Setting 4: Jet Transversality

Short Lecture

01/26

Lecture 7

Next step - incorporating

A very basic minimalist version

$J^1X = T^*X \times \mathbb{R} \longrightarrow X$

$j^1f = (df, f) : X \to J^1X$ a section

$Z \subset J^1X$ a proper submanifold

Thm 4 $j^1 f \pitchfork Z$ for an open and dense set of $f \in C^k(X)$ $k \geq 2$

Cor Can replace J^1X by T^*X: $df \pitchfork Z \subset T^*X$ for an open and dense set $f \in C^k(X)$ $k \geq 2$

Pf Replace $Z \subset T^*X$ by $Z \times \mathbb{R} \subset J^1X$

df by j^1f

Rmk much more limited class of maps:

Sections of $T^*X = \Omega^1(X)$

$\alpha \in \Omega^1(X)$:

$\alpha = df \Rightarrow \alpha$ is closed $\Rightarrow \alpha$ is exact

$\boxed{d\alpha = 0}$ a condition on the derivative

Pf of the thm on Morse functions:

- set $X = P$, $Z = P \subset T^*P$ zero section
- f Morse $\Longleftrightarrow$ $\underbrace{df \pitchfork Z = P \text{ in } T^*P}_{\text{open \& dense condition in } C^{k\geq 2}(P)}$

◁

More details on transversality approach, by passing Jet Transversality:

Banyaga – Hurtubise

based on the idea in this pf.

On the pf of ordinary transversality

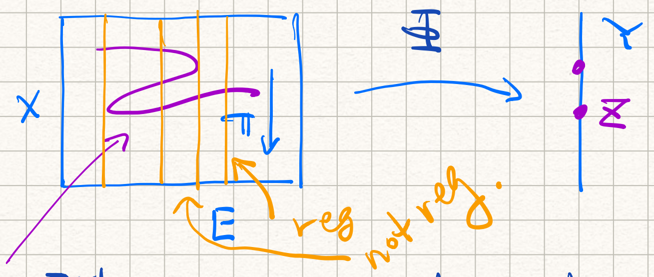
Following Guillemin-Pollock

Assume X & Y are closed for the sake of simplicity

Lemma (Ex) $\Phi : X\times E \longrightarrow \underset{\underset{Z}{\cup}}{Y}$, $\Phi_e = \Phi|_{X\times e}$

$$\Phi \pitchfork Z \Rightarrow \Phi_e \pitchfork Z \text{ for almost all } e\in E$$

Hint:



$M = \Phi^{-1}(Z)$ = smooth submanifold of $X\times E$

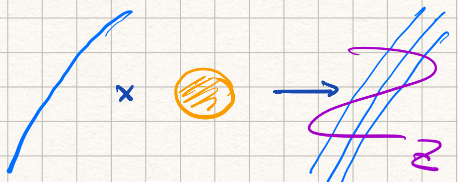
Claim: $\Phi_e \pitchfork Z \Leftrightarrow e$ is a reg value of $\pi|_M$

◁

Lemma ⇒ Transv. thm

$Y = \mathbb{R}^n$, $E = \mathbb{R}^n$,

$\Phi(x,e) = F(x) + e$



Φ is a submersion (already on the second factor)

$\Rightarrow \Phi \pitchfork Z \quad \forall z$

$\Rightarrow \Phi_e \pitchfork Z$ for almost all e

Take e close to 0 !

Now when Y is compact

- Take $Y \hookrightarrow \mathbb{R}^n$
- Care only about small $e \in B^n(\varepsilon)$
- Take $F : X \longrightarrow Y$ and extend to to a submersion

$$\begin{tikzcd}
\Phi : X \times B^n \arrow[rr, Rightarrow] \arrow[dr] & & Y \\
& U \arrow[ur, "\pi"'] &
\end{tikzcd}$$

smth like $F(x) + e$ is good

- Finish the pf as before

$\triangleleft$

§ 13 Existence of Morse functions
Part II: via Height Functions

Lecture 8
0/28

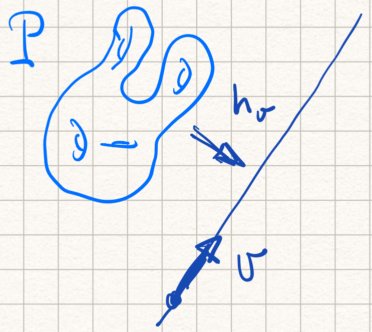
Recall

P closed manifold, $k \geq 2$ e.g. ∞

> **Thm 1** Morse functions form an open and dense set in $C^k(P)$

$\exists$ $P \hookrightarrow \mathbb{R}^m$, from now on $P \subset \mathbb{R}^m$

Denote by h_v the proj to $v \cdot \mathbb{R}$:
$v \in S^{m-1}$



E.g. The heigh function h

> **Thm 2** For an open dense (full measure) subset of S^{m-1}, h_v is Morse

Thm 2 ⇒ Thm 1:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, hook, "j"'] & \mathbb{R} & \arrow[r, phantom, "\rightsquigarrow"] & {} & P \arrow[r, hook, "{(j,f)}"'] \arrow[rr, bend left, "f"] & \mathbb{R}^{m+1} \arrow[r, "h"'] & \mathbb{R} \\ \mathbb{R}^m \end{tikzcd}$$

$h \leftarrow$ height

Take $v \in S^m$ close to $e_{m+1} \subset \mathbb{R}^{m+1}$ so that $h_v : P \to \mathbb{R}$ is Morse

$\Rightarrow \quad h_v \overset{C^\infty}{\approx} h$

$$\begin{tikzcd} f = h : P \arrow[r, "ss"'] & \mathbb{R} \\ h_v : P \arrow[r] & \mathbb{R} \end{tikzcd}$$

$\triangleleft$

Rmk. Milnor's "Morse theory":

A related but different argument

$$\begin{tikzcd} M \arrow[r, "\mathrm{dist}_p^2"'] \arrow[d, phantom, "\cap"] & \mathbb{R} \\ \mathbb{R}^m \end{tikzcd}$$

$\mathrm{dist}_p^2(x) = \| x - p \|^2$

is Morse for almost all p.

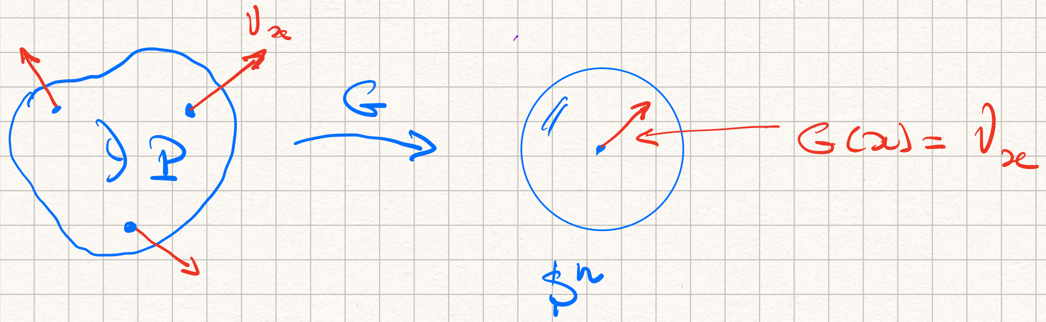
Pf of Thm 2

* Particular case P = hypersurface:

$$P^n \subset \mathbb{R}^{n+1}$$

$G : P \longrightarrow S^n$ the Gauss map

$x \longmapsto \nu_x$ ← unit outer normal



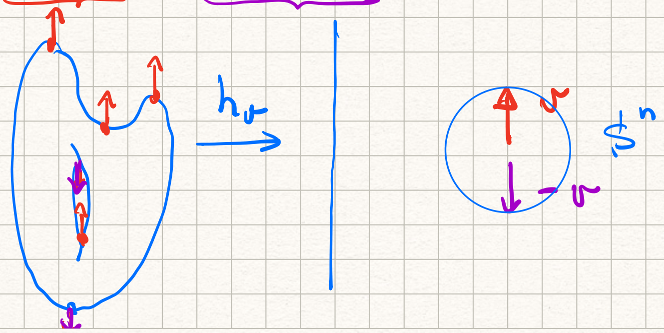
Prop h_v is Morse ⟺ $\pm v$ are reg values of G

⇓ ← Sard's

Thm 2 when P is hypersurface

Pick: $v \in S^n$, $v = e_{n+1}$ ← convenient

$$\underbrace{G^{-1}(v)} \cup \underbrace{G^{-1}(-v)} = \mathrm{Crit}(h_v)$$



Pf of the proposition

$G(p) = v = e_{n+1}$

Suffices to show:

$\boxed{p \text{ is a reg pt of } G} \iff \boxed{d^2 h_v \text{ is non-deg at } p}$

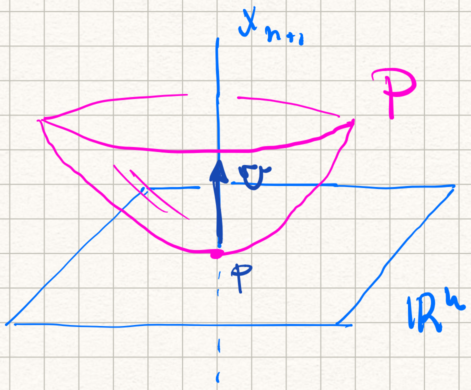
$\Updownarrow$ def

$\boxed{DG : T_p P \to T_v S^n \text{ is onto}}$

From now on

$v = e_{n+1}, \quad h_v = h$

Now we can assume locally



$P = \text{graph}(g : \mathbb{R}^n \longrightarrow \mathbb{R})$

$g = h$

$T_p P = \mathbb{R}^n$

$p = 0$

The second fundamental form

$$(\mathrm{II}_p : T_p P \otimes T_p P \longrightarrow \mathbb{R}) \overset{\text{def}}{:=} d^2_p g$$

$$\mathrm{II}_p = d^2_p h$$

Goal: relate II_p to DG_p

Preliminary steps:

- First of all

  Can assume g is a quadratic form

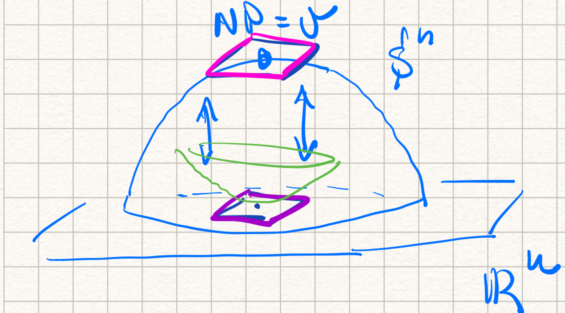
  $$g(x) = \mathrm{II}_p(x) + \underbrace{\dots}_{\text{do not affect } DG}$$

  $\Rightarrow$ Can drop from the calculation

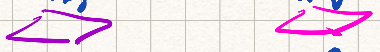
  $$g = \frac{1}{2}\underbrace{\langle Ax, x\rangle}_{\mathrm{II}_p(x)}, \qquad A = A^T$$

  Not really using this...

- Can recycle $\mathbb{R}^n$ as a local chart near $v = NP \in S^n$ using orthogonal proj $\mathbb{R}^{n+1} \to \mathbb{R}$ along x_{n+1}



Now: $DG_p : \underbrace{T_p P}_{\mathbb{R}^n} \longrightarrow \underbrace{T_v S^n}_{\mathbb{R}^n}$



The horizontal $\mathbb{R}^n$

- **Key Claim** $DG_p = -A$

$$\Rightarrow \underbrace{\left(p \text{ is a reg point of } G \Leftrightarrow \mathrm{II}_p \text{ is non-deg}\right)}_{\text{exactly what we need}}$$

Pf of the claim

$$\to \nu_x = \frac{(-\nabla g(x), 1)}{\sqrt{1+|\nabla g(x)|^2}} \quad \leftarrow \text{vector calculus} \quad 23B$$

$\to$ In the $\mathbb{R}^k$-chart

$$\nu_x = -\frac{\nabla g(x)}{\sqrt{1+|\nabla g(x)|^2}} \quad ; \quad g(x) = \frac{1}{2}\langle Ax, x\rangle + \dots$$

$$= -Ax + \dots$$

$\Rightarrow$ at p: $DG = -A$ $\triangleleft$

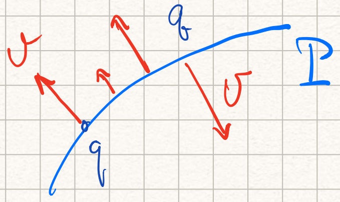
This finishes the pf in the case where P is a hypersurface

General case: $P^n \subset \mathbb{R}^{n+k+1=m}$

The Gauss map and Normal bundle

N_P = normal bundle to $P \subset \mathbb{R}^n \times \mathbb{R}^n$

$= \{(q, v) \in P \times \mathbb{R}^n \mid v \in T_qP$, normal at q to $P\}$



$\dim N_P = m = n+k+1$

$N_P \xrightarrow{\pi} P$ vector bundle

$SN_P = M$ = the unit normal bundle

$= \{ (q, v) \mid \|v\| = 1 \} \subset N_P \xrightarrow{\pi} P$

Should think of M as the boundary of infinitesimally small nbd of P

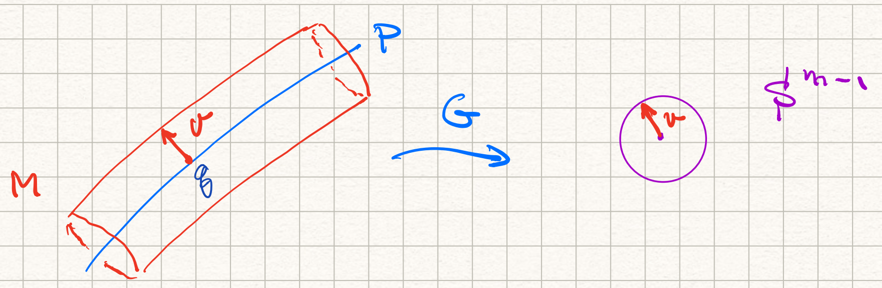
Rmk: P hypersurface

$\Rightarrow$ M = two copies of P: inward & outward

$\dim(M = SN_P) = m-1$

Gauss map: $G: M = SN_P \longrightarrow S^{m-1}$

$(q, v) \xmapsto{G} v$



Prop $\pm v$ are reg values of $G \Leftrightarrow h_v$ is Morse.

$\Downarrow$ ← Sards

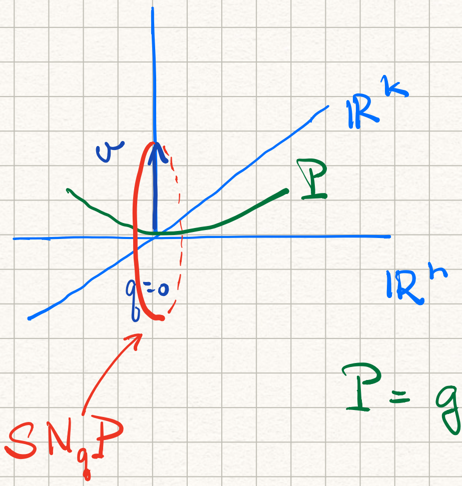
Thm2

$$\underbrace{\pi\left(G^{-1}(v) \cup G^{-1}(-v)\right)}_{\text{base pts}} = \mathrm{Crit}(h_v)$$

Pf (outline)

As before can assume $v = e_m$ ← vertical vec.

$q = 0$.



$T_qP = \mathbb{R}^k$

Locally

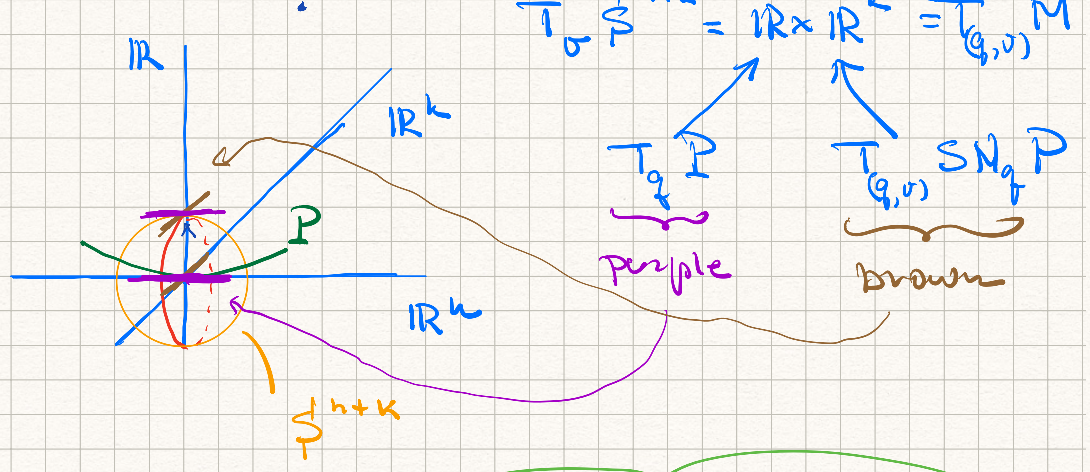
$P = \mathrm{graph}(g: \mathbb{R}^k \longrightarrow \mathbb{R}^k \times \mathbb{R})$

Write $g = (F, h)$ ← the height function

By construction $g(q) = 0$, $Dg(q) = 0$

<u>Note</u>: as before care only about the quadratic part of g

What is DG ?



$$DG_q = \begin{array}{c|c|c} & \mathbb{R}^k & \mathbb{R}^n \\ \hline \mathbb{R}^k & I & DF \\ \hline \mathbb{R}^n & 0 & -A \end{array}$$

$G|_{SN_qP} = id$

DF ← does not matter

$h(x) = \frac{1}{2}\langle Ax, x\rangle + \ldots$

some calculations as for hypersurfaces

$\Rightarrow DG_q$ is onto $\Leftrightarrow$ A is non-deg

$\Leftrightarrow$ $d_q^2 h$ is non-deg

An application (added)

$$\begin{tikzcd} M \subset \mathbb{R}^n \arrow[d, "f"] \\ \mathbb{R} \end{tikzcd}$$

$h_v(x) = \langle x, \underset{\mathbb{R}^n}{\underset{\cap}{v}} \rangle$: proj to $v\mathbb{R}$

What we have shown is that:

For almost all $v \in \mathbb{R}^n$
$f + h_v$ is Morse on M

Cor For a dense set of p
$x \mapsto \|p - x\|^2$ is Morse

Pf. $f(x) = \| x \|^2$

$$f(x) + h_v(x) = \langle x, x \rangle + \langle x, v \rangle$$
$$= \langle x + \tfrac{1}{2} v, x + \tfrac{1}{2} v \rangle - \tfrac{1}{4}\|v\|^2$$

$\Rightarrow \langle x + \tfrac{1}{2} v, x + \tfrac{1}{2} v \rangle$ is Morse

Can replace 0 by any pt. ◁

Next Lecture – Short

§14. Digression: Cup Product in Morse homology

Ref: Jost

skipping details
quite technical

Recall

* P any reasonable space

$\Rightarrow$ $H^*(P)$ is a graded unital alg
Product: cup product "$\cup$"

E.g. • P a manifold, in $H^*_{dR}(P) = H^*(P;\mathbb{R})$

$$[\alpha]\cup[\beta] = [\alpha\wedge\beta]$$

* Assume now P^n is a closed manifold

Intersection product

$$a\cdot b := PD^{-1}(PD(a)\cup PD(b))$$

$\rightarrow$ $|a\cdot b| = |a|+|b|-n$

$\rightarrow$ unit $= [M]$

$$\boxed{H_{*_k}(P)\otimes H_{*_\ell}(P)\longrightarrow H_{*_{\ell+k-n}}(P)}$$

Conceptually $a=[A]$, $b=[B]$ $\rightsquigarrow$ $a\cdot b = [A\cap B]$

Key pt: the intersection product has a simple interpretation in Morse theory.

Even two

setting: $f: P \longrightarrow \mathbb{R}$ morse

$\Rightarrow$ $CM_*(f)$ morse complex

$HM_*(f) = H_*(P)$ morse hom

Interpretation I: $HM_*(f)$ is a module over $H_*(P) \overset{PD}{\cong} H^*(P)$

$$\boxed{H_*(P) \otimes HM_*(f) \longrightarrow HM_*(f)}$$

Define an "action" of $H_*(P)$ on $HM_*(f)$

$b \in H_*(P)$, $b = [B]$ a cycle

$a \in HM_*(f)$, $a = [A]$, $\underbrace{A = \sum \beta_i x_i}_{\text{morse cycle}} \in CM_*(f)$

$$B \circ A := \sum \beta_i \underbrace{(B \cdot x_i)}$$ need to def this

Think of B as a generic, immerse submanifold: of codim

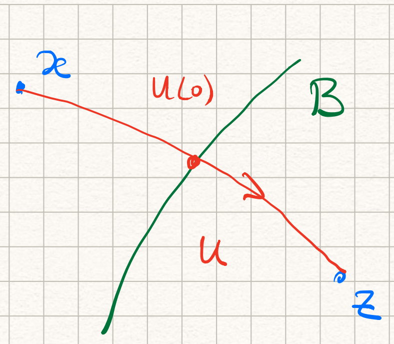
Discuss

$$B \cdot x = \sum_{z} \langle B \cdot x, z \rangle z$$

$$\langle B \cdot x, z \rangle = \text{"} |\mathcal{M}(x,z) \cap B| \text{"}$$

\# of constraints

$$\boxed{\mu(z) = \mu(x) - \operatorname{codim} B} \quad (*)$$



$\langle B\cdot x, z\rangle$

$= \underbrace{\left|\{x \overset{u}{\rightsquigarrow} z \mid u(0)\in B\}\right|}_{\text{a finite set when (*) holds \& B is generic}}$

Then

$$\partial_M(\underset{\uparrow}{B}\cdot w) = B\cdot\partial_M w$$

a cycle

$\Rightarrow$ $B\cdot$ descends to $HM_*(f)$

The resulting map depends only on $[B]=b$

<u>Geometrically and very informally</u>

$$\text{"} CM_* \longrightarrow C_* \text{"}$$

$$x \longmapsto W^u(x)$$

$$B\cdot x = \sum \underbrace{(\qquad)}_{|\mathcal{M}(x,z)\cap B|} W^u(z)$$

$$|(W^u(x) \underset{\pitchfork}{\cap} W^s(z)) \underset{\pitchfork}{\cap} B|$$

Interpretation II: Product on $\underset{H_*(P)}{\underbrace{HM_*(f)}}$

$= $ intersection product

Product on $CM_*(f)$

Need to define $\quad$ Take two functions f_1 & f_2 for transversality

$$CM_*(f_1) \otimes CM_*(f_2) \longrightarrow CM_*(f_1+f_2)$$

$$x \cdot y = \sum \langle x \cdot y, z \rangle z$$

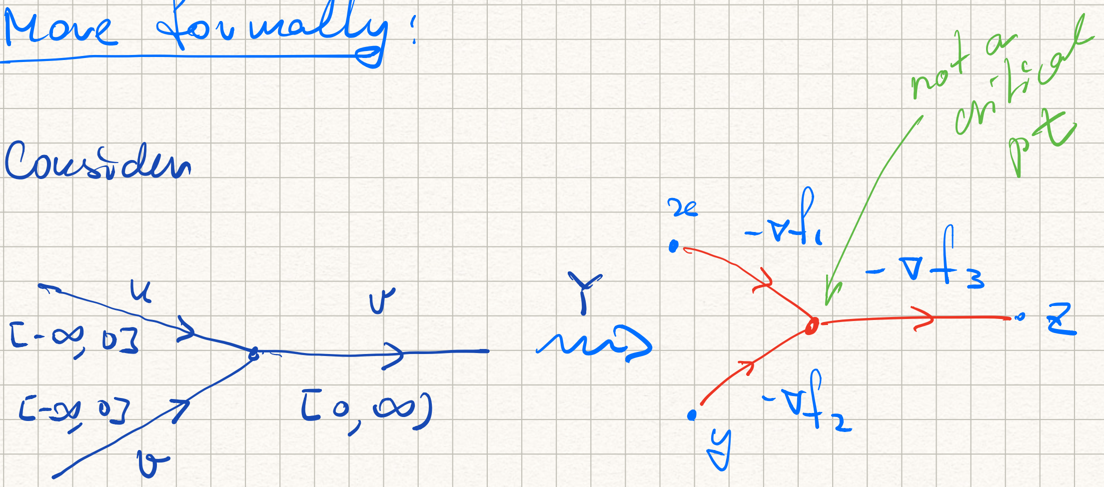
can take f_3 here

$$\mu(z) = \mu(x) + \mu(y) - n$$

Idea: Replace B in the previous constr. by $W^u(y)$ ← informally the chain y gives rise to.

More formally:

Consider



$$\mu(z) = \mu(x) + \mu(y) - n$$

For a generic metric ∃ finitely many such Y shapped traj and their # is $\langle x\cdot y, z\rangle$ (with signs)

As defined it's not associative, but

$$\partial_M(x\cdot y) = \partial_M x\cdot y \pm x\cdot \partial_M y$$

(I think)
and on the level of homology
we have

$$\begin{array}{c} HM_*(f_1)\otimes HM_*(f_2)\longrightarrow HM_*(f_3)\\ \| \qquad\qquad\qquad \| \qquad\qquad\qquad \| \\ \underbrace{H_*(P)\otimes H_*(P)\longrightarrow H_*(P)}_{\text{intersection product}}\end{array}$$

<u>Rmk</u> Other homological features of P also have similar descriptions via <u>graph flows</u>

<u>Rmk</u>: Want:

$$H_*(P_a)\otimes H_*(P_b)\longrightarrow H_*(P_{a+b})$$

$$\Rightarrow \quad f_3 \leqq f_1+f_2$$