§19 Connecting with LS Theory

Lecture 11

02/09

- Critical Value Selectors

Review minimax

This is the first application of of the Min/max Principle

Goal: reprove

$$\boxed{|\mathrm{Crit}(f)| \geq \mathrm{cl}(P) + 1}$$

To be more precise we'll prove

$$\boxed{\mathrm{Crit}(f) \text{ isolated} \Rightarrow \mathrm{cv}(f) \geq \mathrm{cl}(P)+1}$$

By a much more flexible construction

- As before $f : P \xrightarrow{C^\infty} \mathbb{R}$, P closed
- Fix $\mathbb{F}$ ← the background ring (or field)

$H_*(P) = H^{n-*}(P)$ when a field

Critical value selectors

- $\alpha \in H_k(P) = H^{n-k}(P)$
- Stick to some construction of $H_*(P)$ as the homology of $C_*(P)$

$$\mathcal{F}_\alpha = \{ A \mid [A] = \alpha \}$$

Rmk
- A is not necessarily a subset of P but rather a combination of maps to P ($A : M \longrightarrow P$ or simpl. chain) and then we mean the image of this map
- Or it can be literally a subset:
  - pseudo-cycles [McDuff–Salamon]

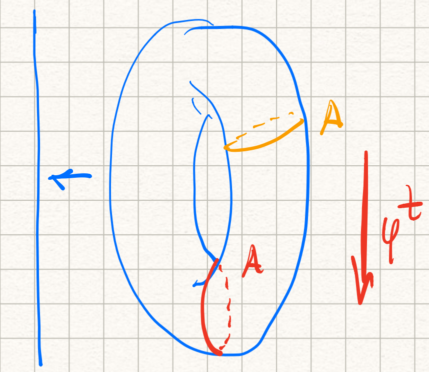
Def 1 Critical value selectors

$c_\alpha(f) := c_{\mathcal{F}_\alpha}(f) :=$ min/max for $\mathcal{F}_\alpha = \{ A \mid [A] = \alpha \}$

$$\boxed{c_\alpha(f) = \inf_{[A]=\alpha} \max_A f}$$

Ex.

$P = \mathbb{T}^2$



$\alpha =$ homology class of these loops

$c_\alpha(f)$ is attained by pushing down A by $\varphi^{t>0}$ till it "hangs on" a critical pt

Def 2 • Assume f is Morse

a finite collection

$$A = \sum m_i x_i \quad , \quad [A] = \alpha \text{ in } CM_*(f)$$

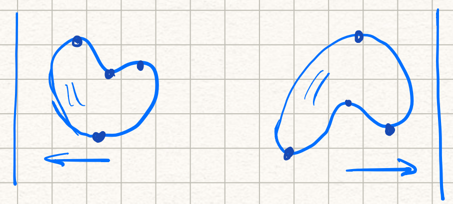
$$c_\alpha(f) = \inf_{A} \max_{m_i \neq 0} f(x_i)$$

min

• If f is not Morse, approximate it by Morse $\tilde{f} \xrightarrow{C^\infty} f$ and set

$$c_\alpha(f) = \lim_{\tilde{f} \to f} c_\alpha(\tilde{f})$$

Examples
- $c_{[M]}(f) = \max f$
- $c_{[pt]}(f) = \min f$



Def. 3

$$c_\alpha(f) = \inf\{\underset{\nwarrow \text{regular}}{a} \mid \alpha \in \mathrm{im}\left(H_*(P_a) \longrightarrow H_*(P)\right)\}$$

Pf of equiv: Def 2 $\underset{\text{by def}}{\Longleftrightarrow}$ Def 3 $\underset{\text{by def when } f \text{ is Morse}}{\Longleftrightarrow}$ Def 1

Properties

- Criticality ⇐ Min/max : $c_\alpha(f) = \text{crit value of } f$
- Monotonicity : $f \le g \Rightarrow c_\alpha(f) \le c_\alpha(g)$ By def
- C^0 continuity : $|c_\alpha(f) - c_\alpha(g)| \le \underbrace{\|f-g\|_{C^0}}_{\max|f-g|}$

Hint : use $f - \varepsilon \le g \le f + \varepsilon$ and monotonicity

Rmk $c_\alpha(f) = f(x)$, $x \in \mathrm{Crit}(f)$

Cannot make x const in f

- <u>Sub-additivity</u> $\quad \alpha, \beta \in H_*(P)$

(1) $\boxed{c_{\alpha\cap\beta}(f+g) \leq c_\alpha(f) + c_\beta(g)}$

In particular $\Downarrow$ Take $g = 0$

(2) $\boxed{c_{\alpha\cap\beta}(f) \leqslant c_\alpha(f)}$

Informal explanation: for (1)

cycles

$\alpha = [A], \quad \beta = [B], \quad \alpha\cap\beta = [A \pitchfork B]$

$$c_{\alpha\cap\beta}(f+g) = \inf_{[C]=\alpha\cap\beta} \max(f+g)\big|_C$$

to be $C = A\cap B$
$$\leq \max(f+g)\big|_{A\cap B}$$
$$\leq \max f\big|_A + \max g\big|_B$$

Now take inf over all A & B

- Can be turned into a pf once "cycles" and $A \cap B$ are defined

(2) is even easier:

$$\max f\big|_{A\cap B} \leq \max f\big|_A$$

Pf of (1) using Y-graph flows

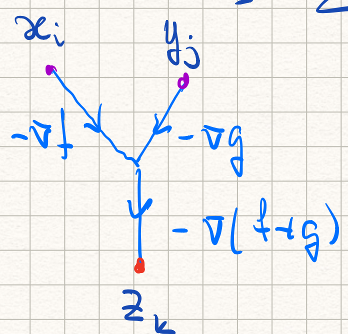
Can assume that f & g are Morse

$\alpha = HM_*(f) = H_*(P)$, $A \in CM_*(f)$

$\beta \in HM_*(g) = H_*(P)$, $B \in CM_*(g)$

$$A = \sum a_i x_i, \quad x_i \in \mathrm{Crit}(f) \qquad\qquad B = \sum b_i y_i, \quad y_i \in \mathrm{Crit}(g)$$

$$\begin{aligned} \alpha \cap \beta &= [A \cdot B] \\ &= \sum a_i b_j \, x_i \cdot y_j \\ &= \underbrace{\sum a_i b_j \, \# \left\{ \begin{smallmatrix} x_i & & y_j \\ & \curlyvee & \\ & z_k & \end{smallmatrix} \right\} z_k} \end{aligned}$$



Pick A & B so that (exists)

$$c_\alpha(f) = \max f(x_i)$$

$$c_\beta(g) = \max f(y_j)$$

one of these C

$$\underline{(f+g)(z_k) \le f(x_i) + g(y_j)}$$

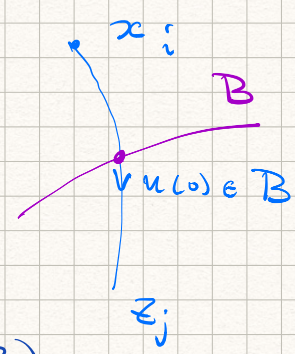
Then
$$\begin{aligned} c_{\alpha \cap \beta}(f+g) &= \inf_{C = \sum c_k w_k} \max (f+g) \\ &\le \max f(z_k) \\ &\le \max f(x_i) + \max g(y_j) = c_\alpha(f) + c_\beta(g) \end{aligned}$$

$\triangleleft$

Ex. Give a direct pf of (2)
using the $H_*(P)$-mod-str on $HM_*(f)$

$$HM_*(f) \otimes H_*(P) \longrightarrow HM_*(f)$$

$\alpha = [A] \qquad \beta = [B]$



$$A = \sum a_i x_i$$

$$B \cdot A = \sum a_i (B \cdot x_i)$$

$$= \sum a_i \, \#\{ \ \}_B \, z_j$$

f decreasing along $u \Rightarrow (2)$

$\triangleleft$

Cor $\min f \leq c_\alpha(f) \leq \max f$

A more subtle result

Thm ("LS inequality")

Assume
- $\mathrm{Crit}(f)$ isolated
- $|\beta| < n$: $\beta \neq [I]$

$\Rightarrow$ $c_{\alpha\cap\beta}(f) < c_\alpha(f)$

strict

I don't know a very simple pf

Explanation:

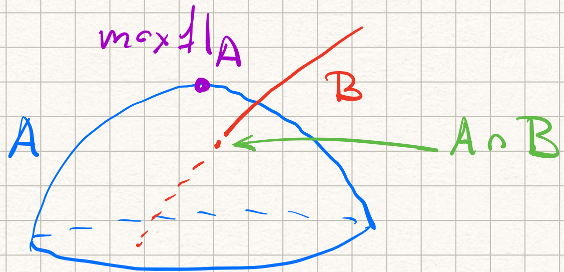
Assume $c = \inf_{[A]} \max f|_A$ attained on some A (Not essential)

$\Rightarrow$ $c = \max f|_A$ and $\max f|A$ is attained at one pt on A $\Leftarrow$ $\mathrm{Crit}(f)$ isolated

$|\deg\beta| < n \Rightarrow \mathrm{codim}\, B \geq 1$

$\Rightarrow$ B does generically not pass through max of $f|A$

$\Rightarrow$ $\max f|_{A\cap B} < \max f|_A$ $\lhd$



Not hard to turn into a a real pf

Now we are ready to re-prove

Prop (LS) $f: P \xrightarrow{C^2} \mathbb{R}$
$\mathrm{Crit}(f)$ isolated

$$\Rightarrow \quad \boxed{|\mathrm{cv}(f)| \geq \mathrm{cl}(P) + 1}$$

Pf. $k = \mathrm{cl}(P)$, use intersection product $\cap$

$$\underbrace{\beta_1 \cap \dots \cap \beta_k}_{\text{necessarily } [pt] \text{ by PD}} \neq 0 \quad \text{in} \quad H_*(P)$$

$|\beta_i| < n$
not ~ units

$$\Rightarrow \underbrace{[M]}_{\alpha_0}, \underbrace{[M] \cap \beta_1}_{\alpha_1 = \alpha_0 \cap \beta_1}, \underbrace{[M] \cap \beta_1 \cap \beta_2}_{\alpha_2 = \alpha_1 \cap \beta_2}, \dots, \underbrace{[M] \cap \beta_1 \cap \dots \cap \beta_k}_{\alpha_k = \alpha_{k-1} \cap \beta_k}$$

Prop

$$\Rightarrow \underbrace{\underbrace{c_{\alpha_0}(f)}_{\max f} > c_{\alpha_1}(f) > \dots > \underbrace{c_k(f)}_{\min f}}_{k+1 \text{ distinct values}}$$

◁

Refinement:

Q. what happens when we have an equality of two crit value selectors?

$$c_{\alpha \frown \beta}(f) \overset{\leq}{=} c_\alpha(f)$$

$<$ — if isolated $|\beta| < n$

Let K = crit set on the level

$f = c$, $c = c_{\alpha\frown\beta}(f) = c_\alpha(f)$

$\alpha, \beta \in H_*(P)$

Prop (Ex) The restriction of $PD(\beta) \in H^*(P)$ is $\neq 0$ in $H^*(K)$

Ex. $\beta = [P] \Rightarrow PD(\beta) = 1 \in H^0(P)$

$$H^0(P) \longrightarrow H^0(K)$$

$$1 \xmapsto{\text{onto}} 1 \qquad \neq 0 \text{ always}$$

and $\alpha \frown \beta = \alpha$

$c_{\alpha\frown\beta}(f) = c_\alpha(f)$ $\triangleleft$

Cor $|\beta| < n$ ($\Leftrightarrow |PD(\beta)| > 0$)

$\Rightarrow$ card(K) = continuum (cannot be countable)

Cor $|cv(f)| \leq cl(P)$

$\Rightarrow$ continuum of crit pts

<u>Rmk</u> Need to be careful about the def of $H^*(K)$, for K could be a very bad set.

§ 20 Another Application: the Courant-Fischer Thm

Lecture 12
02/11

Setting

- $Q(x) = \langle Ax, x\rangle : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, $A^T = A$,
  a quadratic form
- $f = Q|_{S^n} : S^n \longrightarrow \mathbb{R}$.
  Also know as the Rayleigh-Ritz quotient:
  $$f(x) = \frac{\langle Ax, x\rangle}{\langle x, x\rangle} : \mathbb{R}^{n+1} \setminus 0 \longrightarrow \mathbb{R}$$
  Here we think of f as $S^n \longrightarrow \mathbb{R}$
- $\lambda_0 \leq \ldots \leq \lambda_n$ the eigenvalues of Q (or A)

Thm (Courant - Fischer)

$$\lambda_k = \inf_{\{L_k\}} \max f|_{L_k},$$

E.g.
$\lambda_0 = \min f$
$\lambda_n = \max f$

where $L_k = S^n \cap \{(k+1) \text{ dim lin space}\} \cong S^k$
$\uparrow$ $Gr(n+1, k+1)$

Rmk A similar statement for Hermitian forms on $\mathbb{C}^{n+1}$
(Same pf)
$A = A^*$

E.g. $k=1$
$\{L_0\} \cong \mathbb{RP}^n$

Pf.

- $\mathrm{crit}(f) \longleftrightarrow$ unit eigenvectors

  $\mathrm{cv}(f) \longleftrightarrow$ eigenvalues

Pf Lagrange multipliers

Crit of Q on $\{g=c\}$ = sol of $\nabla Q = \lambda \nabla g$

$$\nabla Q = 2Ax \qquad (A^T = A)$$

$$\nabla g = 2x \qquad g = \|x\|^2$$

$$\nabla Q = \lambda \nabla g \ : \quad \underbrace{Ax = \lambda x}_{x \text{ is an eigenvalue}} , \quad \|x\| = 1$$

Then $f(x) = Q(x) = \langle Ax, x\rangle = \lambda$ ◁

Rmk unit eigenvectors (= Crit(f)) come in pairs $\pm x$ with the same crit value λ

- $\mathcal{F} = \{L_k\} \longleftrightarrow Gr(n+1, k+1)$

To check: closed under φ^t (requires a pf)

HW

∇Q is linear on $\mathbb{R}^{n+1}$

$\rightarrow \nabla f =$ proj of $\nabla Q|_{S^n}$ to S^n

$\rightarrow$ Need to check that φ^t sends equators to equators min/max $\Rightarrow$ $\inf_{\mathcal{F}} \max f|_{L_k} = \mu_k$

are critical values

- Routine: $\mu_k \leq \mu_{k+1}$

next page

Get $\mu_0 \leq \dots \leq \mu_n$: $n+1$

$\lambda_0 \leq \dots \leq \lambda_n$:

Need to check that every c.v. is obtained in this way

μ_i's are among λ_j's — clear when distinct — for a generic Q

then pass to a limit

$\Rightarrow$ $\mu_i = \lambda_j$

$\triangleleft$

<u>Rmk.</u> Rather than working on S^n could work on $\mathbb{RP}^n$: $f(-x) = f(x)$

$\to$ Then L_k projects to $\hat{L}_k \subset \mathbb{RP}^n$

linear proj subspace $\cong \mathbb{RP}^k$

$$[\hat{L}_k] = [\mathbb{RP}^k] = \alpha_k \in H_k(\mathbb{RP}^n$$

$\mathcal{F}_k \subset \mathcal{F}_{\alpha_k}$ $\leftarrow$ as defined previously

all cycles in α_k

$$\lambda_k' = \inf_{\hat{L}_k \in \mathcal{F}_k} \max f|_{L_k} \geq \inf_{A \in \mathcal{F}_{\alpha_k}} \max f|_A = \lambda_k$$

$$\begin{matrix} \lambda_0' & \leq & \dots & \leq & \lambda_n' \\ \shortparallel & & & & \shortparallel \\ \lambda_0 & \leq & \dots & \leq & \lambda_n \end{matrix} \Bigg\} \text{ both sequences formed by crit values (generically distinct)}$$

Fixing the gap: all eigenvalues occur.

By continuity, assume λ_i's are distinct

Assume that one of the eigenvalues does not occur

$$\begin{array}{ccc} \lambda_i & < & \lambda_{i+1} \\ \shortparallel & & \shortparallel \\ \mu_i & = & \mu_{i+1} \end{array}$$

Two ways to reason:

1) As in refinement of LS:
   Then Crit(Q) are not isolated

2) By passing to $\mathbb{RP}^n$
   Then $\{Q < \lambda_{i+1}\}$ is contractible to $\mathbb{RP}^i$ and
   $\{Q < \lambda_{i+1}\} \supset \mathbb{RP}^{i+1}$
   impossible $\rightarrow\leftarrow$

§21 History Digression:

Lusternik-Schnirelmann Thm

The origins of LS theory is

Thm (LS, 1929)

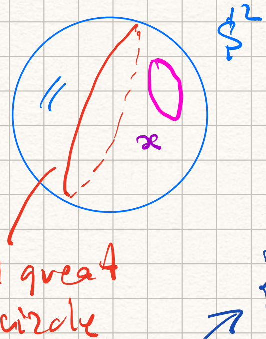
Any metric on S^2 has ≥ 3 simple closed geodesics

embedded (no self-intersections)

- very difficult
- Complete detailed pf: Ballmann 1978

Idea of the pf $\longleftrightarrow$ LS theory

- Λ = smooth embedded loops in S^2 (unoriented, unparametrized)



$$S^1 \hookrightarrow S^2$$

For instance

$\{\text{great circles}\} \subset \Lambda$

$\{S^2 \cap V\}$ ← 2-dim lin. subspace

$Gr(3,2) \cong \mathbb{RP}^2$

$$\mathbb{RP}^2 \xhookrightarrow{\sim} \Lambda$$ ← Homotopy equivalence

(absolutely non-obvious)

- $L : \Lambda \longrightarrow \mathbb{R}$ the length functional

By def $\quad Crit(L) =$ simple closed geodesics

- Punch line : $\quad cl(\Lambda) = cl(\mathbb{RP}^2) = 2$

If we could do LS on Λ we would have

$$|Crit(L)| \geq cl(\Lambda) + 1 = 2 + 1 = 3$$

One can do an analogue of the LS theory on Λ (that was the starting pt for them), but this is very difficult.

$\lhd$

§22 Getting rid of compactness: the Palais-Smale condition

<u>Goal</u>: Find a good replacement for the condition that P is compact

Too restrictive.

<u>Background assumption</u>:

P = finite dimensional (but possibly not compact)
(preferable)

on

Hilbert or Banach manifold
(never compact)

$f : P \longrightarrow \mathbb{R}$ sufficiently smooth

<u>Def</u> f satisfies the Palais-Smale condition (PS) if

- every seq $\{x_i\} \subset P$ s.t.

$$\boxed{|f(x_i)| < C < \infty \quad \text{and} \quad df(x_i) \longrightarrow 0 \quad (\Leftarrow\Rightarrow \ \nabla f(x_i) \longrightarrow 0)}$$

contains a convergent subseq

<u>Rmk</u> $df(x_i) \longrightarrow 0 \Longleftrightarrow \nabla f(x_i) \longrightarrow 0$

when $\exists$ a R.m.: $\dim P < \infty$ or P Hilbert

but not when P is Banach.

Rmk PS condition does not involve the flow of $-\nabla f$. One can have PS satisfied without the flow

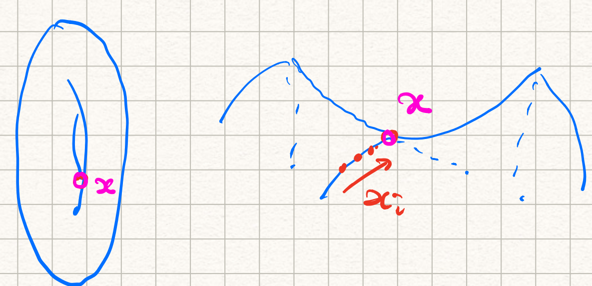
Important: PS condition easily breaks down

Don't need anything exotic

Construction: • Start with $f : \underset{\text{closed}}{P} \longrightarrow \mathbb{R}$

• $\mathring{P} = P \setminus \{ \text{a crit pt} \}$

$\Rightarrow$ $f : \mathring{P} \longrightarrow \mathbb{R}$ does not satisfy the PS but everything else is fine (φ_t is defined for all t)



$|f(x_i)|$ bounded

$\nabla f(x_i) \to 0$

but x_i does not conv

$x_i \to x \notin \mathring{P}$

Upshot: Change of topology from P_a to P_b

Without PS $\not\Rightarrow$ existence of critical pts

Basic example - application - illustration

Assumptions:

- f bounded from below
- f satisfies PS
- The flow for anti-grad-like vector field is defined for all $t \geq 0$

$\Rightarrow$ f has a critical pt

actually min is attained

X s.t. $L_X f \leq 0$

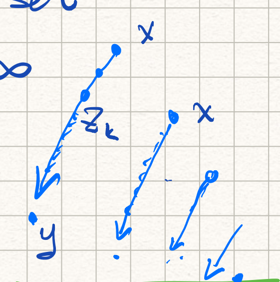
$$L_X f(x) = 0 \iff x \in \mathrm{Crit}(f)$$

E.g. $X = -\nabla f$.

For many purposes as good as $-\nabla f$ but more flexible & does not need metric

Pf
- Start with some x and set

$$z_k = \varphi^k(x_0) \qquad k \to \infty$$



PS + bounded from below

$$\Rightarrow \quad z_k \to y \leftarrow \text{crit pt}$$

- To get min: take x_i s.t. $f(x_i) \to \inf f$. Apply this process and get y_i

$$df(y_i) = 0 \;\&\; f(y_i) \to \inf f$$

$$\overset{PS}{\Rightarrow} \quad y_i \to y \;\&\; f(y) = \inf f$$

$\triangleleft$

Back to Minimax

Setting

with some care
can replace by
anti-grad-like

(*)
- P as above
- $f : P \longrightarrow \mathbb{R}$ satisfies PS
- the anti-grad flow φ^t of f is defined for all times $t \geq 0$

(**)
- $\mathcal{F}$ is closed under φ^t
- f on $\mathcal{F}$ is bounded from below:
$$\inf_{A \in \mathcal{F}} \max f|_A > -\infty$$

Thm (Minimax, II) $\quad c = \inf\limits_{A \in \mathcal{F}} \max f|_A$ is a crit value

Ex. Take $\mathcal{F}$ = pts of P in the previous example.

The pf is very similar to the example

<u>Pf</u>

- Bounded from below $\Rightarrow$ $\underset{\mathbb{R}}{\overset{\in}{c}} > -\infty$

- Will prove: $\forall\, \varepsilon > 0 \;\; \exists\, x$ s.t.

$c - \varepsilon < f(x) < c + \varepsilon$ & $|\nabla f(x)| < \varepsilon$

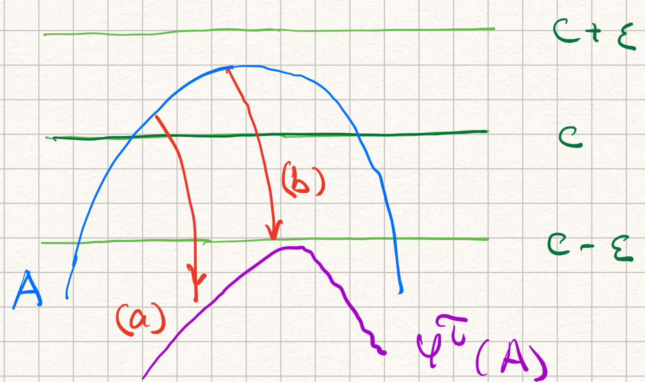
- Arguing by contradiction, assume that:

$$\|\nabla f(x)\| > \varepsilon \quad \text{when } c - \varepsilon \le f(x) < c + \varepsilon$$

for some $\varepsilon > 0$

- $\exists$ $A \in \mathcal{F}$ s.t.

$$\max f|_A < c + \varepsilon$$



- <u>claim</u> $\underline{\tau = \frac{2}{\varepsilon}} \Rightarrow \max f|_{\varphi^\tau(A)} < c - \varepsilon$

Indeed, take $x \in A$

(a) if $f(\varphi^t(x)) < c - \varepsilon$ for some $t \in [0, \tau)$ we're done for $f(\varphi^t(x)) \searrow$

(b) if $c-\varepsilon < f(\varphi^t(x)) < c+\varepsilon$ for $\underline{t \in [0, \varepsilon]}$, we have

$$|\nabla f(\varphi^t(x))| > \varepsilon$$

$$\Rightarrow \quad f(\varphi^t(x)) \underbrace{< c+\varepsilon - \varepsilon^2 t}$$

$$\frac{d}{dt} f(\varphi^t(x)) \leq -\langle \nabla f, \nabla f \rangle \leq -\varepsilon^2$$

$$\Rightarrow \quad f(\varphi^\tau(x)) < c+\varepsilon - 2\varepsilon < c-\varepsilon$$

$$\Rightarrow \quad \max f|_{\varphi^\tau(A)} < c-\varepsilon$$

$\rightarrow\leftarrow$ $\triangleleft$