Part III: Geodesics and all that

Toward ∞-dim applications

Lecture 13
02/16

§22 Preliminaries

Review of relevant notions from Differential Geometry – minimalist

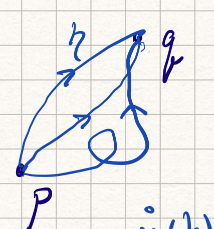
M^n = closed manifold with a fixed Riemannian metric

"Def 1" A geodesic $\gamma : [a,b] \longrightarrow M$ is a curve locally minimizing length: if t_0 & t_1 are close to each other

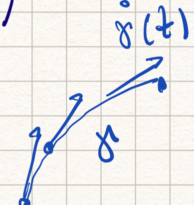
$$\ell(\gamma|_{[t_0,t_1]}) = \rho(\gamma(t_0), \gamma(t_1))$$

Here

$$\ell(\gamma) = \int_a^b \|\dot\gamma(t)\|\, dt$$

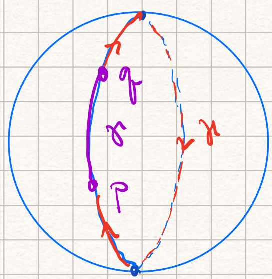


$$\rho(p,q) = \inf_{\substack{\eta \\ \text{connects } p \,\&\, q}} \ell(\eta)$$



Rmk
- $\ell(\gamma)$ is independent of parametrization or orientation
- ρ is a metric & M is complete

Ex.



A geodesic $p \overset{\gamma}{\rightsquigarrow} q$ need not minimize dist between p & q.

Rmk $\circ$ compact (complete as a R.M.)
$\Rightarrow$ any p & q can be connected by a geodesic (Hopf–Rinow Thm)

Ex. $\exists$ a minizing geodesic
use Arzela–Ascoli, also will be clear later

In what follows we will be interested in two types of geodesics: connecting two fixed pts p & q and closed geodesics
$\gamma: S^1 \xrightarrow{C^\infty} M$ $\quad S^1 = \mathbb{R}/\mathbb{Z} = [0,1]/\sim$
$\dot\gamma(0) = \dot\gamma(1)$

Set
sufficiently smooth
$\Lambda = \Lambda(p,q) = \{\gamma : [0,1] \to M \mid \gamma(0) = p,\ \gamma(1) = q\}$
$\Lambda \qquad = \{\gamma : S^1 \to M\}$

Def 2 A geodesic is a "critical pt" of
$\ell : \Lambda \to \mathbb{R}$
Get geodesics connecting p & q, etc

Rmk Not quite satisfactory:

- ℓ is not smooth: $\| \ \|$ not smooth
  $x \mapsto y = |x|$ not smooth

  But smooth at immersions:

  $$\dot\gamma(t) \neq 0 \quad \forall t$$

- ℓ is independent of parametrization
  $\Rightarrow$ huge sets of critical pts
  along with γ contains $\gamma \circ \varphi$
  $[0,1] \circlearrowleft$

Def 3 A geodesic is a critical pt of the energy functional:

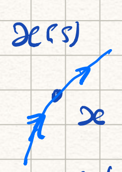
$$E : \Lambda \longrightarrow M$$

Depends on parametrization $\longrightarrow$ $$E(\gamma) = \frac{1}{2}\int_0^1 \|\dot\gamma(t)\|^2 \, dt$$ ignore

An issue to deal with: what is a crit pt

Recall $f : P \longrightarrow \mathbb{R}$ x is a crit. pt if

$$df_x = 0 \iff \underbrace{(L_v f)(x)}_{df_x(v)} = 0 \quad \forall v$$



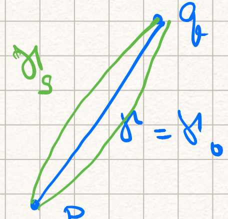
$$\iff \overbrace{\underbrace{\frac{d}{ds} f(x(s))\Big|_{s=0}}_{=:(L_v f)(x)}}^{df_x(v)} = 0 \quad \forall \gamma$$

$x(0) = x$

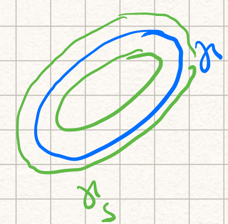
$\frac{d}{ds} x(0) = v$

Def $\gamma \in \mathrm{Crit}(E)$ if $\forall$ any variation γ_s of γ we have $\gamma_0 = \gamma$

$$\boxed{\frac{d}{ds} E(\gamma_s) \Big|_{s=0} = 0}$$



connecting p & q

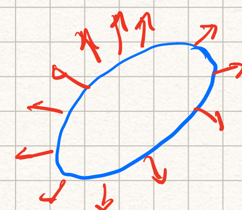


closed loops

$$\frac{d\gamma_s}{ds}\Big|_{s=0}$$

Should think

$T_\gamma \Lambda$ = v.f. along γ (vanishing at p & q)



$$v = \frac{d}{ds}\gamma(s)$$

$$(L_v E)(\gamma) = dE_\gamma(v) := \frac{d}{ds} E(\gamma(s))\Big|_{s=0}$$

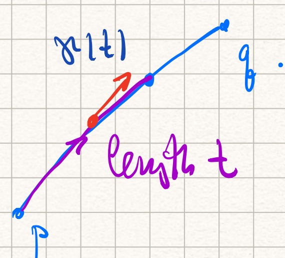
Fact: • geodesics in the sense of Def 3 also satisfy Def's 1 & 2: locally length minimizing and critical pts of L (unless $\gamma \equiv pt$)

• and $\dot\gamma(t) \neq 0$ for any t unless $\gamma \equiv pt$

Fact: As a critical pt γ comes with a natural parametrization:
(E pics up a parametrization)
γ is parametrized proportionally to arc length to get from p to q or close up in time 1.

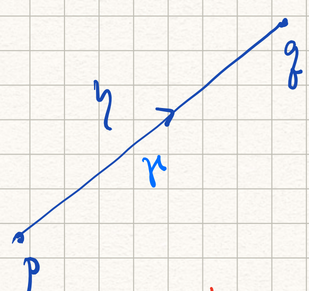
Arclength parametrization:

$$\ell(\eta|_{[0,t]}) = t \Rightarrow \|\dot{\eta}\| = 1$$



Likewise for closed geodesics

$\ell = \rho(p,q) = L(\eta)$
η parametrized by arc length: $[0, \ell]$



$$\gamma(t) = \eta(\ell \cdot t)$$

$$\Rightarrow \boxed{\|\dot{\gamma}(t)\| = \ell}$$

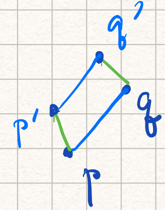
$$\Rightarrow \boxed{E(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|^2 dt = \ell^2}$$

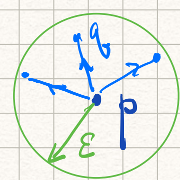
For any γ parametrized by $[0,1]$
$\sim$ arclength

Fact: • $\exists\ \varepsilon > 0$ s.t. $\forall\ p, q \in M$

(injectivity radius) | with $\rho(p, q) < \varepsilon$
$\exists$ unique minimal geodesic $\gamma_{p,q}$
connecting p to q : $\ell(\gamma_{p,q}) = \rho(p,q)$

• Moreover, $\gamma_{p,q}$ depends smoothly on $(p, q) \in M \times M$





Rmk. γ is a geodesic (in the sense of Def 3)
$\Longleftrightarrow$ $\nabla_{\dot\gamma}\dot\gamma = 0 \leftarrow$ acceleration
This is how geodesics are usually defined in R.G.

Rmk In the next two sections we follow Bott's "Lectures on M.T."

§23 Closed Geodesics, I: Cartan's Thm

M^n closed Riemannian manifold

Recall: free homotopy classes $S^1 \xrightarrow{C^\infty} M$

$= [S^1, M]$

$=$ conjugacy classes in $\pi_1(M)$

$\Rightarrow$ $\pi_1(M) \neq 1 \iff \exists$ non-contractible loops

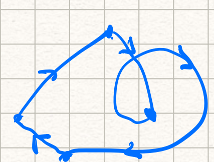
Thm (Cartan) $\pi_1(M) \neq 1$
for any non-trivial free homotopy class $[\alpha] \in [S^1, M]$ $\exists$ a closed geodesic in the class $[\alpha]$.

Idea: take the shortest loop in α as this geodesic or minimize E over all loops in α

Minor adjustment to the setting:

$\Lambda =$ continuous piece-wise smooth loops

Note: E, l etc are still defined ...



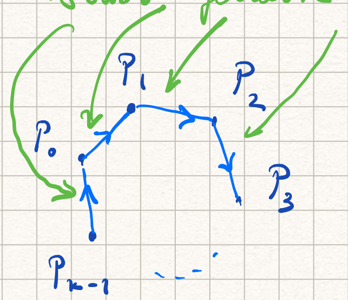
Recall: $[S^1, \Lambda] \longleftrightarrow \pi_0(\Lambda)$ = connected components in Λ

Λ_α = the connected component $\ni [\alpha] \neq 1$

Pf

• Def Geodesic polygons = broken geodesics

$\varepsilon <$ inj. rad (with short edges)



collection of pts

$$\vec{p} = (p_0, \dots, p_{k-1}) \text{ s.t. } \rho(p_i, p_{i+1}) \leq \varepsilon \quad (p_k = p_0)$$

inj. radius

$\vec{p}$ gives rise to a well-defined element in Λ, "broken geodesic" (terminology)

• Every loop α can be approximated by broken geodesic (with large k) in the same free homotopy class.

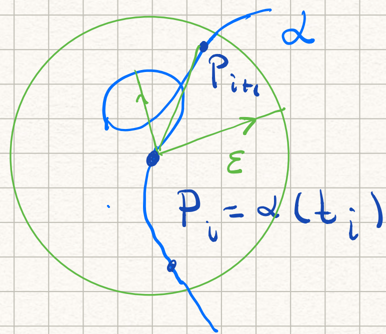
→ Partition $[0,1] = S^1$ into

$$0 = t_0 < t_1 < \dots < t_{k-1} < t_k = 1 = 0$$

$$\begin{array}{cccc} \alpha(t_0) & \alpha(t_1) & \alpha(t_{k-1}) & \alpha(t_k) \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ p_0 & p_1 & p_{k-1} & p_0 = p_k \end{array}$$

so that $\rho(p_i, \alpha(t)) < \varepsilon \ \forall t \in [t_i, t_{i+1}]$

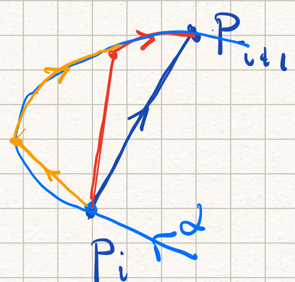
$p_i = \alpha(t_i)$

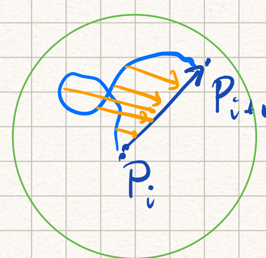


$\Rightarrow$ broken geodesic $\vec{p} = (p_0, \dots, p_{k-1}) = \xi$
C^0-approximating α

parametrized by $[0,1]$ ~ arc length

<u>Homotopy from α to ξ:</u>





Not clear how to do this explicitly

<u>Cor</u>: Every Λ_α contains a broken geodesic

<u>Next</u>: $P_k \subset \underbrace{M \times \dots \times M}_{k}$ s.t.

$$E(\vec{p}) = \rho(p_0, p_1)^2 + \dots + \rho(p_{k-1}, p_0)^2 \leq \varepsilon^2$$

Important:

(1) $\vec{p} \in P_k \Rightarrow \rho(p_i, p_{i+1}) < \varepsilon$

$\Rightarrow$ get a broken geodesic ξ $\forall \vec{p} \in P_k$

$\Rightarrow P_k \subset \Lambda$

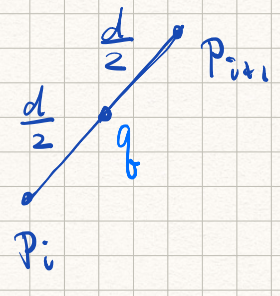
(2) ξ is not necessarily short:

any α can be approximated by an element in P_k ← large

Pf. Approximated α for some k & δ (in place of ε)

$$\rho(p_i, p_{i+1}) < \delta$$

and start subdividing



$$\rho(p_i, p_{i+1}) = d < \delta$$

$$\rho(p_i, q) = \rho(q, p_{i+1}) = \frac{d}{2}$$

$\rho(p_i, p_{i+1})^2 = d^2$ gets replaced by

$$\rho(p_i, q)^2 + \rho(q, p_{i+1})^2 = \left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2 = \frac{d^2}{2}$$

Can make the total sum $\sum \rho(p_i, p_{i+1})^2$ arbitrarily small

Lecture 14
02/28

Instructive: relation between $\mathcal{E}(\vec{p})$ & $E(\xi)$

$$\mathcal{E}(\vec{p}) = \sum \rho(p_i, p_{i+1})^2$$

- depends on partition and parametrization of ξ
- can be really small

E.g. $\ell(\xi) = \ell$, $\rho(p_i, p_{i+1}) = \frac{\ell}{k}$

$$\Rightarrow \mathcal{E}(\vec{p}) = k \cdot \left(\frac{\ell}{k}\right)^2 = \frac{1}{k} \ell(\xi)^2 \approx \frac{1}{k} E(\xi)$$

if arclength

overall depends on where p_i's are:

$$\boxed{E(\xi) = \sum_{i=0}^{k-1} \frac{\rho(p_i, p_{i+1})^2}{t_{i+1} - t_i}}$$

More generally

when parametrized by $[0,1]$, arclength on each $[t_{i-1}, t_i]$

Rmk • P_k approximates $\{E<a\}$ in Λ very well (homotopy eq) when k is large

Upshot $P_{k,\alpha} = \Lambda_\alpha \cap P_k \neq \emptyset$ (k large)

smooth manifold with boundary : $\mathcal{E} = \varepsilon \leftarrow$ generic

$\Rightarrow$ $P_{k,\alpha}$ = union of some connected components of P_k

Idea: Replace E by $\mathcal{E}$ on P_k

- soon see $\underbrace{\mathrm{Crit}(\mathcal{E}) = \mathrm{Crit}(E)}_{\text{closed geodesics}}$

But not crucial.

Then: minimize $\mathcal{E}$ on $P_{k,\alpha}$

Rmk • $\mathcal{E}|_{\partial P_k} = \max \mathcal{E}|_{P_k} = \varepsilon^2$

• $\mathcal{E} \geq 0$

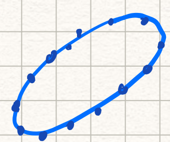
• $\min \mathcal{E}|_{P_{k,\alpha \neq 1}} > 0$ $\leftarrow$ don't need this

Lemma $\quad \gamma \in \mathrm{Crit}(\mathcal{E}) \Rightarrow \gamma$ is a closed geodesic

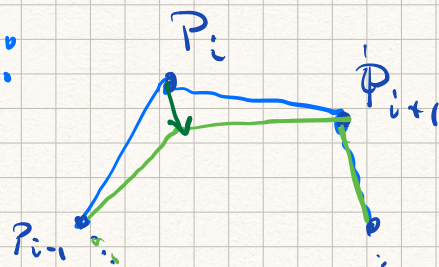
$(\Leftarrow)$

$\Downarrow$

Cartan's Thm



Idea: changes $\mathcal{E}(\vec{p})$



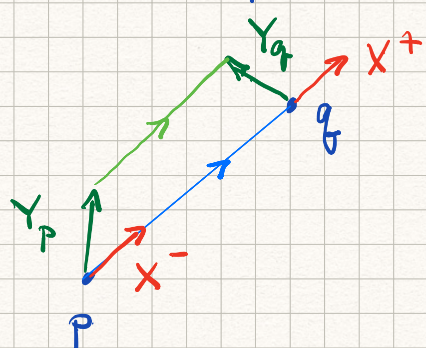
Pf of the lemma

Step 1 $\quad (p,q)$ near diagonal in $M\times M$:

$$\rho(p,q) \leq \varepsilon < \text{inj. radius}$$

How does $\rho(p,q)$ changes when we move p & q?

$X^{\pm}$ = unit tangents



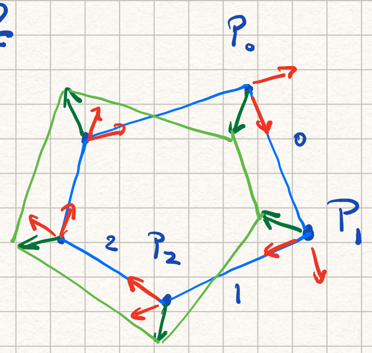
$$\rho^2 = \rho(p,q)$$

$$T_{(p,q)} M\times M = \underset{\substack{\Downarrow \\ Y_p}}{T_p M} \times \underset{\substack{\Downarrow \\ Y_q}}{T_q M}$$

$$L_{(Y_p, Y_q)}\, \rho^2 = 2\left(\langle X^+, Y_q\rangle - \langle X^-, Y_p\rangle\right)\cdot \rho \quad (*)$$

Put the pf of $(*)$ aside for now and finish the pf of the lemma

Step 2



At every p_i we have X_i^+ & X_{i-1}^- and Y_i

Summing up (*) for each segment we get

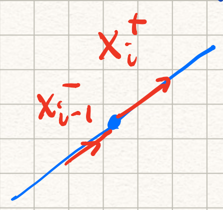
$$L_{(Y_0,\dots Y_{k-1})}\,\mathcal{E}(\vec{p})$$

$$= 2\sum_{i=0}^{k-1}\left(\langle Y_{i+1}, X_{i+1}^+\rangle - \langle Y_i, X_i^-\rangle\right)\rho_i$$

$$= 2\sum_{i=1}^{k}\langle Y_i, X_i^+\rho_i - X_{i-1}^-\rho_{i-1}\rangle$$

$$\Rightarrow \quad \vec{p}\in \mathrm{Crit}(\mathcal{E}) \iff X_i^+\rho_i = X_i^-\rho_{i-1}\quad \forall i$$

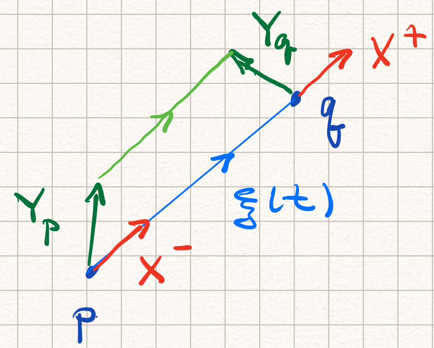
$\iff$ no corner at at p_i and parametrizations of the edges match.



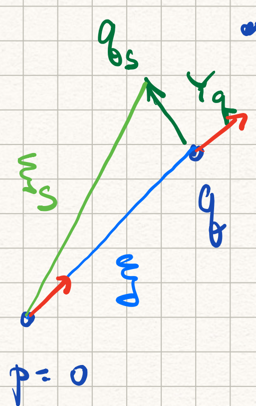
Remains to prove (*)

Pf of (*) in $\mathbb{R}^k$

Similar in general but need a bit more D.G.
See Milnor's book p. 71.



Can assume one of the vectors Y_p or $Y_q = 0$ (by additivity). Say $Y_p = 0$



• $p = 0 \Rightarrow \xi(t) = qt$

$$X^+ = X^- = q/\|q\|$$

$$q_s = q + sY_q$$

$$\rho^2(s) = \|q_s\|^2 = \|q + sY_q\|^2$$

$$\frac{d}{ds}\rho^2(s)\Big|_{s=0} = \frac{d}{ds}\|q + sY_q\|^2\Big|_{s=0}$$

$$= 2\langle q, Y_q\rangle = 2\langle \underbrace{\frac{q}{\|q\|}}_{X^+}, Y_q\rangle \underbrace{\|q\|}_{\rho}$$

$$= 2\langle X^+, Y_q\rangle \cdot \rho$$

◁

Rmk what happens if $[\alpha]=1$
The argument goes through
but gives a <u>point geodesic</u>

Rmk Some argument shows
the existence of a minimizing
geodesic between any two
points.

Rmk Could have worked with E
on P_n, just need to be a bit
more careful near ∂P_n

§24 Closed Geodesics, II: Lusternik - Fet Thm

But what if $\pi_1(M) = 1$?

$$\boxed{\begin{array}{l}\underline{\text{Thm (Lusternik - Fet)}} \\ \quad \pi_1(M) = 1 \implies \exists \text{ a non-const closed geodesic}\end{array}}$$

The argument builds on the machinery we developed in the prev. section

<u>Note:</u> Now Λ is connected

<u>Pf</u>

1) First need a bit on top of Λ.

Evaluation map

$$\left.\begin{array}{l} e: \Lambda \longrightarrow M \\ \quad \gamma \longmapsto \gamma(0) \end{array}\right\} \ \underline{\text{Serre fibration}}$$

$$\begin{tikzcd} \Omega \arrow[r, "j"] & \Lambda \arrow[d, shift left, "e"] \\ & M \arrow[u, shift left, "s"] \end{tikzcd}$$

$\Omega = e^{-1}(p)$
$= \text{loops through } p$
$= \{\gamma \mid \gamma(0) = p\}$

make loops

a section $\quad p \longmapsto \text{constant loop } \gamma(t) \equiv p$

$$es = id \quad : \quad M \hookrightarrow \Lambda$$

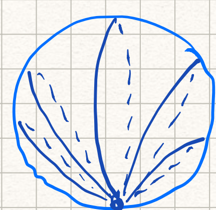
Fibration $\Rightarrow$ long exact seq in homotopy gps

$$\pi_{i+1}(\Omega) \xrightarrow{0} \pi_i(\Omega) \xrightarrow{j_*} \pi_i(\Lambda) \underset{s_*}{\overset{e_*}{\rightleftarrows}} \pi_i(M) \xrightarrow{0} \pi_{i-1}(\Omega) \xrightarrow{j_*} \cdots$$

$es = id \Rightarrow e_* s_* = id$

$\Rightarrow \pi_i(\Lambda) = \pi_i(\Omega) \oplus \pi_i(M)$

Also $\pi_i(\Omega) = \pi_{i+1}(M)$



$\longleftrightarrow$ loop in Ω

$$A \in \pi_2(M) \qquad\qquad \in \pi_1(\Omega)$$

$i \geq 1$

$$\Rightarrow \boxed{\pi_i(\Lambda) = \pi_{i+1}(M) \oplus \pi_i(M)}$$

$\Rightarrow \pi_i(\Lambda) \neq 0$ for some $\underline{i \geq 1}$

Take min i so that $H_i(M; \mathbb{Z}) \neq 0$

$$\begin{matrix} H_i(M;\mathbb{Z}) \neq 0 \\ \| \\ \pi_i(M) \neq 0 \end{matrix}$$

$$\boxed{\begin{array}{l} \underline{\text{Moreover}}\ \exists\, i \geq 1 \text{ such that} \\ \pi_i(\Lambda) \neq 0 \ \&\ \pi_i(M) = 0 \end{array}}$$ (*)

2) Need state exactly how P_k approximates Λ:

<u>Lemma</u> For any i_0 $\exists$ k_0 such that

$$\pi_i(P_k) = \pi_i(\Lambda) \quad \text{for} \quad \begin{array}{l} i \le i_0 \\ k \ge k_0 \end{array}$$

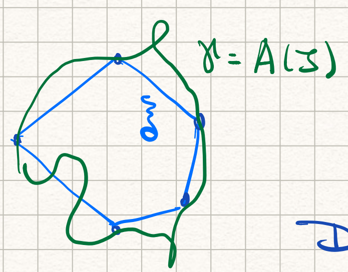
<u>Pf</u>

Here we have

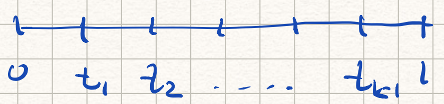
$$\begin{tikzcd} P_k \arrow[r, hook] & \Lambda \\ & S^i \arrow[u, "A"', "\sim"] \arrow[ul, dashed, "\text{need}"] \end{tikzcd}$$

• The pt $A(\xi)$, $\xi \in S^i$ is a loop,
Need to contract it continuously (in ξ) to a loop in P_n

•



We have constructed such a contraction for an individual γ
Depending <u>only</u> on a sufficiently fine partition



$t_j = \frac{j}{k}$

The value k is determined by

$$\max_{t \in S} \|\dot{\gamma}(t)\| = \|\dot{\gamma}\|_{C^0}$$

Now it suffices to take the same k for all $A(\zeta)$:

$$\max_{\zeta \in \xi^i} \left\| \frac{d}{dt} A(\zeta) \right\|_{C^0} \rightsquigarrow k$$

Gap

Rmk • In this construction and the original one, we did not specify how to parametrize ξ.

• Can parametrize ~ arclength

$$\Lambda \underset{\sim}{\Longleftrightarrow} \{\text{loops parametrized} \sim \text{arclength}\}$$

• Deform γ to ξ with whatever parametrization, reparametrize

3) Punchline

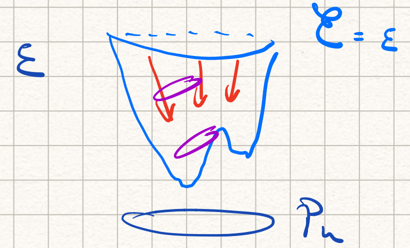
- Take $i \geq 1$ so that $\pi_i(\Lambda) \neq 0$ and $\pi_i(M) = 0$ By $(*)$ in 1)

Take k so large that

$\pi_i(P_k) \neq 0$ ← exists by 2)

- $\mathcal{F} = \{A : S^i \to P_k \hookrightarrow \Lambda\}$, $[A] = \alpha \neq 0$ in $\pi_i(P_k)$

Closed under the positive anti-grad flow for $\mathcal{E}$ on P_n



$\mathcal{E} > 0$

P_n has boundary but it does not matter!

$\mathcal{E}|_{\partial P_n} = \max$

- Apply the Minimax Principle for $\mathcal{F}$

Get a critical value attained on some γ:

$$c = \mathcal{E}(\gamma) = \inf_{A \in \mathcal{F}} \max \mathcal{E}\big|_A \geq 0$$

- Need to know that $c > 0$

$\Rightarrow$ γ is non-trivial.

But if not, A gets contracted into

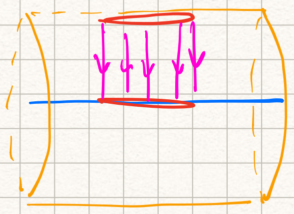
const loops $\longrightarrow$ $M \subset P_n \subset \Lambda$ by the anti-grad flow

$\mathcal{E} = 0$

$\Rightarrow A$ is represented in M

But $[A] \neq 0$ in $\pi_i(P_n)$

and $\pi_i(M) = 0$



$\longrightarrow \longleftarrow$

$\triangleleft$