§25 Infinite-dim approach

Lecture 15
02/23

Very briefly

Recall

Sobolev space (not cohomology) of H^1-functions on S^1: $H^1(S^1, \mathbb{R})$

Abs. continuous functions f such that f' is L^2

Hilbert space with the inner product

$$\langle f, g\rangle_{H^1} = \langle f, g\rangle_{L^2} + \langle f', g'\rangle_{L^2}$$

Rmk Can also define as

$$\langle f, g\rangle'_{H^1} = f(0)g(0) + \langle f', g'\rangle_{L^2}$$

could be $\bar{f}\bar{g}$

Different product but "equivalent"

Likewise: f can take values in $\mathbb{R}^n$

Def Λ = H^1-maps $\gamma : S^1 \longrightarrow M$

M: closed Riemannian manifold

γ is abs. continuous
$\dot{\gamma}$ is L^2:

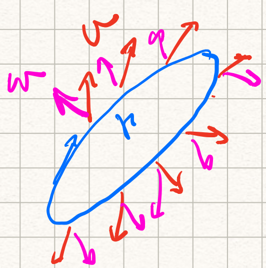
$$\int_{S^1} \|\dot{\gamma}(t)\|^2 dt < \infty$$

Fact Λ is a Hilbert manifold (modeled on $H^1(S^1 \to \mathbb{R}^n)$)

Tangent space:

$T_\gamma \Lambda =$ H^1- v.f. along γ

$= \{ v \mid v$ abs cont & $\nabla_{\dot\gamma} v$ is $L^2 \}$



Hilbert space

$$\langle v, w \rangle_{H^1} = \langle v, w \rangle_{L^2} + \langle \nabla_{\dot\gamma} v, \nabla_{\dot\gamma} w \rangle_{L^2}$$

Ref Klingenberg "Lectures on closed geodesics"
Chap 1 & 2 (ex caution)

Important

$E : \Lambda \longrightarrow \mathbb{R}$ is defined & smooth

$$E(\gamma) = \frac{1}{2} \int_0^1 \|\dot\gamma\|^2 \, dt$$

Elements of Λ have just enough smoothness

Key pt: one can do basic LS for E on Λ

Steps (without pfs)

- Anti-grad flow for $\nabla_{H^1} E$ is defined on Λ for $t \geq 0$
- E satisfies the PS condition

$\Downarrow$

$\Rightarrow$ min E is attained on every connected component

$\Rightarrow$ Cartan's theorem

and

$\Rightarrow$ minimax holds

$\Rightarrow$ Lusternik-Fet thm

Important rmk

- E would still be defined if we required more smoothness for elements of Λ $\quad$ $H^{k>1}$ - Sobolev space
- But PS condition would then break down

Underlying Principle:

In all problems of this type require as little smoothness as necessary to have the function defined.

E.g. For the action functional one would have to work with $H^{1/2}$.

Important underlying features of E that make things work

- E is bounded from below
- At every $\gamma \in \mathrm{Crit}(E)$ = closed geodesic $d^2_\gamma E$ has
  - → finite # of neg squares (finite index)
  - → finite nullity

$V \subset T_\gamma E$ s.t. $d^2_\gamma E|_V \le 0 \Rightarrow \dim V < \infty$

Not always true: e.g. fails for the action functional on $H^{1/2}$

⇒ Huge difficulties

Rmk (Bumpy Metrics)

- In general, E or $\mathcal{E}$ are not Morse
- A closed geodesic γ is non-deg if its non-deg as a periodic orbit of the geodesic flow on its level

  $\Longleftrightarrow$ $d^2_\gamma E$ is non-deg in the Morse-Bott sense

- For a C^∞-generic metric all closed geodesics are non-deg ($\Longleftrightarrow$ E or $\mathcal{E}$ is Morse)
  Such metrics are called Bumpy Metrics

  Abraham (1970) - without complete pf
  Anosov (1983) - complete pf

Application: $\boxed{\operatorname{rk} H_k(\Lambda) \xrightarrow[k\to\infty]{} \infty}$ usually satisfied

$\Rightarrow$ a bumpy metric has ∞ many prime geodesics

can be eliminated

(Gromoll-Meyer c73?)

Not true for S^n. Not known if every metric on S^3 has ∞ many prime geodesics.

§ 26 A quick glance at other calculus of variations questions

Recap: what we have learned

M a R. m.: $\langle\, , \rangle$ = R. m.

$L : TM \longrightarrow \mathbb{R}$ ← a Lagrangian

$(q, v) = \frac{1}{2} \langle v, v \rangle_q$ Fix $T > 0$

$$E(\gamma) = \int_0^T L(\gamma, \dot{\gamma})\, dt$$

$E : \Lambda \longrightarrow \mathbb{R}$, $\Lambda = \{ S = \mathbb{R}/T\mathbb{Z} \longrightarrow M \}$

Crit(E) = closed geodesics parametrized by $[0, T]$ ~ arc length

solutions of:

$\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ ← acceleration

Governs the motion of a unit mass particle confined to M

Cartan & Lusternik - Fet

⇒ Existence

Other types of Lagrangians

(1) $$L(q,v) = \frac{1}{2}\langle v, v\rangle_q - V(q)$$

Potential energy $V : M \to \mathbb{R}$ or $V : M \times S^1 \to \mathbb{R}$, $S^1 = \mathbb{R}/T\mathbb{Z}$

$$\begin{cases} \text{Same } \Lambda \\ \mathcal{L}(\gamma) = \displaystyle\int_0^T L(\gamma, \dot\gamma)\,dt \end{cases}$$

$$\mathrm{Crit}(\mathcal{L}) = \quad \begin{array}{l} \gamma : S^1 \to M \\ \nabla_{\dot\gamma}\dot\gamma = -\nabla V \leftarrow \text{Force} \end{array}$$

Governs the motion of a unit mass particle on M, ext. force $F = -\nabla V$.

$$\frac{1}{2}\langle v, v\rangle + \langle v, g^{-1}(\alpha)\rangle - V$$

(2) $$L(q,v) = \cancel{\frac{1}{2}\langle v-\alpha, v-\alpha\rangle_q - \nabla V}$$

V the same: potential energy

$\alpha \in \Omega^1(M)$: magnetic potential

$$\mathrm{Crit}(\mathcal{L}) = \quad \begin{array}{l} \gamma : S^1 \to \mathbb{R} \\ \nabla_{\dot\gamma}\dot\gamma = -\nabla V + \underbrace{F(\gamma, \dot\gamma)}_{\text{Lorentz Force}} \end{array}$$

$$F(q,v) := g^{-1} i_v d\alpha$$

Rmk: only $\omega = d\alpha$ matters

$$F: \quad TM \xrightarrow{i_v} T^*M \xrightarrow{g^{-1}} TM$$
$$(v,q) \longmapsto i_v d\alpha \longmapsto F$$

$d\alpha =$ magnetic field

<u>Ex</u> a) B = magnetic field in $\mathbb{R}^3$

$$\omega = i_B \underbrace{(dx\wedge dy\wedge dz)}_{dVol} \leftarrow \text{closed}$$

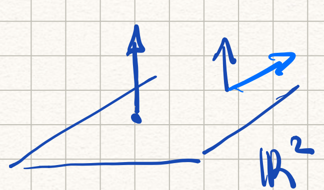
$\Uparrow$ Maxwell

$\omega = d\alpha$

$$F(q,v) = B\times v \quad \text{at } q$$

b) $\omega = B\, dx\wedge dy$ in $\mathbb{R}^2$, $B: \mathbb{R}^2 \to \mathbb{R}$
$= d\alpha$

$$F(q,v) = B\cdot Jv \quad \text{at } q$$



Magnetic field $\perp$ $\mathbb{R}^2$

$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Governs the motion of a unit mass, unit charge particle on M, ext. force $F = -\nabla V$, and magnetic field $d\alpha$.

<u>Ex.</u> Determine the flow on $\mathbb{R}^2$ when B = const.

3) More generally:

$L : TM \longrightarrow \mathbb{R}$ s.t.

or $L : TM \times \underbrace{S^1}_{\mathbb{R}/T\mathbb{Z}} \longrightarrow \mathbb{R}$

E.g.
$L(q,v) \geq A\|v\| - B$

a) fiberwise convex
b) quadratic or superlinear growth as $v \longrightarrow \infty$

exact conditions vary (Tonelli Lagr)

<u>Rmk.</u> When L is time-dependent and T-periodic in time,
$\gamma : S^1 = \mathbb{R}/T\mathbb{Z} \longrightarrow M$ some T.
Can set $T=1$, but don't want to yet.

<u>Goal</u>: Existence of Crit $(\mathcal{L})$

$$\mathcal{L}(\gamma) = \int_0^T L(\gamma, \dot{\gamma})\,dt$$

Solutions $\gamma : S^1 \longrightarrow M$ of the Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial v}(\gamma, \dot{\gamma}) = \frac{\partial L}{\partial q}(\gamma, \dot{\gamma})$$

Thm (Benci, 86) ← sample result

- $\pi_1(M) \neq 1 \Rightarrow$ sol in every $\neq$ trivial free homotopy class
- $\pi_1(M) = 1 \Rightarrow$ inf many contractible T-periodic sol's γ_k

Moreover: $\underbrace{\mathcal{L}(\gamma_k) \longrightarrow \infty}$

this is what makes it more interesting
In some sense γ_k's are distinct
(loose)

Rmk: → no longer have the notion of trivial (const) vs non-trivial (non-const) sol

→ For geodesic flow get reparametrization of the same iterated geodesic:

$\gamma_1, \gamma_2, \gamma_3, \dots$

$\gamma_2(t) = \gamma_1(2t), \gamma_3(t) = \gamma_1(3t), \dots$

On the pf

→ Fix a background R.M.

→ The H^1-anti-grad-flow for $\mathcal{L}$ is defined for all $t \geq 0$

→ P.S. condition is satisfied

- $\pi_1(M) \neq 1$: Exactly as for geodesics in § 25: minimize $\mathcal{L}$ on $\Lambda_{\alpha \neq 1}$

- $\pi_1(M) = 1$. To get just one solution could again just minimize

To get ∞ many, find a seq of classes

$$\beta_k \in H_{m_k}(\Lambda; \mathbb{R}) \qquad m_k \to \infty$$

$\overset{PS}{\Rightarrow}$ minimax orbits γ_k

Moreover β_k's are not supported in the lower energy regions of Λ:

energy for some R.M.

$$\beta_k \notin \operatorname{im}\left(H_*\{E < e_k\} \to H_*(\Lambda)\right)$$

for some $e_k \to \infty$

$$\Rightarrow \mathcal{L}(\gamma_k) \to \infty$$

◁

Rmk

A quick digression: A fixed energy problem

- So far we looked for periodic orbits of a fixed period
  But one can look for periodic orbits with "fixed energy" – dual problem (and much harder)

- Assumption:
  L = independent of time

set $\quad H(q,v) = \frac{\partial L}{\partial v} \cdot v - L$

Then $\quad E\text{-}L \Rightarrow \frac{d}{dt}\left(H(\gamma, \dot{\gamma})\right) = 0$

$\Rightarrow$ Trajectories on the same level of H (compact) $\leftarrow$ Ex

Q! Find per solutions of E-L eq of unspecified period on a fixed level $E = c$ or some specified levels

Much harder

Further ref + names!

Rabinowitz, Benci, Figalli, Mañé Abbondandolo, ...

§27 Geodesics connecting two points – Morse Theory

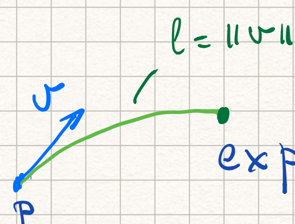
Lecture 16
02/25

- Very briefly
- Skipping most of R.G.
- Following Milnor – a great intro to R.G.

• A Brief Review: exp map & conj. points

$$\exp : T_pM \longrightarrow M$$
$$v \longmapsto \gamma_v(1)$$

γ_v = geodesic starting at p with $\dot{\gamma}_v(0) = v$



$\exp(v) = \gamma_v(1)$

M compact $\Rightarrow$ M is complete

$\Rightarrow$ exp is defined ($\gamma_v : \mathbb{R} \longrightarrow M$)

and onto

Def • q is conjugate to p (along $\gamma = \gamma_v$) if q is a critical value of exp:

$\ker D\exp \neq 0$ at $v : q = \gamma_v(1)$

• multiplicity of q is

$\dim \ker D\exp|_v$

Ex. • $\mathbb{T}^n = \mathbb{R}^n/\Gamma \leftarrow$ lattice e.g. $\Gamma = \mathbb{Z}^n$

$\Gamma = e_1\mathbb{Z} + .. + e_n\mathbb{Z}$

a basis

$\Rightarrow$ no conj. pts   covering map $\mathbb{R}^n \to \mathbb{T}^n$

• More generally

sectional curv ≤ 0

$\Rightarrow$ no conj. pts

• S^n with round metric

$\to$ $q = -p$ conj to p

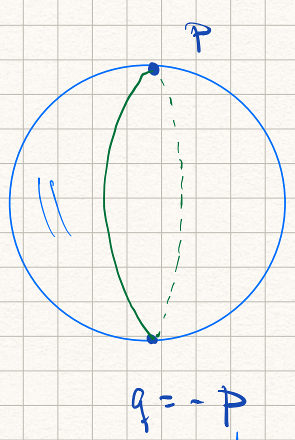
along half-meridian

mult $= n-1$

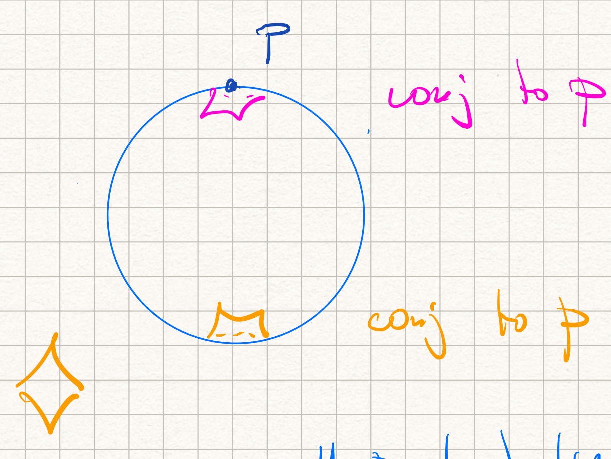
$\to$ p is conj to itself $q = p$

along whole meridian

mult $= n-1$



round metric



a small perturbation of a round metric

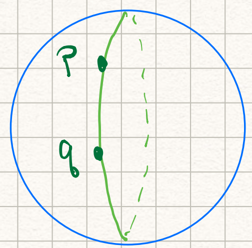
Sard's $\Longrightarrow$

The set of pts conj to p has zero measure in M

Rmk The set of pts conj to p need not be closed because T_pM is not compact

Cor For almost all $(p,q) \in M \times M$ p & q are not conjugate

E.g. In the round S^2 take any pair so that $q \neq \pm p$



• Morse Theory set up : Ω and E

p.w. C^∞

$\to$ $\Omega = \Lambda(M, p, q)$

$= \{ \gamma : [0,1] \longrightarrow M \mid \gamma(0) = p, \gamma(1) = q \}$

a better notation

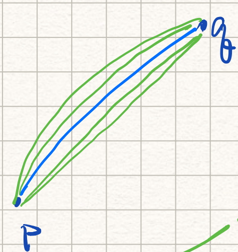
$\to$ $E : \Omega \longrightarrow \mathbb{R}$ the energy

$$E(\gamma) = \int_0^1 \|\dot\gamma(t)\|^2 dt$$

$\to$ γ is a critical pt of E if

$$\frac{d}{dt} E(\gamma_s)\Big|_{s=0} = 0 \quad \forall \text{ any variation } \gamma_s \text{ of } \gamma$$

$$\gamma_s(t) : \underset{s}{(-\varepsilon, \varepsilon)} \times \underset{t}{[0,1]} \longrightarrow M$$

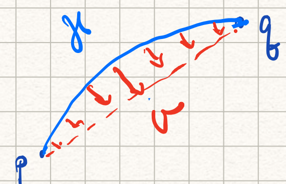


Exactly as for closed geodesics

Fact: $\gamma \in \mathrm{Crit}(E) \iff \gamma$ is a closed geodesic connecting p to q and parametrized by $[0,1]$ $\sim$ arc length

Tangent space

$T_\gamma \Omega =$ v.f. along γ vanishing at p & q



Think $v = \frac{\partial \gamma_s}{\partial s}(t)\Big|_{s=0}$

Every v comes from γ_s

$\gamma_s(t) = \exp_{\gamma(t)}(sv)$

Rmk. We do not make Ω into an inf dim manifold ... This is just an interpretation.

The Hessian

In finite dim's: $f : P \to \mathbb{R}$.

$v, w \in T_x P$, $x \in \mathrm{Crit}(f)$

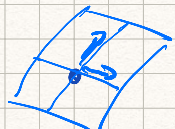
Consider $\underset{s}{(-\varepsilon, \varepsilon)} \times \underset{s'}{(-\varepsilon', \varepsilon')} \xrightarrow{u} P$, $u(s, s')$

s.t. $u(0,0) = x$

$$\frac{\partial u}{\partial s}\Big|_{\substack{s=0\\ s'=0}} = v, \quad \frac{\partial u}{\partial s'}\Big|_{\substack{s=0\\ s'=0}} = w$$

$$d^2_x f(v, w) = \frac{\partial^2}{\partial s \partial s'} f(u(s,s'))\Big|_{\substack{s=0\\ s'=0}}$$

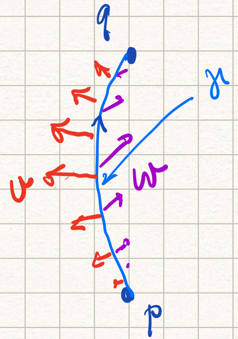
Ind of u but only v, w



$u(s, s')$ Pr: Taylor exp.

Likewise in inf dimensions

$v, w \in T_\gamma \Omega$ v.f. along γ



Take $u(s,s',t)$ s.t.

$u(0,0,t) = \gamma(t)$

$$\left.\frac{\partial u}{\partial s}\right|_{\substack{s=0\\ s'=0}} = v(t)$$

$$\left.\frac{\partial u}{\partial s'}\right|_{\substack{s=0\\ s'=0}} = w(t)$$

E.g. $u(s,s',t) = \exp_{\gamma(t)}(sv + s'w)$

Set

$$d^2_\gamma E(v,w) := \left.\frac{\partial^2 E(u(s,s',\cdot))}{\partial s \partial s'}\right|_{\substack{s=0\\ s'=0}}$$

Ind of the variation
and is completely det by v, w.
when $\gamma \in \mathrm{Crit}(E)$, symmetric

- $\mathrm{Ker}\, d^2_\gamma E = \{ v \mid d^2_\gamma E(v,w) = 0 \ \forall w \}$
  $\gamma \in \mathrm{Crit}(E)$ is non-deg : $\mathrm{Ker} = 0$

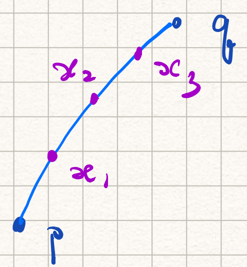
- $\mathrm{index}(\gamma) = \max \dim V$
  over all V s.t.
  $d^2_\gamma E|_V \leq 0$

Thm (See Milnor's book)

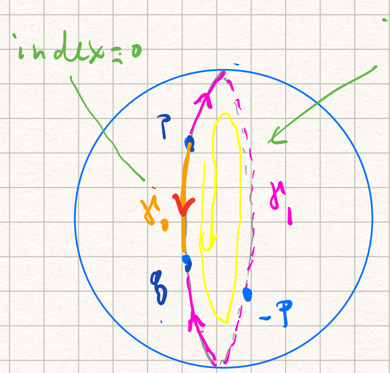
1) p, q not conjugate

$\Rightarrow$ E is Morse, i.e. all $\gamma \in \mathrm{Crit}(E)$ are non-deg. Assume so

2) $\mathrm{index}(\gamma) = \#\{x \text{ conj to } p \text{ on } \gamma\}$ counted with mult.



Ex.



γ_0 index = 0 (min)

γ_1 index = n - 1

γ_2 index = 2(n-1)

....

conj to p: $x_1 = -p$

$x_2 = p$

p to q to -p to p to q

Cor index $< \infty$!

Rmk Closed geodesics

- All the def extend word-for-word
- Assume M is given a bumpy metric
- Part 1 of Thm is vacuous
- Part 2 is almost but not quite true
- One has to be careful:

  $\ker \neq 0$ but $\dim \ker = 1$.

Rmk Can equip Ω with the str of a Hilbert manifold by considering H^1-paths

- Finite-dim approximation

- $P_k \subset \underbrace{M \times \dots \times M}_{k}$ the space of all broken geodesics with $\sum \rho(p_i, p_{i+1})^2 < \varepsilon^2$ sufficiently small connecting p & q

$p_1 = p$

$p_k = q$

Finite-dim manifold with ∂

- Fix a fine partition
$$0 = t_1 < t_2 < \dots < t_k = 1$$
of S^1

- $\vec{p} \in P_k \rightsquigarrow$ a broken geodesic ξ with $\xi(t_i) = p_i$ parametrized proportionally to the arc-length on $[t_{i-1}, t_i]$

$$\Rightarrow \quad P_k \subset \Omega$$

- $$E\big|_{P_k} = \sum \frac{\rho(p_{i-1}, p_i)^2}{t_i - t_{i-1}}$$

- Set $\Omega^c = \{\gamma \in \Omega \mid E(\gamma) \leq c\}$
$$\cup$$
$$P_k^c = \{\xi \in P_k \mid E(\xi) \leq c\}$$

Rmk $\quad E(\gamma) \leq c \Rightarrow \ell(\gamma) \leq c^{1/2}$

Pf $\quad \int_0^1 \|\dot\gamma\| dt = \int_0^1 \|\dot\gamma\| \cdot 1 \, dt$

$$\leq \left[\int_0^1 \|\dot\gamma\|^2 dt\right]^{1/2} \cdot \left[\int_0^1 1^2 dt\right]^{1/2}$$

$$= E(\gamma)^{1/2}$$

So $\gamma \in \Omega^c$ have bounded length

Key pt: P_k^c is a very good approximation of Ω^c

Prop (Milnor's book)

- Fix c = reg. value of E.
- Assume that $\{t_i\}$ is sufficiently fine, ε is sufficiently small and k is large (Depending on c)

$\Rightarrow$ 1) $P_k^c \longrightarrow \Omega^c$ is homotopy eq.

In fact P_k^c is a deformation retract of Ω^c

2) $\mathrm{Crit}(E|_{\Omega^c}) = \mathrm{Crit}(E|_{P_k^c})$

$=$ true geodesics from p to q with $\ell(\gamma) \leq \sqrt{c}$

3) $\forall$ such γ

- $\mathrm{Ker}\, d_\gamma^2 E = \mathrm{Ker}\, d_\gamma^2 E|_{P_k^c}$
- $\mathrm{ind}\, d_\gamma^2 E = \mathrm{ind}\, d_\gamma^2 E|_{P_k^c}$

On the pf

1) We have proved an analogue of 1) for closed geodesics

$E(\gamma) \leq c$ replaces compactness.

2) Similar to closed geodesics

A broken geodesic can be shortened if it has corners

3) Need to work out an explicit formula for $d^2_\gamma E$ (Milnor)

$\triangleleft$

Prop $\Downarrow$ with a bit extra work + Morse theory

Thm Assume p & q are not conj
$\Rightarrow$ Ω has homotopy type of an (infinite) CW-complex with one m-cell for each geodesic from p to q of index m.

Rmk Could have worked with
$\mathcal{E} = \sum \rho(p_i, p_{i+1})^2$
but the calculation of $d^2_\gamma E$ is simpler for E.
(We did not need it in LS theory)

Application to topology
— Homology of Ω

Top digression

Fix p. How does $\Omega(p,q)$ depend on q?
It does not, up to homotopy

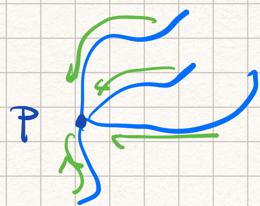
$$\begin{array}{l} \mathcal{P} = \{ [0,1] \xrightarrow{\gamma} M \mid \gamma(0) = p \} \\ \big\downarrow \, ev : \gamma \to \gamma(0) \\ M \end{array}$$

$\leftarrow$ a Serre fibration

$\Rightarrow \Omega(p,q) = ev^{-1}(q) \leftarrow$ all have the same homotopy type

$\Rightarrow$ Can take $q = p$: $\Omega(p,q) = \Omega_p = \Omega$
the based loop space
(This not $\Lambda =$ the space of free loops
and $\underbrace{Crit(E|_{\Omega})}_{\text{geodesic loops}} \neq \underbrace{Crit(E|_{\Lambda})}_{\text{closed geodesics}}$

<u>Rmk</u> P is contractible



Every path contracts along itself to P

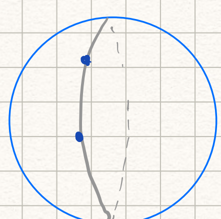
Long exact sequence

$$\to \pi_i(\Omega) \to \cancel{\pi_i(P)}^{\,0} \to \pi_i(M) \xrightarrow{\cong} \pi_{i-1}(\Omega) \to \cancel{\pi_{i-1}(P)}^{\,0} \to \cdots$$

$i > 1$ $\quad \pi_i(M) \cong \pi_{i-1}(\Omega)$ – used before

Understanding $H^*(\Omega)$ and $H_*(\Omega)$ is important in alg. topology.

<u>Application Ω for $M = S^n$</u>



<u>Recall</u> :

one geodesic γ_k
of index $k(n-1)$, $k = 0, 1, \ldots$

$\Rightarrow$ $\boxed{\text{$\Omega$ has homotopy type of a complex with exactly one cell of } \dim = k(n-1),\ k = 0, 1, \ldots}$

$\partial = 0$

Cor $H_i(\Omega) = \begin{cases} \mathbb{F} & i = k(n-1) \\ 0 & \text{otherwise} \end{cases}$ any ring, e.g. $\mathbb{Z}$

$n > 2$

Rmk. The same is true when $n=2$, i.e. for S^2 but the argument is more involved.
Need to show that $\partial = 0$!