Part II Lusternik-Schnirelmann Theory

Lecture 9: 02/02 (Short)

§15 Introduction

Setting: $f : \Sigma \xrightarrow{C^\infty} \mathbb{R}$, $P_a = \{f \leq a\}$ (reg ↓ above a; Σ closed)

Q: Get a lower bound on $|\mathrm{Crit}(f)|$ without non-degeneracy

Underlying idea (both Morse & LS):

top change from P_a to P_b $\Rightarrow$ crit pts between a & b

But Morse & LS process this change differently

Rmk. • LS is more versatile and better adapted to higher dim generalizations

• In dim $< \infty$, Morse is deeper

Comparison – Examples

P	Morse	LS
S^n	2	2
$\Sigma_{g\geq 1}$	$2g+2$	3
T^n	2^n	$n+1$
$\mathbb{C}P^n$ or $\mathbb{R}P^n$	$n+1$	$n+1$

different

don't know this yet

<u>Rmk</u> • In general LS is very far from sharp

<u>But</u>: any P^n (closed) admits $f: P \xrightarrow{C^\infty} \mathbb{R}$ with $|\mathrm{Crit}(f)| \leq n+1$ (Takens)

• Sharpness of Morse Theory $\Longleftrightarrow$ Diff topology; Poincaré conj, h-cobordism etc

<u>Next Goal</u>: Proving LS lower bounds

start with a very traditional approach in topology <u>Eg.</u> Fomenko-Dubrovin-Novikov

§16

Topological Preliminaries: LS Category & Cup-length

X = a reasonably good space

Def: LS category of X:

$$\mathrm{cat}(X) = \min\{k \mid X = \bigcup_{i=1}^{k} A_i\}$$ closed or open

where A_i's are contractible to pt in X

$$\begin{tikzcd} A_i \arrow[r, hook] \arrow[dr] & X \\ & pt \arrow[u] \end{tikzcd}$$

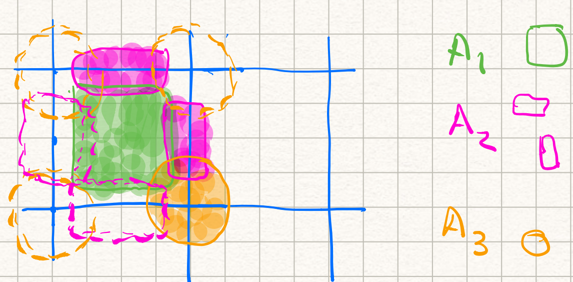
need not be injective

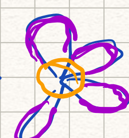
Rmk A_i's need not be connected

Ex.

- $\mathrm{cat}(S^n) = 2$
- $\mathrm{cat}(\mathbb{RP}^n \text{ or } \mathbb{CP}^n) \overset{(=)}{\leq} n+1$ see later
- $\mathrm{cat}(\Sigma_{g\geq 1}) = 3$

T^2



$\Sigma_g \setminus D^2 =$  $2g$

Rmk $\mathrm{cat}(X)$ is in general very difficult to determine. Upper bounds ⇑ expl. constructions

Lecture 10
02/04

more generally

$X =$ n-dim CW complex
$\Rightarrow$ $\text{cat}(X) \leq n+1$

Pf: same argument: proceed inductively

Also note: homotopy invariant

Def The cup-length of X (over $\mathbb{F}$) connected

$$cl(X) = \max\{k \mid \exists\, \alpha_1, \dots, \alpha_k \in H^{*>0}(X)$$
$$\alpha_1 \cup \dots \cup \alpha_k \neq 0\}$$

Clearly: $0 \leq cl(X) \leq \underbrace{\dim X}_{\text{when } X \text{ is a manifold}}$

Rmk:
- depends on the ground field
- $\exists$ more general notions...

Ex.
- $cl(S^n) = 1$
- $cl(\underbrace{\mathbb{RP}^n}_{\text{for } \mathbb{Z}_2} \text{ or } \underbrace{\mathbb{CP}^n}_{\text{any field}}) = n$
- $cl(\mathbb{T}^n) = n \qquad \alpha_i = [dx_i]$
- $X = P^{2n}$ closed symplectic
  $\Rightarrow cl(P) \geq n = \frac{1}{2} \dim P, \quad \alpha_i = [\omega]$
- $cl(\Sigma_{g \geq 1}) = 2$
- More generally P closed $\Leftarrow$ PD
  $cl(P) \geq 2$ over $\mathbb{Z}_2$
  when $H^i(P) \neq 0$, $0 < i < n$ over anything when orientable

Rmk Usually $cl(X)$ is not so hard to calculate — pretty primitive inv.

Prop $$\boxed{cat(X) \geq cl(X) + 1}$$

Rmk The gap "$\geq$" can be huge but examples are not "easy" to construct

not among manifolds you think about at this level

Con-Ex. $cat(\mathbb{RP}^n$ or $\mathbb{CP}^n) = n+1$

Pf

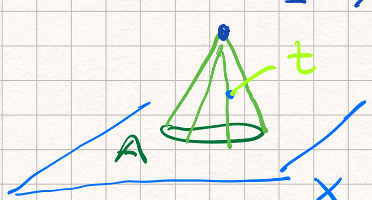
claim $A \subset X$ contractible to a pt
$$\Rightarrow \quad H^*(X,A) \xrightarrow[\text{onto}]{i^*} H^*(X) \qquad * > 0$$

Rmk. In de Rham $H^*(X,A)$ comes from forms vanishing on A

Pf of the claim

$C_A(X)$ = cone over $A \subset X$

$$= X \cup A \times [0,\overset{t}{}1] / \sim \qquad \begin{matrix}(a,0) \sim a \\ A \times 1 \sim 1 \text{ pt}\end{matrix}$$



$$H^{*>0}(X,A) = H^{*>0}(C_A(X))$$
essentially by definition

$f_t : A \longrightarrow X$ contraction to a pt

$f_0 : A \hookrightarrow X$

$f_1(A) = pt$

Define $F : C_A(X) \longrightarrow X$

$$\begin{cases} F(x) = x & \text{for } x \in X \\ F(a,t) = f_t(a) \end{cases}$$

$$X \underset{F}{\overset{i}{\rightleftarrows}} C_A(X)$$

$F i = id \Rightarrow \underbrace{i^* F^* = id \text{ in } H^*(X)}$

$\Downarrow$

i^* is onto

$\triangleleft$

Pf of Prop: Assume the contrary:

$cl(X) \geq cat(X)$

no need to do it by contradiction

$X = A_1 \cup \ldots \cup A_k \leftarrow$ contr in X

$$\begin{array}{ccc} \alpha_1 \cup \ldots\ldots \cup & \alpha_k & \neq 0 \text{ in } H^{*>0}(X) \\ \uparrow & \uparrow & \\ \wr & \wr & \\ \tilde{\alpha}_1 & \tilde{\alpha}_k & \text{in } H^{*>0}(X, A_i) \end{array}$$

Recall $H^*(X,A) \otimes H^*(X,B) \longrightarrow H^*(X, A\cup B)$

E.g. think diff forms vanishing on A & B

$$0 \neq \underbrace{\alpha_1 \cup \dots \cup \alpha_k}_{\in H^*(X)} \leftarrow \underbrace{\tilde{\alpha}_1 \cup \dots \cup \tilde{\alpha}_k}_{\in H^*(X, \underbrace{A_1 \cup \dots \cup A_k}_{X})} = 0$$

$$H^*(X) \leftarrow H^*(X, A_1 \cup \dots \cup A_k) = 0$$

$\rightarrow\leftarrow$ $\lhd$

Ref Handbook of Alg. Top
Edited by I. M. James

Discussion

LS cat is a peculiar notion
"not quite happy with itself"

Some things to keep in mind:

- Not monotone: $X \subset Y \not\Rightarrow \mathrm{cat}(X) \leq \mathrm{cat}(Y)$
- $\exists$ a notion of $\underline{\mathrm{cat}_X(A)}$: covering A, but contractible in X

Then $A \subset B \subset X \Rightarrow \mathrm{cat}_X(A) \leq \mathrm{cat}_X(B)$

- But $A = \text{retract of } X \Rightarrow \mathrm{cat}(A) \leq \mathrm{cat}(X)$

$A \subset X$, $\neq$

- $\mathrm{cat}_X(A)$ is continuous in A
- $F \hookrightarrow E \downarrow B$ $\quad \mathrm{cat}(E) \leq \mathrm{cat}_E(F) \cdot \mathrm{cat}(B)$
- homotopy invariant (not quite obvious)
- $\mathrm{cat}(X \cup Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y)$

§17 Lower bound via LS cat

$f: \underset{\text{closed}}{P} \xrightarrow{C^2} \mathbb{R}$

This is how they do it in topology textbooks

Thm (LS) Assume that $\mathrm{Crit}(f)$ are isolated

$$\boxed{|\underbrace{\text{crit values of } f}_{cv(f)}| \geq \mathrm{cat}(P)}$$

Cor $|\mathrm{Crit}(f)| \geq \mathrm{cat}(P) \geq \mathrm{cl}(P) + 1$

Rmk can be strict

Cor

P	$\lvert\mathrm{Crit}(f)\rvert \geq$
$\mathbb{C}P^n$ & $\mathbb{R}P^n$	$n+1$
$\Sigma_{g\geq 1}$	3
$\mathbb{T}^n$	$n+1$

completing the set of inequalities from p. 101

Pf (outline)

- $f : P \to \mathbb{R}$
  with exactly m critical values $c_1 < \ldots < c_m$ & isolated $\mathrm{Crit}(f)$
  $\Rightarrow$ cover of P by m contr. to pt sets

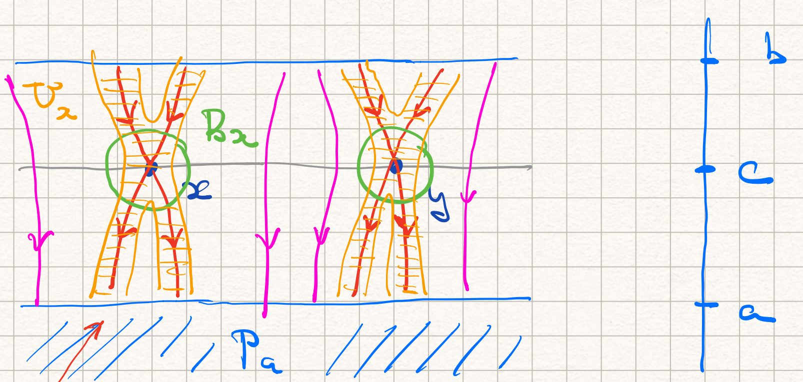
- Do inductively by moving upward

$b \leftarrow$ reg

$c_k \to c \leftarrow$ crit

$a \leftarrow$ reg

Assume the cover for P_a is constructed ($(k-1)$ set
$\Rightarrow$ a cover of P_b by k sets

- 

$$\Sigma_x := \big(\underbrace{W^s(x) \cup W^u(x)}_{\text{stable unstable sets}}\big) \cap \{a \le f \le b\}$$

$U_x := \begin{cases} \text{"flow invariant (in } a < f < b) \\ \text{nbd of } \Sigma_x \end{cases}$

Observations:

→ • U_x is cotrodible by
upward/downward flow
in a small nbd B_x of x

• B_x is contr to x

$\Rightarrow A_k = \coprod_x U_x$ is contr to a pt
in P_b

→ • $P_b \setminus \left(\coprod_x U_x\right)$ is homotopy
equiv to P_a (in fact homeo)

$\Rightarrow$ a cover of $P_b \setminus \left(\coprod_x U_x\right)$
induction
by $k-1$ sets $A_1, \dots, A_{k-1}$

Together $A_1, \dots, A_{k-1}, A_k$ the required
cover of P_b

$\triangleleft$

Q But how few crit pts can a function on P have?

Thm (Takens, Inventiones, 1968)

dim $P = n$

$\Rightarrow \exists\, f : P \xrightarrow{C^\infty} \mathbb{R}$ with $|\mathrm{Crit}(f)| \leq n+1$

Cor $\mathrm{cat}(P) \leq \dim P + 1$

↑ • also know because P = CW of dim $\leq n$

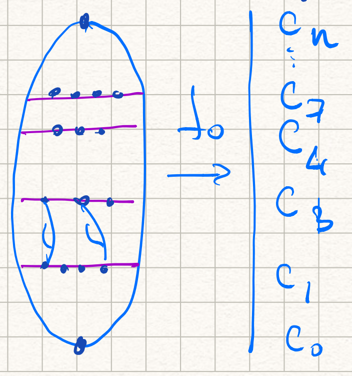
or • selecting disjoint subsets in a cover

Outline of the pf

- Start with $f_0 : P \to \mathbb{R}$ a Morse function with one max & one min
- Sliding handles $\Rightarrow$ can have all critical pts of index k on one level $f = c_k$ and

$c_0 < c_1 < \ldots < c_n$ ← can skip some values

P



- Ex - show that

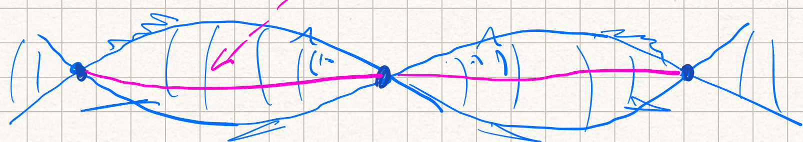
  $\{f = c_k\}$ is connected

Rmk Need to treat the case

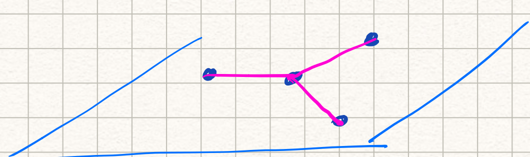
$n = 2$ separately

- Pick a $\underbrace{\text{tree}}$ $\Gamma_k \subset \{f = c_k\}$

a union of smooth arcs intersecting only at their ends homeo to a tree



containing all critical pts on $f = c_k$



- Contract all Γ_k to pts

  $P \underset{\text{diffeo}}{\cong} \underline{P} / \underline{\amalg} \Gamma_k$ more precisely

$\exists$ a smooth map $\underline{P} \xrightarrow{G} P$ X can skip for i's

s.t. $\underline{P} \setminus \underline{\amalg} \Gamma_k \longrightarrow \underline{P} \setminus \underbrace{\{x_0, \dots, x_m\}}$

is a diffeo & $\Gamma_i = G^{-1}(x_i)$

- Now $f_1 = f_0 \circ G^{-1} : P \xrightarrow{C^0} \mathbb{R}$

  is smooth outside X and only C^0 at the pts of X

- Modify f_1 near each x_i to make it smooth and have only one critical pts (Takens, Thm 2.7 – elementary but non-obvious) $\triangleleft$

§ 18. The min/max principle

And this is how they do it in dynamics / calculus of variations

- $f : P \longrightarrow \mathbb{R}$ anti-grad flow

  closed manifold $\longleftarrow$ this condition can be significantly relaxed

- $\mathcal{F}$ = a class of compact subsets of P closed under $\varphi^{t \geq 0}$

<u>Ex.</u>
- Fix $\alpha \in \pi_k(P)$

$$\mathcal{F} = \{ S^k \xrightarrow{u} P \mid [u] = \alpha \}$$

- Fix $\alpha \in H_*(P)$

$$\mathcal{F} = \{ \text{cycles representing } \alpha \}$$

<u>E.g.</u>
- images of singular cycles (over $\mathbb{Z}_2$)
- maps $\sigma : M \longrightarrow P$ s.t. $\sigma_*([M]) = \alpha$

Set
$$c_{\mathcal{F}}(f) = \inf_{A \in \mathcal{F}} \underbrace{\sup}_{\max} f|A$$

$$\boxed{\text{Thm (Minimax Principle, I)} \quad c_{\mathcal{F}}(f) \text{ is a critical value}}$$

<u>Rmk</u>
- Versatile & important
- conditions can be relaxed
- a lot of applications

<u>Ex.</u>
- $\mathcal{F}$ = collection of all pts in P

$$c_{\mathcal{F}}(f) = \inf_{A \in P} f(A) = \min f$$

- $\mathcal{F} = \{P\}$ ← just one set P itself

$$c_{\mathcal{F}}(f) = \max_{P} f = \max f$$

Pf of the min/max principle: ← nearly obvious

- $c := \inf\limits_{A \in \mathcal{F}} \overset{\max}{\sup\limits_{A}} f|_A$ ← A is compact

  $\mathcal{F} = \{A\}$; A is compact

  $\mathcal{F}$ is closed under $\varphi^{t \geq 0}$

  Assume c is not a critical value

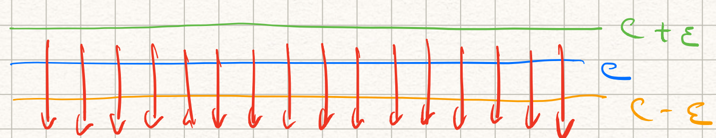
  — a pf by contradiction

- $\exists$ $\varepsilon > 0$ & $\tau > 0$ s.t.

  $f(x) \leq c + \varepsilon \Rightarrow f(\varphi^{\tau}(x)) \leq c - \varepsilon$

  ⇑

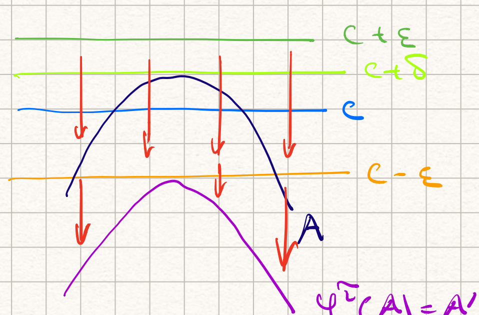
  compactness & c is not a critical value



φ^t = the flow of $-\nabla f$

- As said assume c is not a critical value

  take A so that $\max\limits_{A} f = c + \delta < c + \varepsilon$



$\max\limits_{\underbrace{\varphi^{\tau}(A)}_{A' \in \mathcal{F}}} f = c - \varepsilon < c$ ⇄

because $\mathcal{F}$ is $\varphi^{t \geq 0}$ invariant

◁

- Our next goal is to illustrate how Min/max works by several simple applications of the Idea.

- We'll keep on coming back to it over & over