§5 Some alg topology: CW-complexes

Generalities

References: Any good alg topology book, e.g. Hatcher

- Attaching a cell – procedure

X = a reasonable ~~top~~ space (metrizable, compact or loc. compact...)

$\psi : S^{n-1} \longrightarrow X$ cont map

Def $Y = X \cup_\psi D^n$; $S^{n-1} = \partial D^n$

closed ball

is obtained from X by attaching an n-cell ($\mathrm{int} D^n$) along ψ

$$Y = X \sqcup D^n / \sim \qquad \underset{S^{n-1}=\partial D^n}{x \in} \sim \underset{X}{\psi(x) \in}$$

This is again a reasonable top space

In what follows we are interested is spaces/maps up to homotopy

Ex-Fact (Hatcher): $\boxed{\varphi_0 \sim \varphi_1 \text{ (homotopic)} \Rightarrow X \cup_{\varphi_0} D^n \simeq X \cup_{\varphi_1} D^n}$:

changing φ within its homotopy class does not effect the homotopy type of $X \cup_\varphi D^n$

Def A CW-complex (or a cell complex) is obtained inductively from

X_0 = finite collection of pts

by attaching cells of increasing dimension:

$$X_0 \subset \underset{\text{graph}}{X_1} \subset \dots \subset X_n = X$$

where X_{k+1} is obtained from X_k by attaching one or several k-cells

Rmk • Here all cell-complexes are finite – finite # of cells

• One is allowed to skip steps: $X_k = X_{k+1}$

• $\dim X$ is max n that occurs

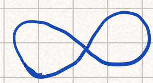
Ex.

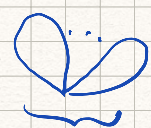
Visualize →

- $S^n = pt + D^n$
- $\mathbb{R}P^n = pt + D^1 + D^2 + \dots + D^n$
- $\mathbb{C}P^n = pt + D^2 + D^4 + \dots + D^{2n}$

} obvious notation

} not-standard

- T^2 :  $+ D^2$

- Σ_g :  $+ D^2$

  $\bigvee_{2g} S^1$

- $\dim X = 1 \overset{\text{def}}{\iff} X$ is a graph

- more generally a simplicial complex is naturally a CW complex and k-cells = k-simplices

A good ref for this & the next §
is Dubrovin–Fomenko–Novikov, Part III

$\boxed{\begin{array}{l}\underline{\text{Prop}} \quad X = \text{CW complex}, \quad Y = X \cup_\varphi D^{k+1} \\ \Rightarrow Y \text{ has hom type of a CW complex}\end{array}}$

Not quite obvious:
In a CW complex have cells of higher dim attached after lower dim:

$$\begin{tikzcd} X_0 \subset X_1 \subset \cdots \subset X_k \subset X_{k+1} \subset \cdots \subset X_n \\ & S^k = \partial D^{k+1} \arrow[u, "\varphi"'] \arrow[ul, "\psi", "\sim"'] \end{tikzcd}$$

<u>Pf</u>

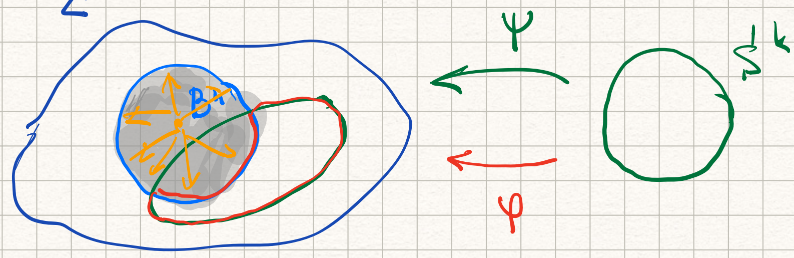
<u>Need</u> $\underline{\varphi \sim \psi : S^k \longrightarrow X_k}$

<u>Idea</u>: • $Z \supset B^n$
top space   <u>open</u> ball

• $\varphi : S^k \longrightarrow Z$, $\underline{k < n}$

$\Rightarrow \exists \ \psi : S^k \longrightarrow Z \sim \varphi$ s.t.
$\psi(S^k) \cap B = \emptyset$

Z



Then use induction:

$$X = X_{n-1} \cup B^n \cup \dots \cup B^n \leftarrow \text{open balls}$$

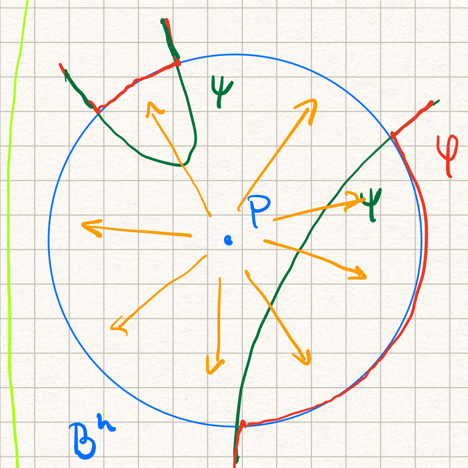
Push ψ outside B^n's, etc.

Some minor conventions:

- $B^n \subset Z$ replaced by
  $\underset{\text{closed ball}}{F: D^n \longrightarrow Z}$ s.t. $F: B^n \hookrightarrow Z$ homeo on its image, $B^n \subset D^n = \overline{B^n}$
  $F(D^2)$ is a "nice" compactification of B^n

- Need to know that
  $\exists\ p \in B^n$, s.t. $p \notin \psi(S^k)$
  (If so can assume $p = 0 \in B^n$) — Assume for now

$$R : D^n \setminus p \longrightarrow S^{n-1}$$



$$\varphi(x) = \begin{cases} \psi(x), & \psi(x) \notin B^n \\ \underbrace{F \circ R \circ \psi(x), \quad \psi(x) \in B^n}_{\text{defined} \Leftarrow p \notin \psi(S^k)} \end{cases}$$

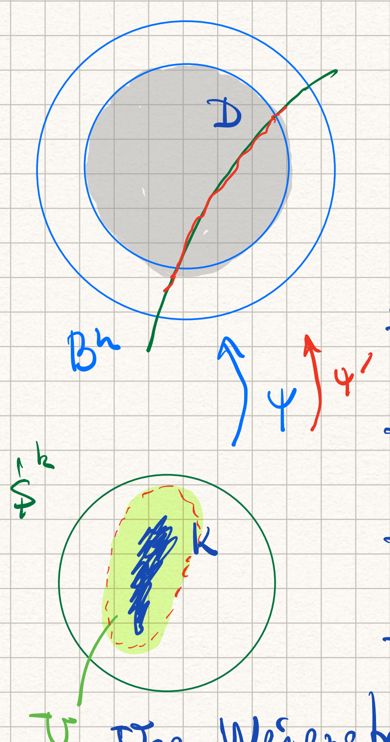
Ex.: Check $\varphi \in C^0$, $\varphi \sim \psi \Leftarrow R$ hom eq.

• Not true in general:
we can have $\psi: S^k \to D^{n>k}$ onto
"Peano curve" $\Leftarrow$ cannot be C^1 by Sards

• But $\exists$ $\psi' \sim \psi$ & C^0-close to it
s.t. $\psi'(S^k) \not\supset B^n$

Idea: approximate ψ by a smooth map or rather a map which is smooth on the part mapped to B^n

Details: $D \subset B^n$ a slightly smaller closed ball



$S^k \supset \underbrace{U \supset \psi^{-1}(D)}_{\text{small nbe}} = K \leftarrow$ compact

$\psi(U) \subset B^n$

$\exists$ $\psi': S^k \to Z$ s.t.

$\to$ $\psi' = \psi$ outside B

$\to$ $\psi' \overset{C^0}{\approx} \psi$ inside B

$\Rightarrow \psi' \sim \psi$

$\to$ $\psi': U \xrightarrow{C^\infty} B \Rightarrow D \not\subset \psi'(S^k)$

Use Weierstrass to construct ψ'... $\triangleleft$

§6 Some Alg Topology: CW complexes Homology

Setting

- $\mathbb{F}$ ground field, e.g.: $\mathbb{F}_2$, $\mathbb{Q}$ or $\mathbb{Z}$
- X a CW complex
  $e_i^k \leftarrow$ k-dim cells, $i = 1, \dots, r_k$
- C_k = free v.s. (module) over $\mathbb{F}$ generated by e_i^k

Key pt: $\exists\ \partial$

$$0 \leftarrow C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} \cdots \leftarrow C_n \leftarrow 0$$

s.t. $H_*(C_*, \partial) = H_*(X; \mathbb{F})$ ← suppress in notation

Assume $\mathbb{Z}$

Construction of ∂

Note • $X_k / X_{k-1} = \bigvee_{r_k} S_j^k$

bouquet of r_k k-spheres labeled by $e_1^k, \dots, e_{r_k}^k$

Want to defined ∂e_i^{k+1}

$$\boxed{\partial e_i^{k+1} = \sum_j m_{ij} e_j^k}$$

need to know what these are

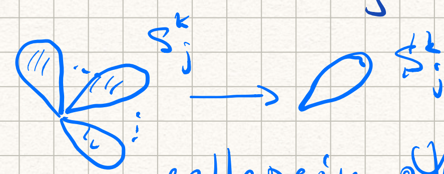
Attaching map for e_i^{k+1}

$$\begin{tikzcd} \psi: D^{k+1} \arrow[r] & X_{k+1} \\ \cup & \cup \\ S^k = \partial D^{k+1} \arrow[r] & X_k \end{tikzcd} \Rightarrow$$

incidence number of e_i^{k+1} & e_j^k

$$\bar{\psi}: S^k \longrightarrow X_k \longrightarrow X_k / X_{k-1} = \vee S^k \longrightarrow S_j^k$$

$$\boxed{m_{ij} = \deg(\bar{\psi})} \in \mathbb{Z}$$



collapsing other spheres

Rmk For other $\mathbb{F}$'s take the image of m_{ij} under $\mathbb{Z} \longrightarrow \mathbb{F}$

Alternatively and more formally:

- $C_k = H_k(X_k/X_{k-1} = \bigvee_{r_k} S^k_i) = \mathbb{F}^{r_k}$

k spheres

- Consider $(\underbrace{X_k/X_{k-2}}_{A}, \underbrace{X_{k-1}/X_{k-2}}_{B})$

Then $A/B \simeq X_k/X_{k-1}$. Long exact seq:

$$\cdots \to H_k(B) \longrightarrow H_k(A) \longrightarrow \underbrace{H_k(A/B)}_{C_k} \longrightarrow \underbrace{H_{k-1}(B)}_{C_{k-1}} \longrightarrow \cdots$$

$$C_k \xrightarrow{\partial} C_{k-1}$$

So we have a connecting map in the long exact seq, and this is:

$$\partial : H_k(X_k/X_{k-1}) \longrightarrow H_{k-1}(X_{k-1}/X_{k-2})$$

- In what follows, it's more convenient to break it down into two steps:

$$\boxed{\partial : H_k(X_k/X_{k-1}) \xrightarrow{\delta} H_{k-1}(X_{k-1}) \longrightarrow H_{k-1}(X_{k-1}/X_{k-2})} \quad (*)$$

connecting map in LES for (X_k, X_{k-1})

Lemma $\partial^2 = 0$

Not difficult but not entirely trivial

Pf (Suffices over $\mathbb{Z}$) From FDN inspired.

Write $X_k' = X_k / X_{k-1}$. By (*), starting with $k+1$

$\partial^2 =$

$$\to \underbrace{\overbrace{H_{k+1}(X_{k+1}')}^{X_{k+1}/X_k}}_{C_{k+1}} \xrightarrow{\delta} \underbrace{H_k(X_k) \to \overbrace{H_k(X_k')}^{X_k/X_{k-1}} \xrightarrow{\delta} H_{k-1}(X_{k-1})}_{=0} \to \underbrace{H_{k-1}(X_{k-1}')}_{C_{k-1}} \to$$

$$C_{k+1} \xrightarrow{\partial^2} C_{k-1}$$

Long exact sequence for (X_k, X_{k-1}):

$$\cdots \to \underbrace{H_k(X_k) \to H_k(\underset{X_k/X_{k-1}}{X_k'}) \to H_{k-1}(X_{k-1})}_{=0} \to \cdots$$

$\Rightarrow \partial^2 = 0$ ◁

Ex-Problem (see, e.g. DFM)

X is a simplicial complex

$\Downarrow$

X is a CW-complex; simplices = cells

$\Rightarrow$ two chain complexes

$$\underbrace{(C^{\Delta}_*, \partial_\Delta)}_{\text{simpl. chain compl}} \quad \& \quad \underbrace{(C^{CW}_*, \partial_{CW})}_{\text{complex intro above}}$$

Same generators.

Prove that $\partial_\Delta = \partial_{CW}$ $\triangleleft$

Key Hom result:

$X =$ CW - complex

Denote: $H_*^{cw}(X) = H_*(C_*, \partial)$

Could possibly depend on the CW-str:

$X_0 \subset X_1 \subset \ldots \subset X_n = X$

$$\boxed{\textbf{Thm} \quad H_*^{cw}(X) = \underbrace{H_*(X)}_{\text{whateverflavor of Hom your pref}}}$$

← homotopy inv

Pf - induction in $\dim X$:

Assume done for $\dim < n$; $\dim = 0$ clear

What does not work: the five lemma

$$\begin{tikzcd}
\arrow[r] & H_*^{cw}(X_{n-1}) \arrow[r] & H_*^{cw}(X) \arrow[r] & H_*^{cw}\left(\frac{X}{X_{n-1}}\right) \arrow[r] & H_{*-1}^{cw}(X_{n-1}) \\
\arrow[r] & H_*(X_{n-1}) \arrow[r] \arrow[u, "\cong"'] & H_*(X) \arrow[r] \arrow[u] & H_*\left(\frac{X}{X_{n-1}}\right) \arrow[r] \arrow[u, "\cong"] & H_{*-1}(X_{n-1}) \arrow[u, "\cong"']
\end{tikzcd}$$

No obvious map $H_*(X) \longrightarrow H_*^{cw}(X)$ making the diagram commute

Instead: a bit more subtle argument still purely formal: uses only axioms for H_*

Actual pf: $X = X_n$

- First note For $k < n-1$

$$H_k(X) = H_k(X_{n-1}) \quad \text{long exact seq}$$

$$\quad\quad \| \leftarrow \text{induction}$$

$$H_k^{cw}(X) = H_k^{cw}(X_{n-1}) \quad \text{by def}$$

- Need to do $k = n-1, n$ for $X_n = X$

LES for (X, X_{n-1}):

$$0 \to H_n(X) \to \underbrace{H_n\left(\frac{X}{X_{n-1}}\right)}_{C_n} \xrightarrow{\delta} \underbrace{H_{n-1}(X_{n-1})}_{Z_{n-1}^{cw}} \to H_{n-1}(X) \xrightarrow{0} \cancel{H_{n-1}\left(\frac{X}{X_{n-1}}\right)} \xrightarrow{} 0$$

$$H_n(X) = \ker \delta$$

$$H_n^{cw}(X) = \ker \partial : \underbrace{H_n\left(\frac{X}{X_{n-1}}\right)}_{C_n} \xrightarrow{\delta} H_{n-1}(X_{n-1}) \to \underbrace{H_{n-1}\left(\frac{X_{n-1}}{X_{n-2}}\right)}_{C_{n-1}}$$

$\boxed{\text{mono}}$

LES for (X_{n-1}, X_{n-2})

$$\underbrace{H_{n-1}(X_{n-2})}_{\overset{\|}{0}} \to H_{n-1}(X_{n-1}) \to \underbrace{H_{n-1}\left(\frac{X_{n-1}}{X_{n-2}}\right)}_{C_{n-1}} \to \cdots$$

$$\Rightarrow \underbrace{\ker\delta}_{H_n} = \underbrace{\ker\partial}_{H_n^{cw}} \quad \& \quad H_{n-1}(X_{n-1}) = Z_{n-1}^{cw} \subset C_{n-1}$$

$$\Rightarrow \quad H_{n-1}(X) = \frac{Z_{n-1}^{cw}}{\partial(C_n)} =: H_{n-1}^{cw}(X)$$

$\triangleleft$

Ex. : work out the details I skipped

Ex. Use cw-str to calculate

- $H_*(\Sigma_g) = \begin{cases} \mathbb{F} & 0 \\ \mathbb{F}^{2g} & 1 \\ \mathbb{F} & 2 \end{cases}$

- $H_*(\mathbb{C}P^n) = \underset{0}{\mathbb{F}}, \underset{1}{0}, \underset{2}{\mathbb{F}}, \underset{3}{0} \dots \underset{2n-1}{0}, \underset{2n}{\mathbb{F}}$

- $H_*(\mathbb{R}P^n; \mathbb{F}_2) = \underset{0}{\mathbb{F}_2}, \dots, \underset{n}{\mathbb{F}_2}$

$H_*(\mathbb{R}P^n; \mathbb{Z}) = ? \leftarrow$ Harder

§7 Main Thm & Morse Inequalities

Lecture 4 01/14

- $f : P \longrightarrow \mathbb{R}$ morse
  closed manifold
- write $P_a = \{f \leq a\}$

Thm $a < c < b$, the only crit value in $[a, b]$

$\Rightarrow$ P_b is obtained from P_a by attaching a k-dim cell for every crit pt x on $\{f = c\}$, where $k = \mathrm{ind}(x)$

Cor P has homotopy type of a CW complex with one cell for each crit pt with dim = ind

Same for P_a or $(\{a \leq f \leq b\}, \{f = a\})$

In essense:

$\exists$ a complex (C_k, ∂) with
C_k generated by $\mathrm{Crit}_k(f)$ over $\mathbb{F}$
and $H_*(C, \partial) = H_*(P; \mathbb{F})$

Cor (Morse inequalities)

any field

$$\underbrace{|\mathrm{Crit}_k(f)|}_{c_k = \dim C_k} \geq \dim H_k(P; \mathbb{F}) = b_k$$

or

$$\mathrm{rk}\, H_k(P; \mathbb{Z})$$

Refining Morse Inequalities

over $\mathbb{F}$ field

set
$$h(t) = \sum \overbrace{\dim H_k(P)}^{b_k} \cdot t^k$$
$$m(t) = \sum \underbrace{|\mathrm{Crit}_k(f)|}_{c_k} \cdot t^k$$

Prop • $\exists$ a pol $r(t)$ with coeff ≥ 0 such that

$$m(t) = (1+t)\, r(t) + h(t) \qquad (*)$$

• $c_k - c_{k-1} + c_{k-2} - \ldots \pm c_0 \geq b_k - b_{k-1} + \ldots \pm b_0$
$\forall k$

Cor $\quad \sum (-1)^k c_k = \sum (-1)^k b_k \quad$ Pf: set $t = -1$ in $(*)$

Pf of Prop

This is a purely algebraic fact

Complex: C_k, $c_k = \dim C_k$

H_k, $b_k = \dim H_k$

Lemma-Ex A complex over a field $\mathbb{F}$ can be decomposed as a direct sum of elementary complexes

$$\ldots 0 \longrightarrow \overset{k}{\mathbb{F}} \longrightarrow 0 \longrightarrow \ldots \quad : H = \mathbb{F}$$

$$0 \longrightarrow \overset{k}{\mathbb{F}} \xrightarrow{\cong} \overset{k-1}{\mathbb{F}} \longrightarrow 0 \ldots : H = 0$$

Essential

$\mathbb{Z}$ not OK

For elementary complexes (*) & ME obviously hold.

Additivity $\Rightarrow$ General case $\triangleleft$

Rmk • Can replace P by P_a everywhere

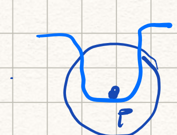
• Or even $(\{a \leq f \leq b\}, \{f = a\})$

Need to take rel. homology

§8. Pf of the Main Thm and further refinements: handle bodies

Some preliminary pts:

- can assume $a = c-\varepsilon < c < c+\varepsilon = b$ so that $\{f=a\}$ & $\{f=b\}$ are very close to $\{f=c\}$
- For the sake of simplicity, assume only one critical pt on $\{f=c\}$. Call it p, $k = \mathrm{ind}(p)$. General case — similar.
- Fix a "morse chart" U near p and a Riemannian metric, Euclidean on U
- $X_0 = -\dfrac{\nabla f}{\|\nabla f\|^2}$ defined on outside crit pts



← cut-off function near p:

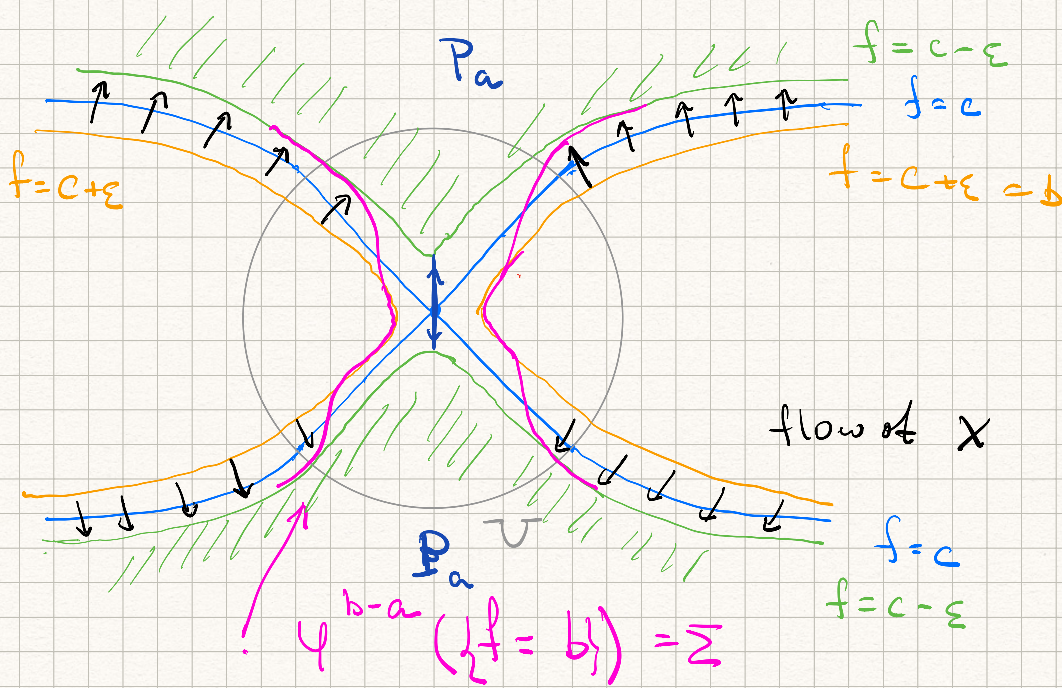
$$h = \begin{cases} 0 & \text{near } p \\ 1 & P \setminus U \end{cases}$$

Replace X_0 by $h \cdot X_0 \cdot \big(\underbrace{\text{cut-off below } a}_{\text{function of } f}\big)$

to get rid of other critical ps

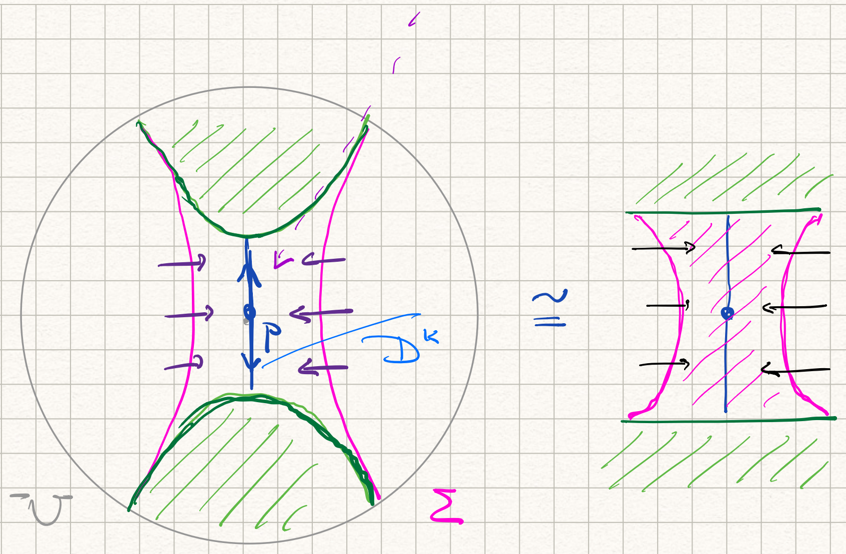
$\Rightarrow$ • the flow of X is defined for all times

• Outside U: "$\varphi^{b-a}: P_b \to P_a$"



$$\varphi^{b-a}(P_b) = P_a \cup \underbrace{\begin{pmatrix}\text{nbd of ustable} \\ \text{manifold of } p\end{pmatrix}}_{D^k}$$

$$\sim P_a \cup D^k$$



◁

In fact, we proved much more:

Attaching handles

= smooth analogue of attaching cells

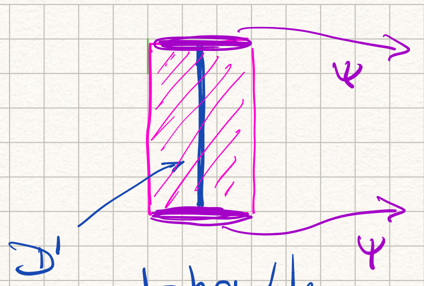
Consider X^n manifold with boundary

$H = D^k \times D^{n-k}$ = k-handle

$\partial(H) = S^{k-1} \times D^{n-k} \cup D^k \times S^{n-k}$

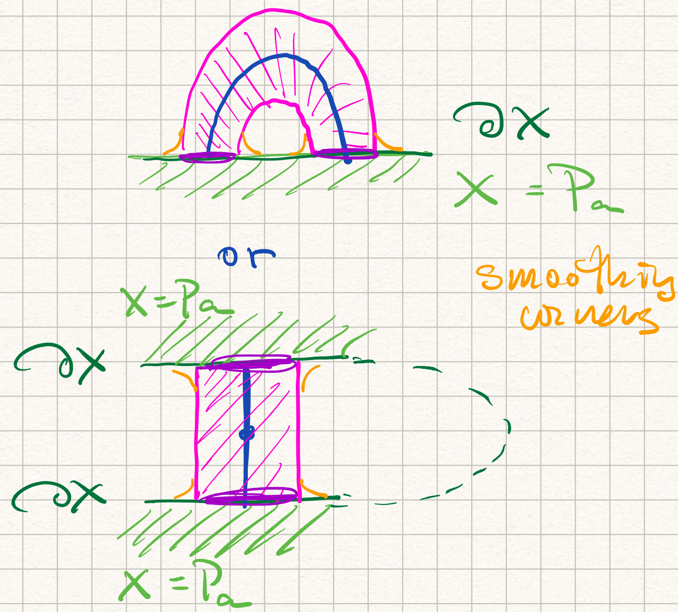
$\psi: S^{k-1} \times D^{n-k} \longrightarrow \partial X$ diffeo

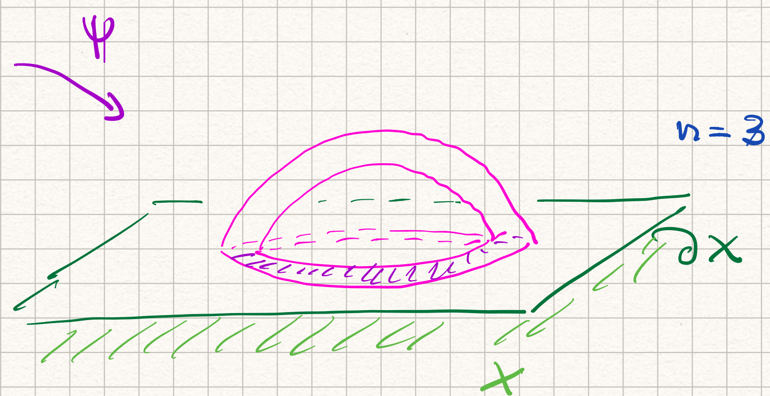
$Y = X \cup_\psi H \leftarrow$ top manifold with ∂ can be made smooth



1-handle

$D^1 \times D^{n-1}$





2-handle

$D^2 \times D^{n-2}$

attaching along

$S^1 \times D^{n-2}$

what we proved is

$\boxed{\textbf{Thm}\ \ P_b \text{ is obtained from } P_a \text{ by attaching a } k\text{-handle for every crit pt of index } k.}$

Cor Need to know that Morse functions exist.

A closed smooth manifold has a "handlebody decomposition"

Rmk Do not require k-handles to be attached before $k+1$-handles, but this can also be achieved.

Ex. Σ_g is obtained from D^2 by attaching $2g$ 1-handles and one 2-handle

Rmk Why Morse Lemma does not matter

Near a non-deg critical pt: $p=0$

$$f(x) = \underbrace{Q(x)}_{\substack{\text{non-deg} \\ d_0^2 f}} + \underbrace{R(x)}_{\text{remainder}}$$

Q dominates R

$$\nabla f = \underbrace{\nabla Q}_{\text{completely dominates } \nabla R} + \nabla R$$

$$\boxed{\text{smooth and top picture of } f \text{ near } p \text{ is completely determined by } Q.}$$

$\Uparrow$ some local dynamics

§9 A Glimpse of Applications to Differential Topology

Skipping Details

- The Key Point:

  Use Morse functions $\rightsquigarrow$ handle body decomposition
  to understand the str of manifolds

- Strategy: starting with $f: P \xrightarrow[\text{Morse}]{} \mathbb{R}$,

  try to simplify f to get as few $\mathrm{Crit}(f)$ as possible & as simple as possible handle-body decomposition

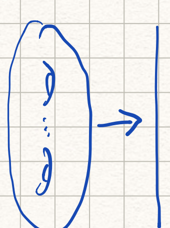
  Background assumption: P has a Morse function — to prove later

- "Lower-bound" to how simple f can be

  $\Leftarrow$ Morse inequalities

Def f is perfect if $\partial = 0$ in the associated complex: $|\mathrm{Crit}_k(f)| = b_k$
(In reality one should also worry about $\mathbb{F}$ in the background.)

Ex.



perfect

Rmk In fact perfect Morse functions are rare

Ideally we want to start with some f and then modify it to make it perfect, but this is rarely possible

- Two preliminary steps - always work

Ex* → Can modify f to "kill" extra max's & min's to make sure ∃ only one max & min

Ex* → Can move lower index pts below higher index pts

$f(x) < f(y) \iff \operatorname{ind}(x) \leq \operatorname{ind}(y)$

(sliding handles)

Similar to CW-complexes

Beyond these two steps things get tricky and one has to impose extra conditions

Ref. J. Milnor "Lectures on the h-cobordism thm"

Applications:

1) Classification of closed surfaces
("Elementary"; see, e.g. Hirsch)
but tedious

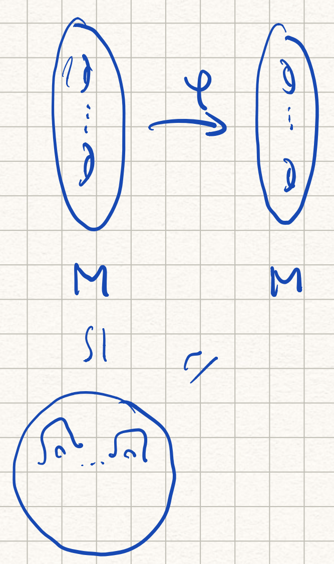
2) The Poincaré conjecture
$P^n \simeq S^n \Rightarrow P \underset{\text{homeo}}{\cong} S^n$

See Milnor's book or many other sources

$n \geq 5$

3) Heegaard Splitting

Thm For every closed orientable 3-manifold P
$\exists$ $\varphi : \Sigma_g \circlearrowleft$ diffeo s.t.



$P = M \cup_{\varphi} M$

$M =$ solid body with g-handles $\partial M = \Sigma_g$

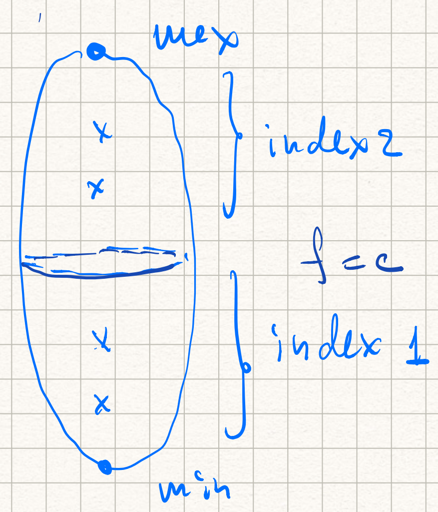
Outline of the pf

- $f : P \longrightarrow \mathbb{R}$ Morse with one max and one min

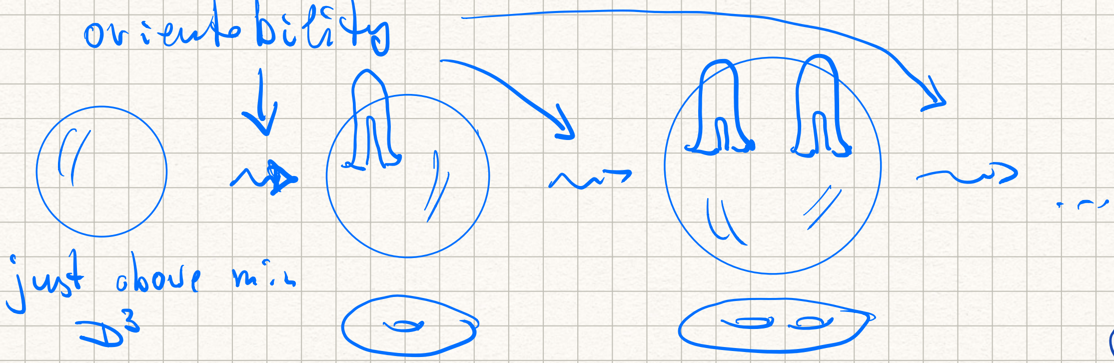
  and index 1 below index 2

- Poincaré Duality $\Rightarrow |Crit_1(f)| = |Crit_2(f)|$

  to be discussed later



Passing through every pt of index 1
= attaching a 1 handle



"$\Rightarrow$" $\{f \le c\}$ is a solid body with some g handles

similarly for $\{f \ge c\}$ (Replace f by $-f$)

<u>Now</u> $P = \underbrace{\{f \le c\}}_{\cong M} \cup \underbrace{\{f \ge c\}}_{M \cong}$

boundary identification

<u>Rmk.</u> This does <u>not</u> lead to a classification of 3-manifolds
<u>Problem</u>: impossible to tell

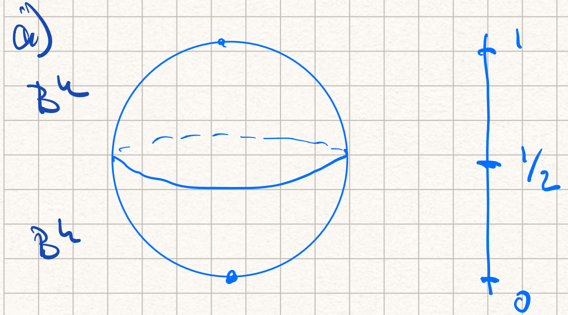
when $\varphi : \Sigma_g \circlearrowleft$ and $\varphi' : \Sigma_{g'} \circlearrowleft$ give rise to the same P!

4) **Thm** P admits a Morse function with exactly two critical pts

$\Rightarrow \quad P \underset{\text{homeo}}{\cong} S^n$ (<u>But not diffeo</u>)

<u>Pf - Outline</u>

a)



Morse lemma

$$\left.\begin{aligned} &f^{-1}([0,\varepsilon]) \cong B^n \\ &f^{-1}([1-\varepsilon, 1]) \cong B^n \end{aligned}\right\} \text{diffeo}$$

$\Rightarrow$ no crit pts

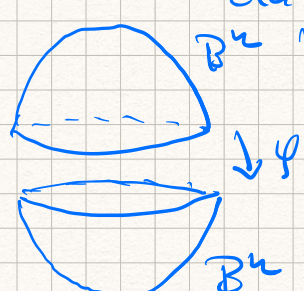
$$\begin{aligned} &f^{-1}([0, 1/2]) \cong B^n \\ &f^{-1}([1/2, 1]) \cong B^n \end{aligned}$$

b) Now we have two copies of B^n

a diffeo $\varphi: S^{n-1} \circlearrowleft$ and

clutching map

$P = B^n \cup_\varphi B^n$:



c) Claim: φ extends to a diffeo $B^n \to B^n$ ← not true in general

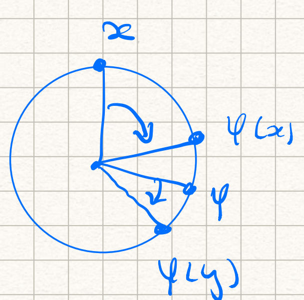
$\Rightarrow P \underset{\text{diffeo}}{\cong} S^n$

↑ Ex ↓

Claim: φ extends to a homeo $B^n \to B^n$ ← always true: use radial ext:

⇓

Thm $\Rightarrow P \underset{\text{homeo}}{\cong} S^n$



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