§10 Floer Theory Perspective

Lecture 5 01/28

$f : P \longrightarrow \mathbb{R}$ Morse

$\Rightarrow$ CW-complex s.t.

$\mathrm{Crit}_k(f) \underset{1-1}{\longleftrightarrow}$ k-cells

$\Rightarrow$ $\exists$ a complex (C_*, ∂) s.t.

- C_k is generated by $\mathrm{Crit}_k(f)$
- $H_*(C_*, \partial) \cong H_*(P)$

Our goal is to describe ∂ explicitly without using alg topology and CW-complex str.

Def. $(C_*, \partial) = (CM_*(f), \partial)$ ← new notation

is called the Morse complex of f

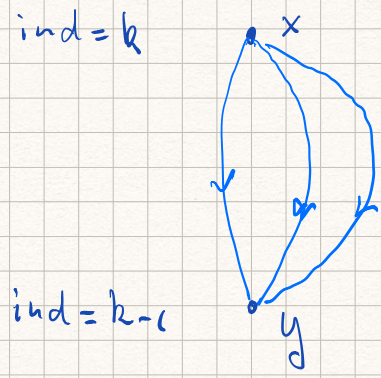
We have

$$\boxed{\underset{\mathrm{Crit}_k(f)}{\overset{\in}{\partial x}} = \sum_{y \in \mathrm{Crit}_{k-1}(f)} m(x,y)\, y}$$

Want to describe the Morse differential explicitly

Rmk Depends on the metric!

Idea



$$\underbrace{\mathrm{ind}(x) - \mathrm{ind}(y) = 1}$$

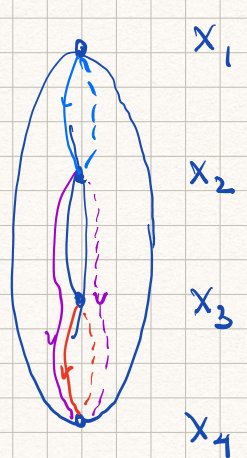
$\Downarrow$ generically

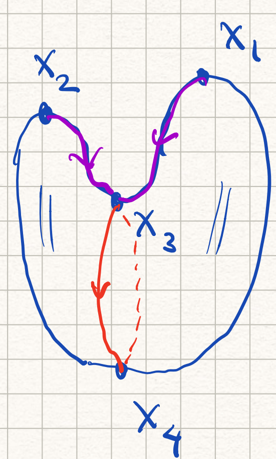
finite number of anti-grad trajectories from x to y

$$\boxed{m(x,y) = \#\text{ of traj } x \rightsquigarrow y}$$

parity over $\mathbb{F}_2 = \mathbb{Z}_2$
or with signs if over $\mathbb{Z}$

Ex Height function on $\mathbb{T}^2$ and on 





$\partial = 0$

$\partial x_1 = x_3 = \partial x_2$

$\partial x_3 = (1-1)x_4 = 0$

In what follows I'll skip most of the pfs, but explain ideas. Usually intuitively clean, but tedious

References:

- Jost
- Audin-Damian
- Banyaga - Hurtubise

§10.1 Construction of the Morse differential ∂: Preliminaries

- While $CM_*(f)$ is completely determined by f, ∂ depends on an extra str: a R. metric on M

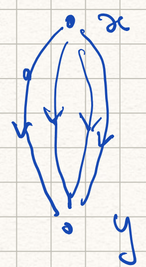
- Fix a R. m. on M (has to be from a certain open and dense set of R.m.'s)

Consider the antigradient flow of f:

$$\dot{x} = -\nabla f(x) \quad : \quad \varphi_t$$

Set $\mathcal{M}(x,y) = \left\{ z \,\middle|\, \varphi_t(z) \begin{array}{l} \to x \quad t \to -\infty \\ \to y \quad t \to +\infty \end{array} \right\}$

x, y crit



Denote the index of x by $\mu(x)$.

Note: "$\dim \mathcal{M}(x,y) \geq 1$" if $\neq \emptyset$

Thm For a generic metric, $\mathcal{M}(x,y)$ is a smooth manifold of dimension $\mu(x) - \mu(y)$

Explanation

$$W^u(x) = \{ p \mid \varphi^t(p) \longrightarrow x,\ t \to -\infty \}$$
$$= \text{unstable manifold}$$

$$W^s(x) = \{ p \mid \varphi^t(p) \longrightarrow x,\ t \to +\infty \}$$
$$= \text{stable manifold}$$

Morse Lemma or just non-deg $\Rightarrow$

$$\Rightarrow \begin{cases} W^u(x) \cong D^k \\ W^s(x) \cong D^{n-k} \end{cases}, \quad k = \mu(x) = \operatorname{ind}(x)$$

$$\mathcal{M}(x,y) = W^u(x) \cap W^s(y)$$

<u>generic metric</u> $\Rightarrow W^u(x) \pitchfork W^s(y)$

requires a pf
but not hard

$\Rightarrow$ $\mathcal{M}(x,y)$ is a manifold &

$$\begin{aligned} \dim \mathcal{M}(x,y) &= n - \dim W^u(x) - \dim W^s(y) \\ &= n - \mu(x) - (n - \mu(y)) \\ &= \mu(x) - \mu(y) \end{aligned}$$

$\triangleleft$

Rmk a) $\dim \mathcal{M} < 0 \Rightarrow \mathcal{M} = \emptyset$

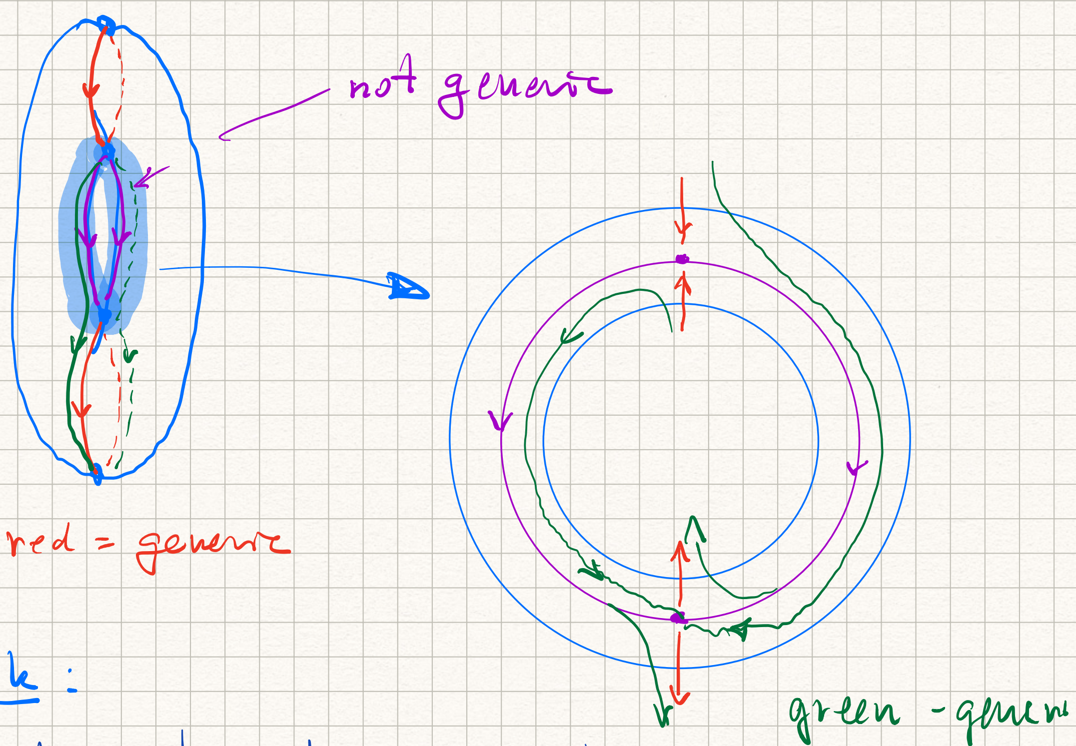
b) $\dim \mathcal{M} \geq 1$: with every pt $\mathcal{M}$ contains a whole trajectory

(Here $x \neq y$)

$\Rightarrow$ c) $\operatorname{ind}(x) \leq \operatorname{ind}(y)$

$\Rightarrow \mathcal{M}(x,y) = \emptyset$

Ex. Height function on $\mathbb{T}^2$



Rmk:

How to achive transversality

Look at $(W^u(x) \cap \{f=c\}) \cap (W^s(y) \cap \{f=c\})$

$f(y) < \underset{\text{reg}}{c} < f(x)$, perturb the metric slightly above c to alter

§10.2 Digression: sliding handles

Recall from §9

Prop: $\exists$ admits a Morse function s.t.

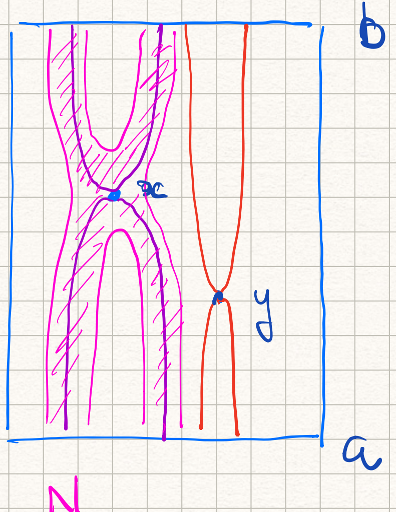
$$f(x) > f(y) \iff \mu(x) \geq \mu(y)$$
$$\forall\, x, y \in \mathrm{Crit}(f)$$

Pf-outline: start with some f and modify it by moving critical pts by each other

Need this:

x, y the only critical pts with
$$a < f(y) < f(x) < b$$
$$\mu(y) > \mu(x)$$
$\Rightarrow$ can modify f in $[a,b]$ so that
$$f(y) \leq f(x)$$

Idea



$$\Sigma_x = \left(W^u(x) \cup W^s(x)\right) \cap f^{-1}([a,b])$$

$$\Sigma_y = \left(W^u(y) \cup W^s(y)\right) \cap f^{-1}([a,b])$$

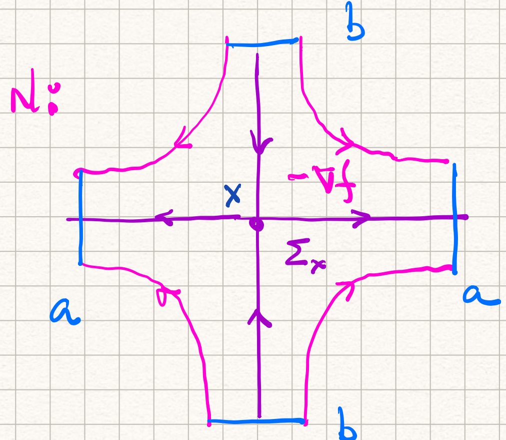
Generically

$$\Sigma_x \cap \Sigma_y = \emptyset$$
$$\Uparrow$$
$$\mu(y) > \mu(x)$$

modify f in a nbd $N \supset \Sigma_x$
(or $N \supset \Sigma_y$)
keeping it the same near $\partial\Sigma$

$\Rightarrow$ $\boxed{\text{can give } x \text{ any value in } (a,b) \text{ by modifying how } f \text{ changes along integral curves of } \nabla f}$

N is sketched unrealistically.
A better way:

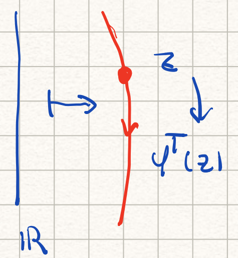


Can change f so that
- remains the same near ∂N
- same $-\nabla f$ up to scaling
- any value $f(x)$ in (a,b)

$\triangleleft$

§10.3 More modern & different perspective

$\mathcal{M}(x,y)$ = the space of parametrized trajectories $t \mapsto \varphi^t(z)$ from x to y



trajectory $\longleftrightarrow$ initial condition

$\varphi^t(z) \longleftrightarrow z = \varphi^0(z)$

Time shift: $t \mapsto \varphi^t(z)$ $\qquad z \mapsto \varphi^T(z)$

$$\begin{array}{c} t \mapsto \varphi^t(z) \\ \wr\!\downarrow \\ t \mapsto \varphi^{t+T}(z) \end{array}$$

$\Rightarrow$ free $\mathbb{R}$-action on $\mathcal{M}(x,y)$, $x \neq y$

Space of unparametrized trajectories

$$\hat{\mathcal{M}}(x,y) = \mathcal{M}(x,y)/\mathbb{R}$$

Cor $\hat{\mathcal{M}}(x,y)$ is a smooth manifold of dim $\mu(x) - \mu(y) - 1$

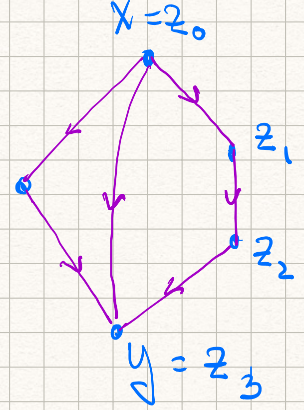
E.g. $\mu(x) = \mu(y) + 1 \Rightarrow \hat{\mathcal{M}}$ is discr

Note
- $\mathcal{M}$ & $\hat{\mathcal{M}}$ are usually non-compact
- Geometrically, $\hat{\mathcal{M}}$ can be identified with $\mathcal{M} \cap \{f = c\}$

$$f(y) < \underset{\substack{\uparrow \\ \text{regular}}}{c} < f(x)$$

For a generic metric:

Thm $\hat{\mathcal{M}}(x,y)$ has a compactification formed by broken trajectories



Such trajectories $x = z_0 \rightsquigarrow z_1 \rightsquigarrow \ldots \rightsquigarrow z_k = y$ form a compact manifold with corners.

Rmk $f(x) > f(z_1) > \ldots > f(y)$

$\mu(x) > \mu(z_1) > \ldots > \mu(y)$

Cor. $\mu(x) = \mu(y) + 1$

$\Rightarrow \hat{\mathcal{M}} = \text{compact} \Rightarrow$ finite collection of pts

$\Leftrightarrow \exists$ finite many traj from x to y (for a generic metric)

§10.4 Definition of ∂

Fix a generic metric so that all the thms hold

$$\mu(x) = \mu(y) + 1$$

- Over $\mathbb{Z}_2$, set

$$\mathbb{Z}_2 \ni m(x,y) = |\hat{\mathcal{M}}(x,y)| \bmod 2$$

$$\boxed{\partial x = \sum_{\substack{y \\ \mu(x) = \mu(y)+1}} m(x,y)\, y} \qquad (*)$$

- Over $\mathbb{Z}$ (and hence any ring)

Need to take into account orientations

Fix orientations of $T_x W^u(x)$ $\forall x$

$\Rightarrow$ coorientations of $T_x W^s(x)$

$\Rightarrow \begin{cases} \text{orientations of } W^u(x) \\ \text{coorientations of } W^s(x) \end{cases}$

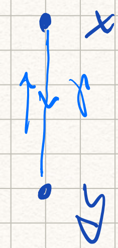
$\Rightarrow$ orientations of

$$\mathcal{M}(x,y) = W^u(x) \cap W^s(y)$$

• when $\mu(x) = \mu(y)+1$

$\mathcal{M}(x,y)$ = disj union of finite # of trajectories

• Each trajectory γ is also oriented by the flow



$\Rightarrow$ Two orientations on γ

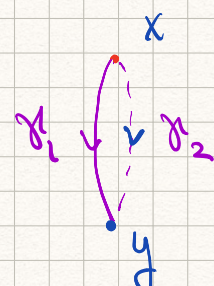
$$\text{sign}(\gamma) = \begin{cases} +1 & \text{orientations agree} \\ -1 & \text{— . — . disagree} \end{cases}$$

And

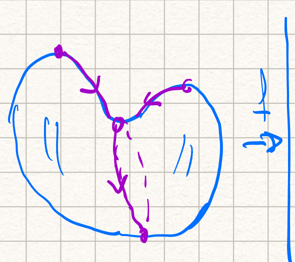
$$m(x,y) = \sum_{x \overset{\gamma}{\rightsquigarrow} y} \text{sign}(\gamma)$$

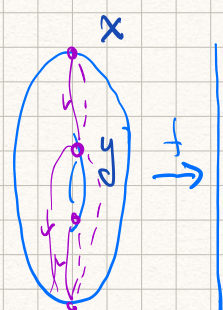
$$*: \quad \boxed{\partial x = \sum m(x,y)\, y}$$

$W^s(y) \cap W^u(x) = \gamma_1 \cup \gamma_2$



Ex. Do these:





$W^s(y) \cap W^u(x)$

§10.5 Checking that $(CM(f), \partial)$ is a complex

$\boxed{\textbf{Thm} \quad \partial^2 = 0}$

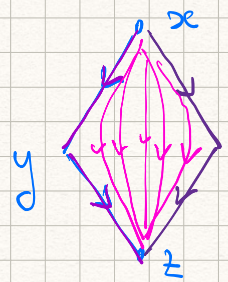
Pf. For the sake of simplicity over $\mathbb{Z}_2$

$$\partial^2 x = \partial \sum_y m(x,y)\, y$$

$\mu(x) = \mu(y)+1$
$\mu(y) = \mu(z)+1$

$$= \sum_y m(x,y) \sum_z m(y,z)\, z$$

$$= \sum_z \Big(\underbrace{\sum_y m(x,y)\, m(y,z)}_{\text{\# of broken trajectories (one break) from } x \text{ to } z \text{ (mod 2)}} \Big) z \quad \text{mod } 2$$



But $\hat{M}(x,y)$ one-dim manifold
its compactification: S^1 or I
closed interval

$\Rightarrow$ broken trajectories come in pairs

$\Rightarrow$ # is even

$$\Rightarrow \sum_y m(x,y)\, m(y,z) = 0 \mod 2$$

$\Rightarrow \partial^2 = 0$ $\triangleleft$

Set $HM_*(f) = H_*(CM(f), \partial)$; fixed coefficients

$$\boxed{\begin{array}{l}\textbf{Thm} \text{ (Morse theory)} \\ HM_*(f) = H_*(M)\end{array}}$$

Rmk • As a consequence, l.h.s is independent of f ← can be proved directly

• We have already seen some consequences: Morse inequalities, etc

Outline of the pf:

Ref
• Audin–Damian
• Banyaga–Hurtubise

"Classical" Morse theory

Morse function f on M $\rightsquigarrow$ Cellular decomposition of M

$\mathrm{Crit}_k(f) \rightsquigarrow W^u(x)$ ← "cells"

$\quad\quad x \in \mathrm{Crit}_k(f)$

Morse complex $CM_k(f)$ $\longleftrightarrow$ Cellular complex of M: $C(M)$

$\partial_M = \partial_{cw}$

$\Rightarrow H_*(CM(f), \partial_M) = \underbrace{H_*(C(M), \partial_{cw})}_{H_*(M)}$ $\triangleleft$

§11. Calculations of $H_*(M)$ using Morse homology; Applications

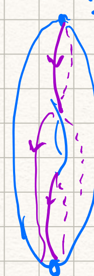
$$H_*(M) = H_*(HM_*(f), \partial_M)$$

very difficult to deal with in general

Works well when $\partial = 0$

Examples (over $\mathbb{Z}$ or $\mathbb{F}$)

1) Σ_g on $\mathbb{R}^2$



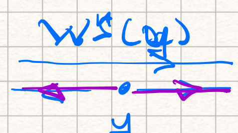
$$\partial x = y_1 + - y_1 + y_2 + - y_2 = 0$$

unstable traj of y_1 & y_2 should come from x

$$\partial y_1 = 0 = \partial y_2$$

$$\Rightarrow \quad H_*(\Sigma_g) = \begin{cases} \mathbb{F} & k = 2 \\ \mathbb{F}^{2g} & k = 1 \\ \mathbb{F} & k = 0 \end{cases}$$

Orientations!



2) $\mathbb{C}P^n$ (over $\mathbb{Z}$ or $\mathbb{F}$)

$$\mathbb{C}P^n = \{ (z_0 : \ldots : z_n) \mid \sum |z_j|^2 = 1 \}$$

$$f(z) = \sum \lambda_j |z_j|^2$$

$$\lambda_0 < \lambda_1 < \ldots < \lambda_n$$

Ex. a) Crit(f) = "coordinate axes"

$$= \{ (0, \ldots, 0, \underset{j}{1}, 0, \ldots, 0) = x_j \}$$

b) In coordinates

$$u = (u_0, \ldots, u_{j-1}, 1, u_{j+1}, \ldots, u_n)$$

At x_j

$$d^2_{x_j} f = (\lambda_0 - \lambda_j)|u_0|^2 + (\lambda_1 - \lambda_j)|u_1|^2 + \ldots \text{ skip } (\lambda_j - \lambda_j)$$
$$\ldots + (\lambda_n - \lambda_j)|u_n|^2$$

$\Rightarrow$ f is morse & $\mu(x_j) = 2j \leftarrow z_j$ is a complex #.

& $\partial = 0$

$$\Rightarrow H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{F} & 0 \le k = 2j \le 2n \\ 0 & \text{otherwise} \end{cases}$$

3) $\mathbb{RP}^n$ over $\mathbb{Z}_2$

Similarly $\mathbb{RP}^n = \{ (y_0 : \dots : y_n) \mid \sum |y_j|^2 = 1 \}$

$$f(y) = \sum \lambda_j |y_j|^2$$

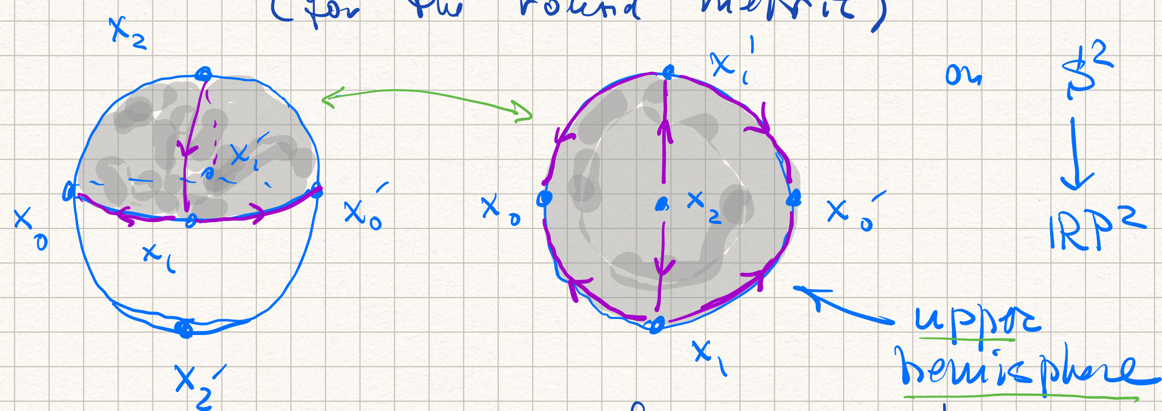
Ex. similarly

a) $x_j = (0, \dots, 0, \underset{j}{1}, 0, \dots, 0) \leftarrow$ Critical pts

b) Hessian : similar — same calculation

$$\Rightarrow \quad \mu(x_j) = j$$

c) $\partial = 0$ over $\mathbb{Z}_2$: exactly two trajectories from x_{j+1} to x_j (for the round metric)



$$\Rightarrow \quad H_k(\mathbb{RP}^n; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

Rmk. Over $\mathbb{Z}$, harder — orientations

Lecture 6
01/21

4) Hamiltonian S^1-actions or torus actions

P a symplectic manifold (closed)

$H: P \longrightarrow \mathbb{R}$ Hamiltonian generating an S^1-action

$$\underbrace{\mathrm{Crit}(H) = \text{Fixed pts} = P^{S^1}}_{\text{assume isolated}} = \{x\}$$

Details:
Last year sympl geometry class

- Equivariant Darboux $\Rightarrow$ $d^2_x H$ have even index

$\Rightarrow$ $\partial_M = 0$ Similar to $\mathbb{CP}^n$

$\dim H_k = |\mathrm{Crit}_k(H)|$

- Works for a lot of interesting manifolds: Flag manifolds, Grassmannians, symplectic toric manifolds, etc.

- Same for $\mathbb{T}^r$-actions with moment map $(H_1, \dots, H_r)$

Set $H = \sum \lambda_j H_j$ – generalization of $\mathbb{CP}^n$

Two Textbook Applications

i) The Künneth formula for manifolds:

P_0, P_1 two closed manifolds

$$\Rightarrow \boxed{H_*(P_0 \times P_1) = H_*(P_0) \otimes H_*(P_1)}$$

over a field

Pf. • f_0 a Morse function on P_0
• f_1 — . — . — . — . P_1
• pick generic metrics $\Rightarrow$
the product metric on $P_0 \times P_1$
is good for us

$$CM_*(f_0 + f_1) = CM_*(f_0) \otimes CM_*(f_1)$$

as complexes

Algebraic Künneth ← needs a field

$$\Rightarrow \underbrace{HM_*(f_0+f_1)}_{H_*(P_0 \times P_1)} = \underbrace{HM_*(f_0)}_{H_*(P_0)} \otimes \underbrace{HM_*(f_1)}_{H_*(P_1)}$$

◁

Rmk. Likewise over $\mathbb{Z}$ or a ring
but the formula is more involved
⇐ algebra

2) Poincaré Duality

P smooth closed manifold, $\dim P = n$

$$\boxed{H^k(P) \cong H_{n-k}(P)}$$

canonical

- over $\mathbb{Z}_2$
- over other fields if orientable

<u>Cor</u>: $b_k = b_{n-k}$

<u>Pf</u>

f morse $\iff$ $-f$ is Morse

$\nabla f \qquad\qquad \nabla(-f) = -\nabla f$

integral curves $\qquad$ same curves but traveled backward

$$\mathrm{Crit}_k(f) = \mathrm{Crit}_{n-k}(-f)$$

$$\Rightarrow \quad CM^*_k(f) = CM_{n-k}(-f)$$

$$\partial^*_M \qquad\qquad \partial_M(-f)$$

dual

use the basis

$$H^*(C) := H_*(C^*) = H_*(C)^* \quad \text{over a field}$$

$$\Rightarrow \quad \underbrace{H_*(CM^*(f), \partial^*_M)}_{H^*(P)} = \underbrace{H_*(CM_{n-*}(-f), \partial_M(-f))}_{H_{n-*}(P)}$$

Morse Theory for $(-f)$! $\qquad \triangleleft$

§ 12 **Existence of Morse functions**
Part I: via Transversality thm

P closed manifold, $k \geq 2$ e.g. ∞

Thm Morse functions form an open and dense set in $C^k(P)$

Cor Every manifold admits a Morse function. Moreover, every function can be C^k-approximated by Morse functions.

We'll give two pfs

Pf 1: based on Transversality
Pf 2 (Outline) - direct

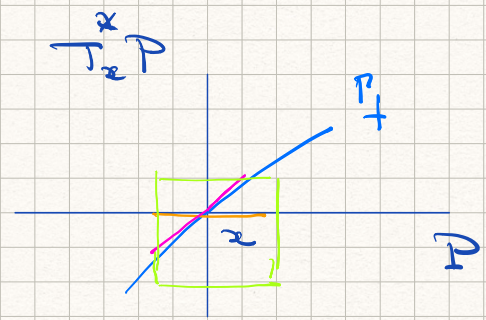
Revisiting the definition of Morse function

$f: P \to \mathbb{R} \rightsquigarrow df$ = section of T^*P

$\Gamma_f = \text{graph}(df)$

$x \in \text{Crit}(f) \iff df(x) = 0$

$\iff x \in \Gamma_f \cap P$ ← zero section

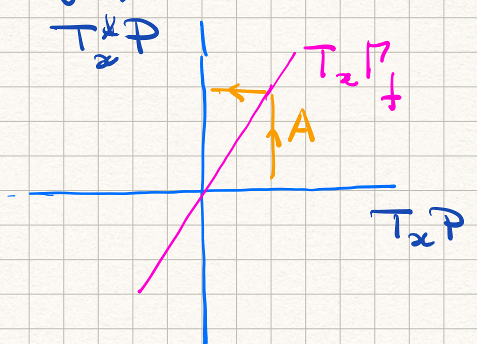


Claim x is non-deg $\iff \Gamma_f \pitchfork P$ at x

$$T_x\Gamma_f + T_xP = \underbrace{T_{(x,0)}T^*P}_{T_xP \oplus T_x^*P}$$

Pf.

- $T_x\Gamma_f = \text{graph}(A: T_xP \to T_x^*P)$



A = linearization of $s: y \to df(y)$ at x

- $T_x\Gamma_f \pitchfork P$ at $x \iff \operatorname{Ker} A = 0$

$\iff A$ is onto

$\iff A$ is non-deg

- A is essentially the Hessian

$$d_x^2 f(v, w) = \underbrace{A(v)}_{T_x^* P}(w), \quad v, w \in T_x P$$

$$\underbrace{A \text{ is non-deg}}_{\iff T_x\Gamma_f \pitchfork P} \iff d_x^2 f \text{ is non-deg}$$

$\triangleleft$

$\boxed{\text{Cor } f \text{ is Morse } \iff \Gamma_f \pitchfork P}$

Idea of the pf of $\exists$ Morse functions

- start with some f_0. Need to approx by a Morse function f
- Transversality thm $\Rightarrow df_0$ can be approximated by $\alpha \in \Omega^1(P)$ = sections of T^*P so that $\Gamma_\alpha \pitchfork P$. Difficulty: Need $\alpha = df$

Transversality Theorems: Review (without pfs) & Pf I

Ref. E.g. Arnold-Varchenko-Gusein-Zade or many other textbooks

Based on Sard's Lemma

Definitions

$$X \xrightarrow{F} Y \supset Z$$

$F \pitchfork Z$ if

$$\boxed{DF(T_x X) + T_{F(x)} Z = T_{F(x)} Y}$$

$\forall\ x \in X$ with $F(x) \in Z$

Open condition in C^1-top

Ex $\dim X + \dim Z < \dim Y$, $F \pitchfork Z$

$$\Rightarrow \quad F(X) \cap Z = \emptyset$$

Key pt: Almost all $F \pitchfork Z$ often even extra constraints on F

Ex. $L_0, L_1 \subset \mathbb{R}^3$ two embedded loops

For almost all $v \in \mathbb{R}^3$

$$(L_0 - v) \cap L_1 = \emptyset$$

Hint: • $L_0 \times L_1 \xrightarrow{\Phi} \mathbb{R}^3$

$(x, y) \longmapsto x - y$

• $(L_0 - v) \cap L_1 \neq \emptyset \iff \exists\, x, y$ s.t. $x - v = y$

$\iff x - y = v$

$\iff v \in \Phi(L_0 \times L_1)$

• $\dim(L_0 \times L_1) = 2$, $\dim \mathbb{R}^3 = 3$

Sard's $\Rightarrow$ $\Phi(L_0 \times L_1)$ = zero measure
nowhere dense
closed

$\Rightarrow$ For an open full measure set in $\mathbb{R}^3$
$(L_0 - v) \cap L_1 = \emptyset$

$\triangleleft$

A sequence of settings increasing constraints on F: transversality theorems

Setting 1.

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ & & \cup \\ & & Z \end{array}$$

at least proper

X closed manifold

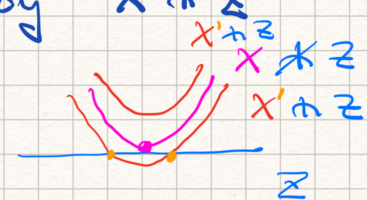
Thm 1 $F \pitchfork Z$ for open and dense set in $C^k(X, Y)$

$1 < k \le \infty$

Rmk • Here and below it's clear that the condition is open in C^k top.

• Pf is based on Sard's lemma

• If $F : X \hookrightarrow Y$ we say $X \pitchfork Z$

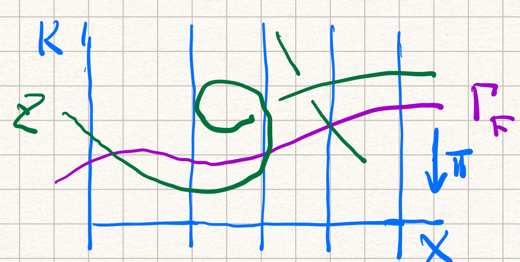


Setting 2

$Y = X \times K \supset Z$, $\quad F : X \to K$

$\Gamma_F : X \hookrightarrow Y \qquad x \mapsto (x, F(x))$

Thm 2 $\Gamma_F \pitchfork Z$ For an open & dense set in $C^k(X, K)$



Much more restrictive

class: $\Gamma_F : X \hookrightarrow Y = X \times K$

$$\underbrace{\pi \circ \Gamma_F = id}_{\text{section of } X \times K \to X}$$

Setting 3 : same but

$Z \subset Y$

$$\begin{tikzcd} K \arrow[r] & Y \arrow[d, "\pi"] \\ & X \arrow[u, bend left, "s"] \end{tikzcd}$$

fiber bundle

$\underbrace{\pi \circ s = \mathrm{id}}_{\text{section}}$

Tm 2

$\Uparrow$

Thm 3 $s \pitchfork Z$ For an open and dense set of C^k-sections of π.

Rmk. The class of sections is much smaller than all $C^k(X, Y)$

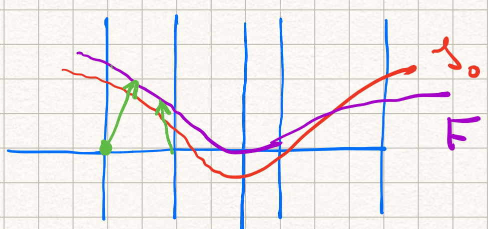
Outline of Thm 1 $\Rightarrow$ Thm 3

Need density

$s_0 : X \to Y$ a given section

$F : X \to Y$, $F \pitchfork Z$ $C^{k \geq 1}$-close to s_0

not a section : $\pi \circ F(x) \neq x$



$\varphi : x \longmapsto \pi \circ F(x)$

$X \longrightarrow X$

Claim (Ex) $\varphi : X \to X$ a diffeo, C^k close to id

Replace F by $s := F \circ \varphi^{-1} \pitchfork Z$

$\approx s_0$ $\triangleleft$

Rmk It is essential that

$$\varphi \overset{C^{k\geq 1}}{\approx} id$$

Not enough $\varphi \overset{C^0}{\approx} id$

Ex $\varphi(x) = x + \varepsilon \sin(\omega x)$

id