



$\exp(v) = \gamma_v(1)$

M compact $\Rightarrow$ M is complete
$\Rightarrow$ exp is defined ($\gamma_v : \mathbb{R} \longrightarrow M$)
and onto

Def • q is conjugate to p (along $\gamma = \gamma_v$) if q is a critical value of exp:
$\ker D\exp \neq 0$ at v : $q = \gamma_v(1)$

• multiplicity of q is
$\dim \ker D\exp|_v$

Ex. • $\mathbb{T}^n = \mathbb{R}^n/\Gamma \leftarrow$ lattice e.g. $\Gamma = \mathbb{Z}^n$

$\Gamma = e_1\mathbb{Z} + \dots + e_n\mathbb{Z}$

a basis

$\Rightarrow$ no conj. pts   covering map $\mathbb{R}^n \to \mathbb{T}^n$

• More generally

sectional curv ≤ 0

$\Rightarrow$ no conj. pts

• S^n with round metric

$\to$ $q = -p$ conj to p

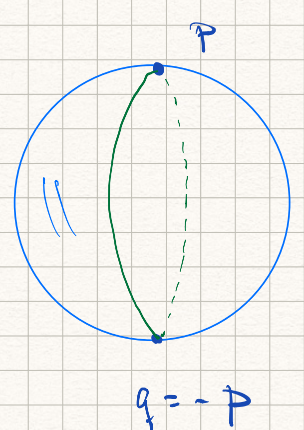
along half-meridian

mult $= n-1$

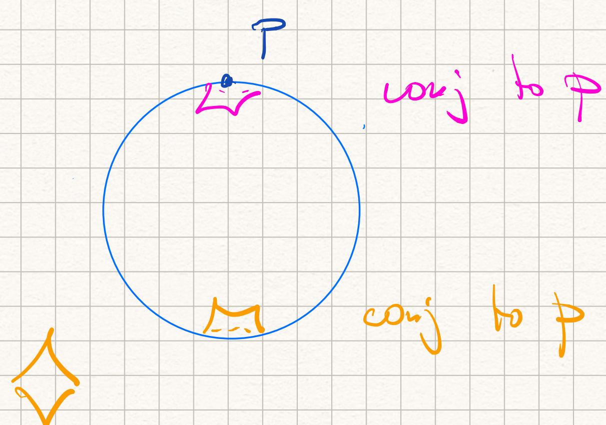
$\to$ p is conj to itself $q = p$

along whole meridian

mult $= n-1$



round metric



a small perturbation of a round metric

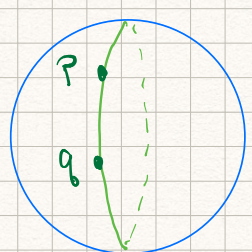
Sard's $\Longrightarrow$

The set of pts conj to p has zero measure in M

<u>Rmk</u> The set of pts conj to p need not be closed because T_pM is not compact

<u>Cor</u> For almost all $(p,q) \in M \times M$ p & q are not conjugate

<u>E.g.</u> In the round S^2 take any pair so that $q \neq \pm p$



- Morse Theory set up : Ω and E

→ $\Omega = \Lambda(M, p, q)$ — p.w. C^∞

$= \{\gamma : [0,1] \longrightarrow M \mid \gamma(0) = p, \gamma(1) = q\}$

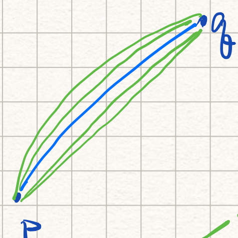
a better notation

→ $E : \Omega \longrightarrow \mathbb{R}$ the energy

$$E(\gamma) = \int_0^1 \|\dot\gamma(t)\|^2 dt$$

→ γ is a critical pt of E if

$$\frac{d}{dt} E(\gamma_s)\Big|_{s=0} = 0 \quad \forall \text{ any variation } \gamma_s \text{ of } \gamma$$



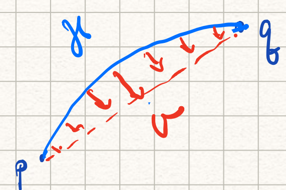
$\gamma_s(t) : \underset{s}{(-\varepsilon, \varepsilon)} \times \underset{t}{[0,1]} \longrightarrow M$

Exactly as for closed geodesics

Fact : $\gamma \in Crit(E) \Leftrightarrow \gamma$ is a closed geodesic connecting p to q and parametrized by $[0,1]$ ~ arc length

Tangent space

$T_\gamma \Omega =$ v.f. along γ vanishing at p & q



Think $v = \left.\dfrac{\partial \gamma_s}{\partial s}(t)\right|_{s=0}$

Every v comes from γ_s

$\gamma_s(t) = \exp_{\gamma(t)}(sv)$

Rmk. We do not make Ω into an inf dim manifold ... This is just an interpretation.

The Hessian

In finite dim's: $f : P \to \mathbb{R}$.

$v, w \in T_x P$, $\quad x \in \mathrm{Crit}(f)$

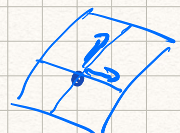
Consider $\underset{s}{(-\varepsilon, \varepsilon)} \times \underset{s'}{(-\varepsilon', \varepsilon')} \xrightarrow{u} P$, $\quad u(s, s')$

s.t. $u(0,0) = x$

$$\left.\frac{\partial u}{\partial s}\right|_{\substack{s=0\\ s'=0}} = v, \quad \left.\frac{\partial u}{\partial s'}\right|_{\substack{s=0\\ s'=0}} = w$$

$$d_x^2 f(v, w) = \left.\frac{\partial^2}{\partial s \partial s'} f(u(s, s'))\right|_{\substack{s=0\\ s'=0}}$$

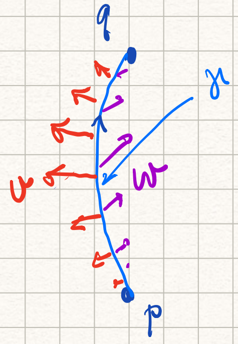
Ind of u but only v, w



$u(s, s')$ Pr: Taylor exp.

Likewise in inf dimensions

$v, w \in T_\gamma \Omega$ v.f. along γ



Take $u(s, s', t)$ s.t.

$u(0,0,t) = \gamma(t)$

$$\left.\frac{\partial u}{\partial s}\right|_{\substack{s=0\\ s'=0}} = v(t)$$

$$\left.\frac{\partial u}{\partial s'}\right|_{\substack{s=0\\ s'=0}} = w(t)$$

E.g. $u(s, s', t) = \exp_{\gamma(t)}(sv + s'w)$

Set

$$d^2_\gamma E(v, w) := \left.\frac{\partial^2 E(u(s,s',\cdot))}{\partial s \partial s'}\right|_{\substack{s=0\\ s'=0}}$$

Ind of the variation
and is completely det by v, w.
when $\gamma \in \mathrm{Crit}(E)$, symmetric

- $\mathrm{Ker}\, d^2_\gamma E = \{ v \mid d^2_\gamma E(v, w) = 0 \ \forall w \}$
  $\gamma \in \mathrm{Crit}(E)$ is non-deg : $\mathrm{Ker} = 0$

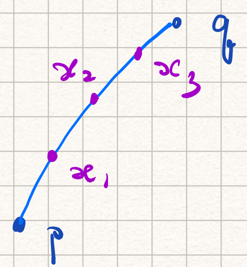
- $\mathrm{index}(\gamma) = \max \dim V$
  over all V s.t.
  $d^2_\gamma E|_V \leq 0$

Thm (See Milnor's book)

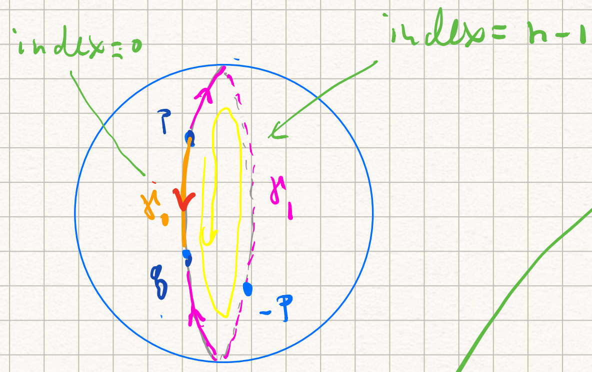
1) p, q not conjugate

$\Rightarrow E$ is Morse, i.e. all $\gamma \in \mathrm{Crit}(E)$ are non-deg. Assume so

2) $\mathrm{index}(\gamma) = \#\{x \text{ conj to } p \text{ on } \gamma\}$ counted with mult.



Ex.



γ_0 index $= 0$ (min)

γ_1 index $= n-1$

γ_2 index $= 2(n-1)$

. . . .

conj to p: $x_1 = -p$

$x_2 = p$

p to q to -p to p to q

Cor index $< \infty$!

Rmk Closed geodesics

- All the def extend word-for-word
- Assume M is given a bumpy metric
- Part 1 of Thm is vacuous
- Part 2 is almost but not quite true
- One has to be careful:

$$\mathrm{Ker} \neq 0 \quad \text{but} \quad \dim \mathrm{Ker} = 1.$$

Rmk Can equip Ω with the str of a Hilbert manifold by considering H^1-paths

• Finite-dim approximation

• $P_k \subset \underbrace{M \times \dots \times M}_{k}$ the space of all broken geodesics with $\sum \rho(p_i, p_{i+1})^2 < \varepsilon^2$ sufficiently small connecting p & q

$p_1 = p$
$p_k = q$

Finite-dim manifold with ∂

• Fix a fine partition

$$0 = t_1 < t_2 < \dots < t_k = 1$$

of S^1

• $\vec{p} \in P_k \rightsquigarrow$ a broken geodesic ξ with $\xi(t_i) = p_i$ parametrized proportionally to the arc-length on $[t_{i-1}, t_i]$

$$\Rightarrow \quad P_k \subset \Omega$$

• $$E\big|_{P_k} = \sum \frac{\rho(p_{i-1}, p_i)^2}{t_i - t_{i-1}}$$

• Set $\Omega^c = \{\gamma \in \Omega \mid E(\gamma) \leq c\}$

$$\cup$$

$$P_k^c = \{\xi \in P_k \mid E(\xi) \leq c\}$$

Rmk $E(\gamma) \leq c \Rightarrow \ell(\gamma) \leq c^{1/2}$

Pf $\displaystyle\int_0^1 \|\dot\gamma\| dt = \int_0^1 \|\dot\gamma\| \cdot 1 dt$

$$\leq \left[\int_0^1 \|\dot\gamma\|^2 dt\right]^{1/2} \cdot \left[\int_0^1 1^2 dt\right]^{1/2}$$

$$= E(\gamma)^{1/2}$$

So $\gamma \in \Omega^c$ have bounded length

Key pt: P_k^c is a very good approximation of Ω^c

Prop (milnor's book)

- Fix c = reg. value of E.
- Assume that $\{t_i\}$ is sufficiently fine, ε is sufficiently small and k is large (Depending on c)

$\Rightarrow$ 1) $P_k^c \longrightarrow \Omega^c$ is homotopy eq.

In fact P_k^c is a deformation retract of Ω^c

2) $\mathrm{Crit}(E|_{\Omega^c}) = \mathrm{Crit}(E|_{P_k^c})$

$=$ true geodesics from p to q with $\ell(\gamma) \leq \sqrt{c}$

3) $\forall$ such γ

- $\mathrm{Ker}\, d^2_\gamma E = \mathrm{Ker}\, d^2_\gamma E|_{P_k^c}$
- $\mathrm{ind}\, d^2_\gamma E = \mathrm{ind}\, d^2_\gamma E|_{P_k^c}$

On the pf

1) We have proved an analogue of 1) for closed geodesics

$E(\gamma) \leq c$ replaces compactness.

2) Similar to closed geodesics

A broken geodesic can be shortened if it has corners

3) Need to work out an explicit formula for $d^2_\gamma E$ (Milnor)

$\lhd$

Prop $\Downarrow$ with a bit extra work + Morse theory

$$\boxed{\begin{array}{l}\textbf{Thm} \quad \text{Assume } p \,\&\, q \text{ are not conj} \\ \Rightarrow \Omega \text{ has homotopy type of an (infinite) CW-complex} \\ \text{with one } m\text{-cell for each geodesic from } p \text{ to } q \text{ of index } m.\end{array}}$$

Rmk Could have worked with

$\mathcal{E} = \sum \rho(P_i, P_{i+1})^2$

but the calculation of $d^2_\gamma E$ is simpler for E.

(We did not need it in LS theory)

Application to topology

– Homology of Ω

Top digression

Fix p. How does $\Omega(p,q)$ depend on q?

It does not, up to homotopy

$$\mathcal{P} = \{ [0,1] \xrightarrow{\gamma} M \mid \gamma(0) = p \}$$

$$\downarrow ev : \gamma \to \gamma(0) \quad \leftarrow \text{a Serre fibration}$$

$$M$$

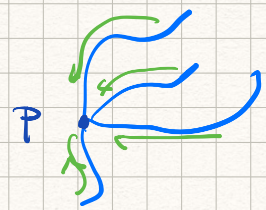
$\Rightarrow \Omega(p,q) = ev^{-1}(q) \leftarrow$ all have the same homotopy type

$\Rightarrow$ Can take $q = p$: $\Omega(p,q) = \Omega_p = \Omega$ the based loop space

(This not $\Lambda =$ the space of free loops

and $\underbrace{Crit(E|_\Omega)}_{\text{'geodesic loops}} \neq \underbrace{Crit(E|_\Lambda)}_{\text{closed geodesics}}$)

Rmk P is contractible



Every path contracts along itself to P

Long exact sequence

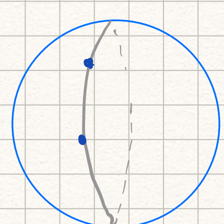
$$\to \pi_i(\Omega) \to \pi_i(\cancel{P}) \xrightarrow{\;} \pi_i(M) \xrightarrow{\cong} \pi_{i-1}(\Omega) \to \pi_{i-1}(\cancel{P})$$

$0 \qquad 0$

$i>1 \qquad \pi_i(M) \cong \pi_{i-1}(\Omega)$ – used before

Understanding $H^*(\Omega)$ and $H_*(\Omega)$ is important in alg. topology.

Application Ω for $M = S^n$



Recall:

one geodesic γ_k of index $k(n-1)$, $k=0,1,\dots$

$\Rightarrow$

> Ω has homotopy type of a complex with exactly one cell of
> dim $= k(n-1)$, $k=0,1,\dots$

// $\partial = 0$

Cor $H_i(\Omega) = \begin{cases} \mathbb{F} & i = k(n-1) \\ 0 & \text{otherwise} \end{cases}$ any ring, e.g. $\mathbb{Z}$

$n > 2$

Rmk. The same is true when $n=2$, i.e. for S^2 but the argument is more involved.
Need to show that $\partial = 0$!

§ 29 Application I: Bott's Periodicity (for $U(n)$)

Lecture 17 03/02

Top Digression

Facts

1) $U(n) \hookrightarrow U(n+1)$

$\rightsquigarrow$ $\pi_k(U(n)) \longrightarrow \pi_k(U(n+1))$

$\cong$ $k \le 2n-1$

$\boxed{\begin{array}{l} U(n) \\ = \{A \mid A^* = A^{-1}\} \\ = \text{unitary transf of } \mathbb{C}^n \end{array}}$

2) $SU(n) \hookrightarrow U(n)$

$$\begin{array}{c} U(n) \\ \downarrow \det \\ S^1 \subset \mathbb{C} \end{array}$$

Long exact seq

$\Rightarrow$ $\pi_k(SU(n)) \cong \pi_k(U(n))$ $k \ge 2$

Pf of 1): $U(n+1)$ acts on $S^{2n+1} \subset \mathbb{C}^{n+1}$

and $U(n) = \text{Stab}(\underbrace{\text{North Pole}}_{P_+})$

$$\Rightarrow \quad \begin{array}{ccc} U(n) \hookrightarrow U(n+1) & \ni & A \\ \downarrow & & \downarrow \\ S^{2n+1} & & AP_+ \end{array}$$

$$\cancel{\pi_{k+1} S^{2n+1}} \to \pi_k U(n) \longrightarrow \pi_k U(n+1) \longrightarrow \cancel{\pi_k S^{2n+1}} \longrightarrow \pi_{k-1} U(n)$$

0 0

$\cong$

$k+1 < 2n+1$

$\triangleleft$

Set $$U = \bigcup U(n) = \varinjlim U(n)$$

$$= \left\{ \begin{pmatrix} \boxed{}^{U(n)} & 0 \\ 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{pmatrix} \right\} \text{ for some } n$$

$$\underbrace{\pi_k U = \pi_k U(n)}_{\text{stable homotopy gps}} ; \quad 2n > k$$

Thm (Bott's periodicity for U)

$$\pi_1 U = \mathbb{Z}, \quad \pi_2 U = 0, \quad \pi_3 U = \mathbb{Z}, \quad \pi_4 U = 0, \ \dots$$

Rmk $\exists$ a similar periodicity for $SO(n)$ but it's 8-periodic, & more involved

Pf.
- Following Milnor
- skipping one diff. geometry step

Note: can replace $U(n)$ by $SU(n)$, $k > 1$
$$\pi_k U = \pi_k SU$$

A bit more topology:

$$Gr_m(2m) = \{ m\text{-dim } \mathbb{C}\text{-subspaces in } \mathbb{C}^{2m} \}$$

Fact: $$\pi_i \, Gr_m(2m) \cong \pi_{i-1} U(m), \quad i \leq 2m$$

Pf || Two fibrations:

a) $$\begin{array}{c} U(m) \hookrightarrow U(2m) \\ \downarrow \\ U(2m)/U(m) \end{array}$$

induces $\cong$ in π_i $i\le \dots$

complex Stiefel manifold: subspaces with frames

$\Rightarrow$ $\pi_i(U(2m)/U(m)) = 0$ $i \le 2m$

b) $$\begin{array}{c} U(m) \hookrightarrow U(2m)/U(m) \\ \downarrow \\ U(2m)/U(m)\times U(m) = Gr_m(2m) \end{array}$$

$\Rightarrow$ $\pi_i Gr_m(2m) \cong \pi_{i-1} U(m)$ $i \le 2m$

$\triangleleft$

Lower-dim calculations

- $\pi_1 U(n) \cong \pi_1(\underset{U(1)}{\overset{\shortparallel}{S^1}}) = \mathbb{Z}$ $\quad n \geq 1$

$\underbrace{k \le 2n-1}_{\text{stable range}}$

- $\pi_2 U(n) \cong \pi_2 \underbrace{SU(2)}_{\overset{\shortparallel}{S^3}} = 0$ $\quad n \geq 2$

Rmk. $\pi_2(\text{any Lie gp}) = 0$

Now the key part: Morse Theory

Metric on SU(n) or U(n)

Equip SU(n) with the Killing metric

- biinvariant
- On $T_I SU(n) = su(n)$ (or $T_I U$):
  $= \{ A \mid A^* = -A, \operatorname{tr} A = 0 \}$

$$\langle A, B \rangle = \operatorname{tr} AB^*$$

Conj. invariant $\Rightarrow$ right (or left) translation is bi-inv

- $\exp(A) = I + A + \frac{1}{2} A^2 + \frac{1}{3!} A^3 + \dots$

$$\Omega(I, -I, SU(2m)) \sim \Omega_I(SU(2m))$$

Lemma 1: The space of minimizing geodesics from I to $-I$ is homeo (diffeo) to $Gr_m(2m)$

Lemma 2: Every non-minizing geodesic from I to $-I$ has Morse index $\geq 2m+2$

Rmk $-I$ is conj to I

index $:= \max_V \dim V$

$d^2_\gamma E|_V < 0$ accounting for deg

- We'll prove Lemma 1.
- Lemma 2 $\longrightarrow$ Milnor (need more diff. geom..)

  Sounds reasonable: think of $SU(2m)$ as something like the sphere: longer geodesics get a lot of conj. pts. (See the pf
- Lemmas 1 & 2 $\Rightarrow$ Bott's periodicity

$$\pi_k\, \Omega(I,-I,SU(2m)) = \pi_k\, \Omega_I(SU(2m))$$

Lemmas $\longrightarrow \cong$ $k \le 2m$

$$\pi_k\, Gr_m(2m) \qquad \qquad \cong \quad \pi_{k+1}\, SU(2m)$$

$\cong$ $k \le 2m$ $\sim$

$$\pi_{k-1}\, SU(2m)$$

Using the calculation of $\pi_k U$ for $k=1$ & 2 we learn what these groups are.

Pf of Lemma 1

$n = 2m$

$\gamma_A(t) = \exp(tA)$

$A \in T_I SU(n) : \quad A^* = -A, \; trA = 0$

$\Rightarrow$ A is diagonalizable by a unitary transf

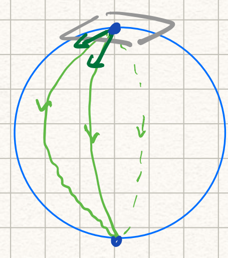
$$BAB^{-1} = \begin{pmatrix} i\pi a_1 & & 0 \\ & \ddots & \\ 0 & & i\pi a_n \end{pmatrix}, \quad B \in U(n)$$

Let's say A itself has this form

$$\gamma_A(1) = \exp(A) = \begin{pmatrix} e^{i\pi a_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\pi a_n} \end{pmatrix}$$

- $\exp(A) = -I \iff$ all a_j = odd integers
- $\ell(\gamma_A) = \int_0^1 \|A\| dt = \|A\|$

$$= \pi\sqrt{a_1^2 + \dots + a_n^2}$$



$\Rightarrow$ γ_A is a minimizing geodesic

$\Leftrightarrow$ all $a_j = \pm 1$

- But $\operatorname{tr} A = \sum a_j = 0$, $n = 2m$

$\Rightarrow$ m of $a_j = -1$ & m are $+1$

<u>Rmk</u> This is the reason to work with $SU(2m)$ but not $U(n)$

$A \rightsquigarrow L \subset \mathbb{C}^{2m}$, $L \in Gr_m(2m)$

$\uparrow$ span of eigenvectors with eigenvalue -1

$$\left\{\begin{matrix}\text{minimizing} \\ \text{geodesics}\end{matrix}\right\}$$

$$\Big\updownarrow$$

$$\{A\} \longleftrightarrow Gr_m(2m) \qquad \triangleleft$$

<u>Rmk</u> This also suggests why Lemma 2 is true: then $|a_j| \geq 3$ for at least one a_j. This turns out to imply $\exists$ mult conj pts along such a geodesic. (Non-obvious)

§ 30 Application II: Lefschetz Hyperplane Section Thm

Also in Milnor but here we do a slightly different pf

Should really be in Part I

Recall from complex analysis:

- $F: \underset{\mathbb{C}^m}{\overset{\cap}{U}} \longrightarrow \underset{\mathbb{C}^n}{\overset{\cap}{V}}$ is holomorphic if

$$DF \circ J = J \circ DF \quad @ \text{ every pt}$$

$$\Leftrightarrow \quad DF \in M_{m \times n}(\mathbb{C}) \qquad \text{"} J = i \text{"}$$

$\Leftrightarrow$ all components of F satisfy CR with respect to every variable (and F is C^1 ?)

- complex manifolds are just like smooth manifolds but smooth maps, charts, etc are replaced by hol maps

<u>Rmk</u> Complex manifold never have boundary: either open or closed.

E.g. 1) The system of equations

$$\begin{cases} f_1 = 0 \\ \vdots \\ f_k = 0 \end{cases}$$

$f_j : \mathbb{C}^n \longrightarrow \mathbb{C}$ hol

gives a complex submanif, provided that df_j are lin. ind at every pt (over $\mathbb{C}$)

2) Same true in $\mathbb{C}P^n$ when f_j a hom. polynomials

$f(\lambda z) = \lambda^d f(z)$

We'll also need

Fact

$M \subset \mathbb{R}^n$

$$\boxed{\text{The function } f(x) = \|x - p\|^2 \text{ is Morse on } M \text{ for allmost all } p \in \mathbb{R}^n}$$

In particular this is true for complex submanifolds of $\mathbb{C}^n$.

In what follows we'll always assume that $p = 0$ and $f(x) = \|x\|^2$ is Morse.

In fact we have proved this

See next page

$$\begin{array}{c} M \subset \mathbb{R}^n \\ \downarrow f \\ \mathbb{R} \end{array} \qquad h_v(x) = \underset{\substack{\cap \\ \mathbb{R}^n}}{\langle x, v \rangle} : \text{proj to } v\mathbb{R}$$

What we have shown is that:

For almost all $v \in \mathbb{R}^n$

$f + h_v$ is Morse on M

But $f(x) = \| x \|^2$

$$f(x) + h_v(x) = \langle x, x \rangle + \langle x, v \rangle$$
$$= \langle x + \tfrac{1}{2} v, x + \tfrac{1}{2} v \rangle - \tfrac{1}{4} \| v \|^2$$

$\Rightarrow$ $\langle x + \frac{1}{2} v, x + \frac{1}{2} v \rangle$ is Morse

Can replace 0 by any pt.

Thm $M^m \subset \mathbb{C}^n$ complex submanifold

$\Rightarrow$ every critical pt of $f(x) = \|x\|^2$ on M has index $\leq m$

Cor Assume that M is proper

($M \cap$ any compact is compact)

$\Rightarrow$ M has homotopy type of m-dim CW complex (perhaps infinite)

Morse theory

Such complex submanifolds of $\mathbb{C}^n$ are called **Stein** manifolds.

Discussion

- Here $m = \dim_{\mathbb{C}} M$, $2m = \dim_{\mathbb{R}} M$

- $\mathbb{C}^n$ has no closed complex submanifolds
  (if it did, Thm would be wrong)
  Max principle: proj of M to any coord is a hol function

  M closed $\Rightarrow$ $f = \text{const}$ by max principle

- The assertion is essentially local
  Only need to have a nbd of a critical pt.

Pf I

Calculation

• $f: M^m \longrightarrow \mathbb{R}$, $\quad f(x) = \|x\|^2$, Morse

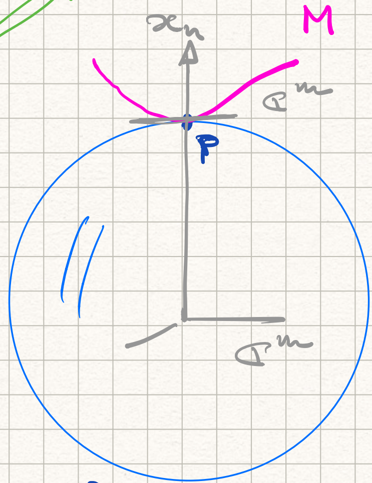
$M^m \subset \mathbb{C}^n$

• $p \in \mathrm{Crit}(f)$

WLOG:

$p = (0, \dots, 1)$ NP

$\|p\| = 1$



$T_p M \subset T_p S^{2n-1}$

complex $\Downarrow$ real hyperplane

$S^{2n-1} = \{\|x\| = 1\}$

$z_k = x_k + i y_k$

$\underbrace{T_p M}_{\mathbb{C}^m} \subset \mathbb{C}^{n-1} \subset T_p S^{2n-1}$

$\{(\underbrace{z_1, \dots, z_{n-1}}, y_n)\}$

$(z_1, \dots, z_m)$

• Near p: M is a graph of a map $\mathbb{C}^m \xrightarrow[\text{hol}]{} \mathbb{C}^{n-m}$

$\zeta = (z_1, \dots, z_m) \longmapsto (\overset{z_{m+1}}{g_1}, \dots, \overset{z_n}{g_{n-m}})$ $\longrightarrow$ to z_n

• $g_k(0) = 0$, $\quad k < n-m \qquad dg_k(0) = 0$

• $g_{n-m}(0) = 1$

$$f(\zeta) = \underbrace{\sum_{j=1}^{m} |z_j|^2}_{Q_0} + \sum |g_k|^2 \;=\; \sum_{j=1}^{n} |z_j|^2$$

$k < n-m \Rightarrow \quad g_k(\zeta) = O(|\zeta|^2)$

$\Rightarrow \quad |g_k|^2 = O(|\zeta|^4)$

$\Rightarrow$ does not contribute to $d^2 f$

$k = n-m, \quad g_{n-m} = 1 + O(|\zeta|^2)$

$$g = g_{n-m} = 1 + \underbrace{\sum c_{eq} z_e z_q}_{} + \dots$$

$H : \mathbb{C}^m \longrightarrow \mathbb{C}$ a complex quadratic form

$$|g|^2 = 1 + \underbrace{H + \overline{H}}_{Q_1} + \dots$$

$Q_1 = H + \overline{H} : \mathbb{R}^{2m} \longrightarrow \mathbb{R}$

$Q_1(z) = H(z) + \overline{H(z)}$

$= 2 \operatorname{Re} H(z)$

$$d_p^2 f = Q_0 + Q_1$$

<u>claim</u> : index $Q_1 \leq m$

To be more precise $\mathbb{R}^{2m} = \underbrace{V_0 \oplus V_+ \oplus V_-}_{\text{ortogonal for } Q_1}$

and $Q_1|V_0 = 0$

$Q_1|V_\pm \gtrless 0$

$\dim V_+ = \dim V_-$

Claim $\Rightarrow$ Thm

$$\underset{\overset{\vee}{0}}{Q_0} + \underset{\overset{\vee\!/}{0}}{Q_1} \Big|_{V_0 \oplus V_+} > 0$$

and $\dim(V_0 + V_+) \geq m$

$\Rightarrow$ $\operatorname{ind}(Q_0 + Q_1) \leq m$

Pf of the Claim

Diagonalize H:

$$BHB^T = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{pmatrix}, \quad B \in GL(m, \mathbb{C})$$

$L =$ eigenspace

- eigenvalue 0: $Q_1|_L = 0$

  L goes into V_0

- eigenvalue 1: $L = \mathbb{C} = \mathbb{R}^2$, $z = x + iy$

  $Q_1(z) = z^2 + \bar{z}^2 = 2\operatorname{Re} z^2$
  $= x^2 - y^2$

  x-axis goes into V_+
  y-axis goes into V_-

$\triangleleft$

$\triangleleft$

Pf II - Symplectic geometrical

- Real symplectic form on $\mathbb{R}^{2n} = \mathbb{C}^n$

$$\omega = \sum dx_k \wedge dy_k = \frac{i}{2} \sum dz_k \wedge d\bar{z}_k$$

$$= d\lambda \quad , \lambda = \frac{1}{2} \sum x_k dy_k - y_k dx_k$$

- $f = \frac{1}{4} \sum (x_k^2 + y_k^2) = \frac{1}{4} \sum |z_k|^2$

$$df = \frac{1}{2} \sum \left(x_k dx_k + y_k dx_k \right)$$

- J acts on $T(\mathbb{R}^{2n} = \mathbb{C}^n)$ as multiplication by i

$$J = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & 0 \\ & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \\ 0 & & \ddots \end{pmatrix}$$

$\Rightarrow J$ acts on $T^*(\mathbb{R}^{2n} = \mathbb{C}^n)$ as $(J\alpha)(v) = -\alpha(Jv)$

$$J df = \frac{1}{2} \sum \left(x_k dy_k - y_k dx_k \right) = \lambda$$

Checking: $(J df)(\partial_{x_k}) = -df(J\partial_{x_k}) = -df(\partial_{y_k})$

$(J df)(\partial_{y_k}) = -df(J\partial_{y_k}) = -df(-\partial x_k)$

$\Rightarrow \omega = d(J df)$ on $\mathbb{C}^n = \mathbb{R}^{2n}$

- $M \subset \mathbb{C}^n$ complex submanifold

$$J : TM \circlearrowleft$$

$\Rightarrow$ $\boxed{\omega|_M = d(Jdf|_M)}$ $\quad \lambda|_M = Jdf|_M$

- Liouville v.f.

X on M for $\lambda|_M$

$$i_X \omega = \lambda$$

$\Rightarrow$ $L_X \omega = \omega$: $L_X\omega = di_X\omega + \cancel{i_X d\omega}^{\,0}$
$= d\lambda = \omega$

The flow φ_t of X stretches ω:

$$\varphi_t^* \omega = e^t \omega$$

- $\lambda = Jdf$: $i_X \omega = Jdf$

$\omega(X, v) = -df(\underbrace{Jv}_{w})$ $\quad \langle \, , \, \rangle = \omega(\cdot, J\cdot)$

$\underbrace{\omega(X, +Jw)}_{\langle X, w \rangle} = +df(w)$

$\langle X, w\rangle = df(w)$: $\langle X, \cdot \rangle = df$

$\Rightarrow$ $\boxed{X = \nabla f}$

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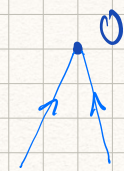
Punchline

$p \in \mathrm{Crit}(f \text{ on } M)$, ω_p

$$D\varphi_t^* \omega_p = e^t \omega_p \quad \text{on } T_pM = \mathbb{R}^{2m}$$



$v, w \in T_pM$

$$\omega(D\varphi_t v, D\varphi_t w) = e^t \omega(v, w) \quad (*)$$

$V \subset T_pM$ be such that $d_p^2 f|_V \leq 0$ } "stable manifold"

$v, w \in V \Rightarrow D\varphi_t v, D\varphi_t w \longrightarrow 0$ if < 0

or at most grow polynomially as $t \to \infty$

$\Rightarrow \omega(D\varphi_t v, D\varphi_t w)$ <u>cannot</u> grow exp in $(*)$

$\Rightarrow \omega(v, w) = 0 \quad \forall\, v, w \in V$

$\Rightarrow V$ is isotropic

$\Rightarrow \dim V \leq m$

$\lhd$

<u>Rmk</u> Both pfs don't need non-deg.

In both cases $\mathrm{ind} := \max_{d^2f|_E > 0} \dim V \geq m$

Geometrically, assuming non-degeneracy

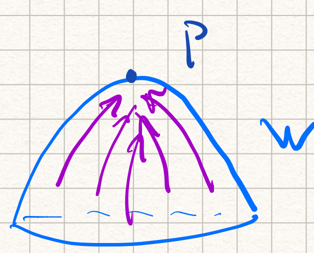
- $W^s(p) =: W$ = stable manifold of p for $X = \nabla f$

- φ_t = grad flow = flow of X

- $\varphi_t^* \omega|_W = e^t \omega|_W$
- But φ^t contracts W to a pt

$\Rightarrow$ $\omega|_W = 0$ i.e. W is isotropic

$\Rightarrow$ $\dim W \leq n$



Back to the Lefschetz hyperplane section thm

Setting

$M^n \subset \mathbb{C}P^n$ complex submanifold (closed) (by def. algebraic)

$\cup$

$H = \mathbb{C}P^{n-1} \leftarrow$ hyperplane

$M \pitchfork H \Rightarrow \underbrace{M \cap H}_{\text{complex submanifolds in } H = \mathbb{C}P^{n-1}} \subset H$

The Lefschetz hyperplane section thm:

Thm M is obtained from $M \cap H$ by attaching cells of $\dim \geq m$

$\Downarrow$

Thm

- $H_k(M \cap H) \xrightarrow{\simeq} H_k(M)$ if $k < m-1$
- $H_{m-1}(M \cap H) \underset{\text{onto}}{\twoheadrightarrow} H_{m-1}(M)$ (both over $\mathbb{Z}$)
- $\pi_r(M, M \cap H) = 0 \quad r < m$

long exact sequence (+ Lefschetz duality for $k = m-1$?)

Pf. • Recall

$\mathbb{C}P^n \setminus (H = \mathbb{C}P^{n-1}) = \mathbb{C}^n$ holomorphically

$\sim B^n$ "metrically" (not literally)

Pick $0 \in \mathbb{C}P^n \setminus H$ (generic)

• $\exists\ h : \mathbb{C}P^n \longrightarrow \mathbb{R}$ such that

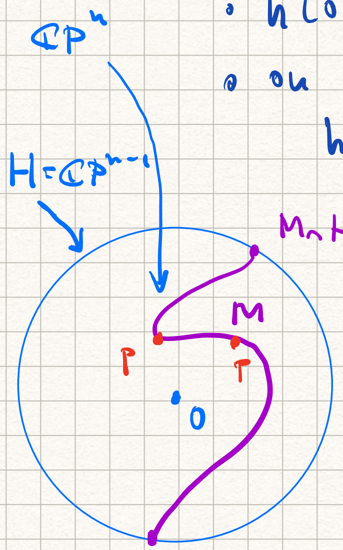
• $h|_H = \min h = 0$

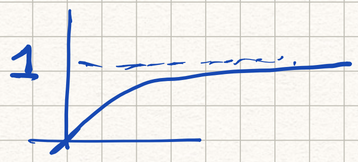
• $h(0) = \max h = 1$

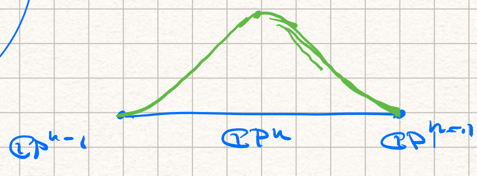
• on $\mathbb{C}^n = \mathbb{C}P^n \setminus \mathbb{C}P^{n-1}$

$h(z) = -g \circ f(z), \quad f(z) = \|z\|^2$

E.g. $(\text{dist to } H)^2$ would do.







• $h|_M$ • attains min on $M \cap H = 0$

• Morse outside $M \cap H$ (for a generic choice of 0)

with only finitely many crit pts.

Rmk Can make sure $h|_M$ is Morse Bott

- M is obtained from

$$\{h \le \varepsilon\} = \text{small tub. nbd of } M \cap H$$
$$\simeq M \cap H$$

by attaching a cell of

$$\dim = \operatorname{index}(h \text{ at } p)$$

for every $p \in \mathrm{Crit}(h \text{ on } M \setminus (M \cap H))$

$$= \mathrm{Crit}(f \text{ on } M \setminus (M \cap H))$$

and $\operatorname{ind} h = 2m - \underbrace{\operatorname{ind} f}_{\le m} \ge m$

$\triangleleft$