Morse Theory, Math 232

2021W

Lecture 1

01/05

A few words about the class:

- Not on Canvas – email Zoom links
- Post lecture notes
  Record and post videos
- No exams or hw
  But mention problems in class
- Presentations (optional)
  – the end of the quarter
- No textbook – discuss sources below
- Prerequisites:
  - The manifolds sequence
  - Basic Differential Geom
  - (Co)homology

Textbooks and other sources:

- "Morse Theory" by J. Milnor (1963) ← Rec
- "Lectures on Morse theory, old and new" R. Bott BAMS 7 (1982), 331–358 ← Rec
- Many Diff Top books: Hirsh, Fomenko-Dubrovin-Novikov
  not so bad

Before Floer theory

- "Riemannian Geometry and Geometric Analysis" J. Jost (Chap 7) 2011 ← Recommended
- "Morse Theory and Floer Homology" M. Audin and M. Damian 2014
- Lectures on Morse Homology A. Banyaga and D. Hurtubise 2004

After Floer Theory

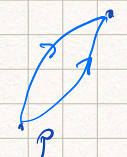
In mid 80s a new way of thinking about Morse Theory – Floer theory – was developed.

It's actually a variant (subset) of Morse theory but it has influenced the theory as a whole.

Our treatment in this class will be modern based on this new perspective but we probably only briefly touch upon Floer theory as such.

Suggested Topics for Presentations

- Morse-Novikov Theory (Erman)
- Morse-Bott & Equiv Morse Theory
- Morse theory for geodesics (connecting two pts) Milnor's book



Milnor

- Morse theory for closed geodesics and Lusternik-Fet thm (Bott's notes)

Bott

- Convex Hamiltonian systems (Periodic orbits a la Ekeland, Fadel-Rabinowitz, pf of Weinstein conj in the convex case
- h-cobordism thm (Milnor)
- Lefschetz hyperplane thm
- Bott periodicity thm (over $\mathbb{C}$)

Appl to top & alg geometry

- Hamiltonian circle actions: symplectic geometry, calculations of cohomology a la $\mathbb{C}P^n$, etc

What Morse theory is about

P = a reasonable space
e.g. finite or inf. dim manifold: a closed manifold, space of closed loops, or paths with fixed pts

$f: P \longrightarrow \mathbb{R}$ reasonably nice "smooth" func.

e.g. a "generic" smooth function on a smooth closed manifold or length or better energy
$x \longmapsto \int |\dot{x}|^2 dt$

Goal: relate critical pts of f to the topology of P

homology or homotopy type
Not pt set topology

E.g. • lower bound on $|\mathrm{Crit}(f)|$ in terms of $H_*(P)$

In particular the existence of $\mathrm{Crit}(f)$
How do we know $\exists$ at least one?

E.g. existence of closed geodesics

Conversely: understand the top of P (e.g. $H_*(P)$) via the str of f and in particular $\mathrm{Crit}(f)$ + more info

Note: Many hugely important objects in math, in physics are critical pts of some functional f — variational principles

Part I: Morse theory for finite-dim manifolds

Setting: $P = M$: $\underbrace{\text{closed}}_{\text{can be relaxed}}$ = compact, $\partial P = \emptyset$

$f : P \xrightarrow{C^k} \mathbb{R}$, $k \geq 2$ usually ∞

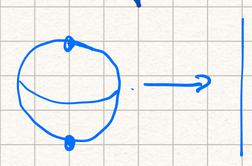
$\mathrm{Crit}(f) = \{ p \mid \underbrace{df(p) = 0}_{L_X f(p) = 0 \ \forall\, X \in T_p P} \}$

the set of critical pts

$$\boxed{\text{Main question: produce a meaningful lower bound for } |\mathrm{Crit}(f)|.}$$

§1 Motivation & Examples:

E.g. $\underbrace{|\mathrm{Crit}(f)| \geq 2}_{\exists \text{ max \& min}}$ — not very interesting



$S^n \subset \mathbb{R}^{n+1}$

Rmk. • But here we can already look at some interaction with top

• what if $|\mathrm{Crit}(f)| = 2$

$\mathrm{Crit}(f) = \{\max, \min\}$

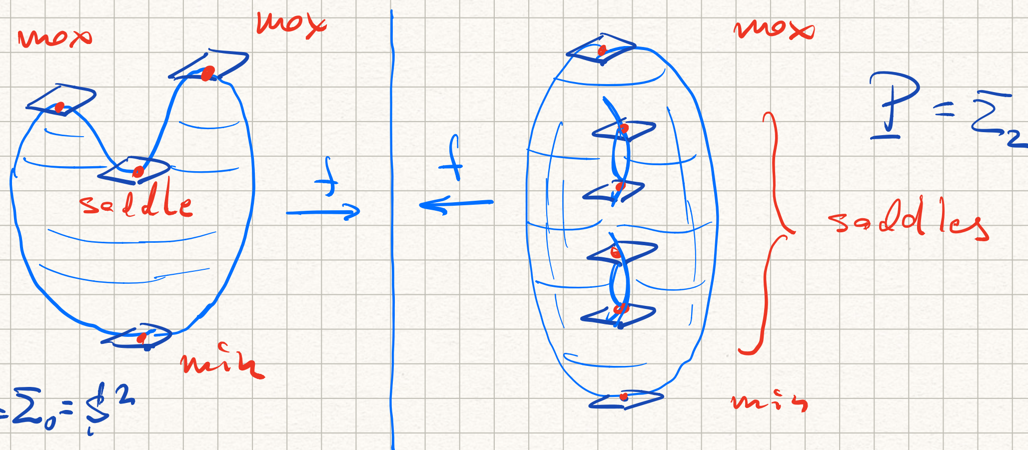
Fact: $P \underset{\text{homeo}}{\simeq} S^n$ but not necessarily diffeo

Return to this later

Ex

$$\text{closed surface} \to P \xrightarrow{\quad} \mathbb{R} \quad \text{height function}$$
$$P \subset \mathbb{R}^3 \to \mathbb{R}$$

$p \in \mathrm{Crit}(f) \iff T_pP$ is horizontal



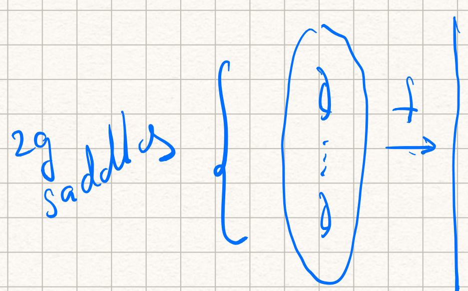
$P = \Sigma_0 = S^2$

$\# = 4 = 2$ min possible

$\# = 6$

These and similar examples suggest that

$$\boxed{\left| \mathrm{Crit}\left(P = \underbrace{\Sigma_g}_{\text{surface of genus } g} \xrightarrow{f} \mathbb{R}\right) \right| \geq \underbrace{2 + 2g}_{\dim H_*(\Sigma_g)}}$$



Not true (other than $g=0$)

But almost true

Ex. $\exists\ f: P = \Sigma_g \xrightarrow{C^\infty} \mathbb{R}$

with exactly 3 crit. pts when $g \geq 1$

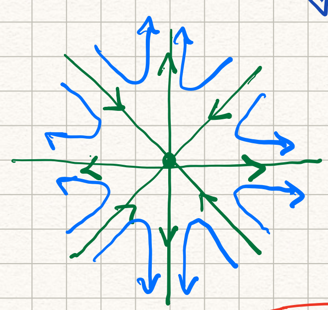
sphere $\longrightarrow$ 2 —— . —— . $\quad g=0$: S^2

(obvious)

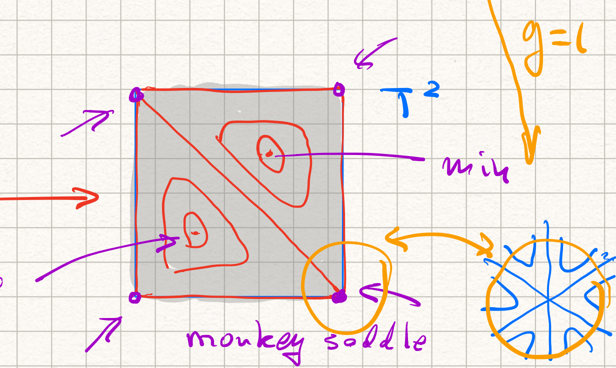
Construct such a function! But not fewer?!

Hint: collapse all $2g$ saddles into one "monkey saddle":

Ex.



smth like $\mathrm{Re}\, z^{2g+1}$



Rmk. This f is not a height function

But true for a broad class of functions = Morse functions

$\Rightarrow$ Key definitions! — p. 8

Rmk - Ex.

- Such a function f cannot be the composition

$$\Sigma_g \xhookrightarrow[\text{embed}]{} \mathbb{R}^3 \xrightarrow{z} \mathbb{R}$$

- But it can be

$$\mathbb{T}^2 \xrightarrow[\text{imm}]{} \mathbb{R}^3 \xrightarrow{z} \mathbb{R}$$, but not $\Sigma_{g>1}$

Ref (From Elijah):

projecteuclid.org/euclid.ijm/1256050732

= T. Banchoff, F. Takens
"Height functions on surfaces with three critical points", Illinois. J. Math
<u>19</u> (1975), 325 - 335.

§2 Basic Definitions — Morse functions and the strategy

$$f: \underset{\substack{\cup\!\!\!\!\!\!\;\;\\ p\, \in\, Crit(f)}}{P} \xrightarrow{C^2} \mathbb{R}$$

Def. The Hessian of f at p

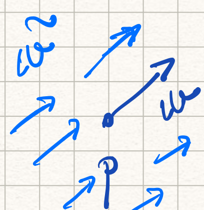
$$d^2_p f: T_pP \times T_pP \longrightarrow \mathbb{R}$$

$$(v,w) \longmapsto (L_{\tilde v} L_{\tilde w} f)(p)$$

directional derivatives

could be v

$\tilde v, \tilde w$ ext of v & w to v.f. near p.



Need to check: independent of $\tilde v, \tilde w$

Prop. a) well-defined ⟵ ind of $\tilde v$ & $\tilde w$

b) symmetric

c) In local coordinates: the ordinary hessian

$$d^2_p f(v,w) = \sum \frac{\partial^2 f}{\partial x_i \partial x_j}(p)\, v_i w_j$$

where $v = \sum v_i \frac{\partial}{\partial x_i}$, $w = \sum w_i \frac{\partial}{\partial x_i}$

$x = (x_1, \dots, x_n)$ ⟵ local coord near p

Rmk. • Condition $p \in \mathrm{Crit}$ is essential:
otherwise $d_p^2 f$ is not well-defined

• Ex. $df(p) \neq 0 \Rightarrow \exists$ a coordinate system $(x_1, \dots, x_n) = x$ near p s.t.

$$f(x) = x_1$$

In fact one can make $f(x) = x_1 + Q(x)$

anything higher order

Pf c) $\Rightarrow$ a) & b)

the dumbest and simplest way to prove the prop

Proving c):

$$L_{\tilde{w}} f = \sum_j \underbrace{\tilde{w}_j}_{\text{functions of } x} \frac{\partial f}{\partial x_j}$$

$$L_v L_{\tilde{w}} f \Big|_p = \sum_{i,j} L_{\tilde{v}_i} \tilde{w}_j \Big|_p \underbrace{\frac{\partial f}{\partial x_j}(p)}_{=0 \Leftarrow d_p f = 0}$$

$$+ \sum_{i,j} \underbrace{\tilde{v}_i(p)}_{v_i} \underbrace{\tilde{w}_j(p)}_{w_j} \frac{\partial^2 f}{\partial x_i \partial x_j}(p)$$

Using more advanced tools of calculus on manifolds:

Coordinate-free pf of b):

$$L_{\tilde v} L_{\tilde w} f - L_{\tilde w} L_{\tilde v} f = L_{[\tilde v, \tilde w]} f$$
$$= df([\tilde v, \tilde w])$$
$$= 0 \quad @ \ p$$

Again the condition $df_p = 0$ is essential

Coordinate-free pf a):

Already know that $d_p^2 f$ is symmetric

Need to show $\tilde w(p) = 0 \Rightarrow d_p^2 f(\tilde v, \tilde w) = 0$

But $d_p^2 f(\tilde w, \tilde v) = L_{\tilde w}(L_{\tilde v} f)(p) = 0$

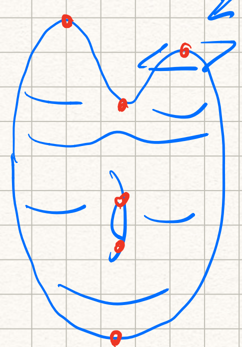
Depends only on $\tilde w(p) = w = 0$

$\triangleleft$

Ex. $P^n \subset \mathbb{R}^{n+1}$

$f =$ projection to some direction

e.g. x_{n+1} - coordinate



$p = 0$ Near p: Locally $P = \mathrm{graph}(f)$

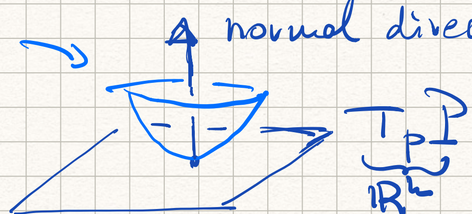
$$f(x) = f(p) + \underbrace{d_p f(x)}_{=0} + \frac{1}{2} d_p^2 f(x) + \cdots$$

the best quadratic approximation of f

The second fundamental form of P

$$\mathrm{II}_p : T_pP \times T_pP \to \mathbb{R}$$

P



$$P = \mathrm{graph}(f)$$

$$\underbrace{(x_1, \dots, x_n)}_{\cong \mathbb{R}^n} \leftarrow \text{local coord on } P$$

Further definitions:

$p \in Crit(f)$

Def. a) p is non-degenerate if $d_p^2 f$ is non-degenerate

b) The Morse index (or just index) of p is the $\underbrace{\text{index of } d_p^2 f}$

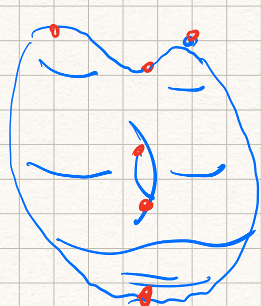
$0 \leq ind_p f \leq \underset{\dim P}{n}$

the # of negative squares when $d_p^2 f = x_1^2 + \dots + x_k^2 - \underbrace{x_{k+1}^2 - \dots x_n^2}_{\text{index}}$

c) f is Morse if all $Crit(f)$ are non-degenerate

Notation: $Crit_k(f) =$ crit pts of index k

Ex



max ind $= 2$

saddles ind $= 1$

min ind $= 0$

$P \subset \mathbb{R}^3$

Or

Ex A monkey saddle is degenerate! unless $k = 1$

Goal

Relate the critical pts of a Morse function to the topology of P.

In particular prove:

Morse Inequalities

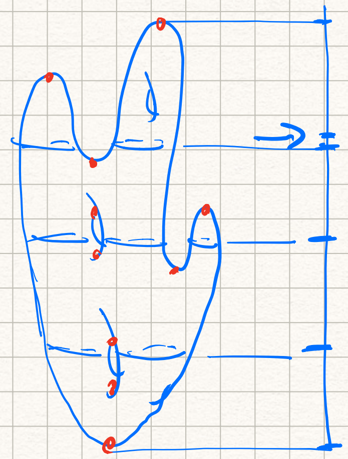
$$\boxed{|\mathrm{Crit}_k(f)| \geq \dim H_k(P)}$$

strategy:

Do this inductively moving

$\min f$ to $\max f$

and looking at how the topology of $\{f \leq c\}$ changes with c



Relevent questions:

- How do we know Morse functions exist on P?
- If so, how common are they?

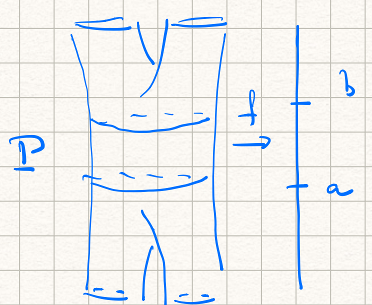
§3 Regular value intervals

<u>Key pt</u>: nothing happens to $\{f \leq c\}$ as long as c remains regular

To be more precise:

- $f : P \xrightarrow{C^2} \mathbb{R}$
  P compact or f is proper
- $[a, b] \subset \mathbb{R}$ contains no critical values

<u>Prop</u> $\{f \leq a\}$ is diffeo to $\{f \leq b\}$



<u>Rmk</u> only matters that $f^{-1}([a, b])$ is compact.

<u>Pf</u>: Use (anti) gradient flow
- key tool in Morse theory

- Fix a Riemannian metric on P:
  $$\langle , \rangle : T_pP \times T_pP \longrightarrow \mathbb{R}$$
  symmetric & pos. def
- ∇f : $\langle \nabla f, \cdot \rangle = df$
  If $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ is orthonormal basis at p:
  $$\text{metric} = \sum dx_i^2 \quad @\, p$$
  ∇f is C^1 $\Leftarrow$ f is C^2

$$\boxed{\nabla f(p) = \sum \frac{\partial f}{\partial x_i}(p) \frac{\partial}{\partial x_i}}$$ Same as on $\mathbb{R}^n$

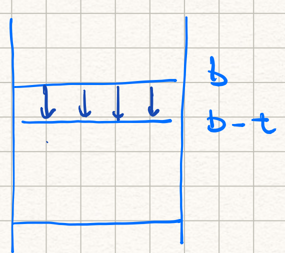
$$L_{\nabla f} f = df(\nabla f) = \langle \nabla f, \nabla f \rangle = \|\nabla f\|^2$$

$$X = -\frac{\nabla f}{\|\nabla f\|^2} \qquad L_X f = -1$$

φ^t = the flow of X, local existence $\Leftarrow \underbrace{\nabla f \; C^1}_{\Uparrow \; f \; C^2}$

$$\underbrace{\frac{d}{dt} f(\varphi^t(x)) = L_X f(x) = -1}_{\text{f decreases along the flow lines of X with unit speed}}$$

$$\Rightarrow f(\varphi^t(x)) = f(x) - t$$



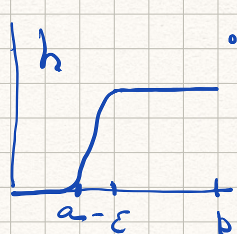
$\varphi^{b-a} : \{f \le b\} \longrightarrow \{f \le a\}$
diffeo

Technical pts:

- compactness $\Rightarrow$ φ^t is defined for all t
- $[a,b]$ regular $\Rightarrow$ $\nabla f \neq 0$ on $f^{-1}([a,b])$ $\triangleleft$

Rmk – Ex. what to change when only $f^{-1}([a,b])$ is compact:

- $f^{-1}([a,b])$ compact $\Rightarrow f^{-1}([a-\varepsilon, b])$ compact
- Replace X by $g \cdot X$:
  $g : P \to \mathbb{R}$ $\quad g = h \circ f$
  cut-off function.



§4. Morse Lemma

Answers the question of what happens at critical pts

But actually more significant than that

Prop (Morse Lemma) $f \in C^3$

$p \in \mathrm{Crit}(f)$ is non-deg $\Rightarrow$

$\exists$ a coord system s.t.

$$f(x) = f(p) + \frac{1}{2}(-x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2)$$
$$= f(p) + \frac{1}{2} d_p^2 f \quad ; \quad k = \mathrm{ind}_p f$$

In other words, by a local change of coordinates near p one can eliminate higher order terms in the Taylor exp of f

Rmk.: Going from $d_p^2 f$ to $-x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots$ is simply diagonalization of $d_p^2 f$ – linear algebra

Or to rephrase: the morse index is the only local invariant of a non-deg critical pt – two crit pts with the same same index are locally diffeo

Morse lemma gives a local picture of what f looks like near a critical pt of index k.

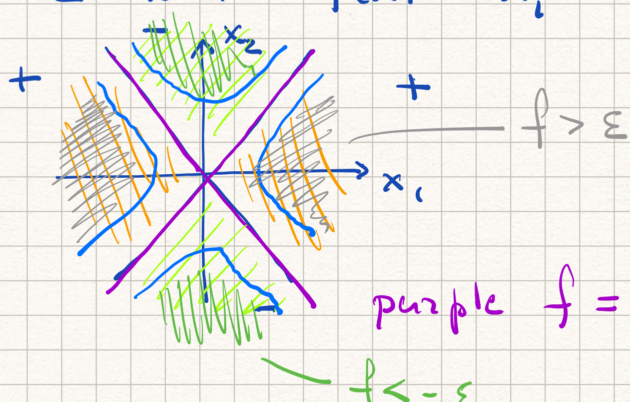
To be used later

Examples

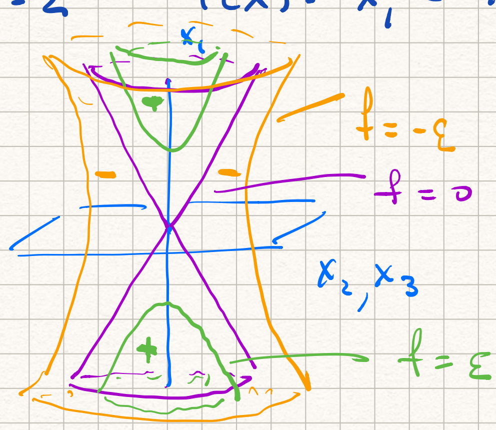
1) $n=2$, $k=0$: $f(x) = x_1^2 + x_2^2$

min

2) $n=2$ $k=1$ $f(x) = x_1^2 - x_2^2$ saddle



3) $n=3$, $k=2$ $f(x) = x_1^2 - x_2^2 - x_3^2$



Preliminaries - of independant interest

Lemma (Hadamard)

$$f : \mathbb{R}^n \xrightarrow{C^r} \mathbb{R}, \quad f(0) = 0$$

$$\Rightarrow \exists\ g_i : \mathbb{R}^n \xrightarrow{C^{r-1}} \mathbb{R} \qquad g_i(0) = \frac{\partial f}{\partial x_i}(0)$$

$$f(x) = \sum g_i(x)\, x_i$$

(near 0)

Rmk. can replace $\mathbb{R}^n$ by a star-shapped domain

Attempts - what does not quite work

$n = 1 \qquad f(x) = x \cdot g(x), \quad g \in C^{r-1}$

1) f is real analytic

$$f(x) = \underbrace{a_0}_{= 0} + a_1 x + a_2 x^2 + \dots$$

$$= x\underbrace{(a_1 + a_2 x + \dots)}_{g(x) \text{ converging}} = x\, g(x)$$

But in general the Taylor exp

- need not converge
  (Any power series is the T. exp of some function - Ex)

- if it converges, need not conv to f.

E.g. It can be identically zero

2) Capitalizing on the assumption that $n=1$ set

$$g(x) = \begin{cases} f(x)/x & x \neq 0 \\ f'(0) & x = 0 \end{cases}$$

Need to show that g is C^{r-1}

Not entirely straightforward

Ex. Prove f $C^1 \Rightarrow g$ C^0

f $C^2 \Rightarrow g$ C^1

Rmk. The condition that

f is C^1

is essential to get g C^0:

and defined at 0

E.g. $f(x) = x^{1/3}$

$$g(x) = \frac{x^{1/3}}{x} = \frac{1}{x^{2/3}} \leftarrow \text{not defined at } 0$$

$$g(0) = \infty$$

<u>Pf</u>

$n=1$

$$f(x) \overset{\text{FTC}}{=} \int_0^1 \frac{d}{dt} f(tx)\,dt \overset{\text{FTC}}{=} f(x) - f(0)$$

Chain rule

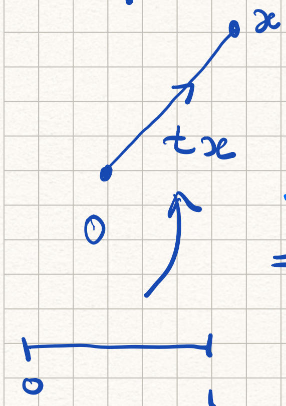
$$= \int_0^1 x f'(tx)\,dt$$

$$= x \underbrace{\int_0^1 f'(tx)\,dt}_{g(x) \quad C^{r-1}} = x\, g(x)$$

when $x=0$

$g(0) = f'(0)$

$n \geq 1$



$$f(x) \overset{\text{FTC}}{=} \int_0^1 \frac{d}{dt} f(tx)\,dt$$

$$= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx) \underbrace{\frac{d(x_i t)}{dt}}_{x_i} \,dt$$

$$= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx)\, x_i \,dt$$

$$= \sum_{i=1}^n x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx)\,dt}_{g_i(x) \quad C^{r-1}}$$

$$= \sum_i x_i\, g_i(x)$$

Again at 0:

$g_i(0) = \frac{\partial f}{\partial x_i}(0)$

Cor Assume $f: \mathbb{R}^n \xrightarrow{C^r} \mathbb{R}$, $r \geq 2$,

$$\nabla f(0) = 0$$

$\Rightarrow$ $h_{ij}: \mathbb{R}^n \xrightarrow{C^{r-2}} \mathbb{R}$ s.t.

- $f(x) = f(0) + \sum_{i,j} h_{ij}(x)\, x_i x_j$
- $h_{ij}(0) = \dfrac{\partial^2 f}{\partial x_i \partial x_j}(0)$

Pf. Apply Hadamard's Lemma twice

Rmk $h_{ij} \overset{?}{=} h_{ji}$ $\leftarrow$ but we don't need this

Pf of the Morse Lemma

The question is local: can work in $\mathbb{R}^n$

$p = 0, \; f(0) = 0$

$f : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}$

nbd of $0 \subset \mathbb{R}^n$

In reality interested in germs

$$f(x) = \underbrace{Q(x)}_{d_0^2 f(x)} + R(x) \leftarrow \text{remainder}$$

the Hessian

Want a diffeo

$$\varphi : (V, 0) \longrightarrow (U, 0)$$

nbds of 0

$$\boxed{f \circ \varphi = Q}$$

Homotopy (Moser) method:

- important
- ref to sympl. geom class (e.g. the pf of Darboux thm, Moser's thm)

But the pf of Morse Lemma is a bit harder

Idea: 1) Set $f_t = Q + tR$
$= (1-t)Q + t \cdot f$
$t \in [0,1]$

Looking for a family of diffeo's
φ_t (near 0) $\qquad \varphi_0 = id$

s.t. $(*)$ $f_t \circ \varphi_t = Q$ $\qquad$ Seemingly a more difficult question

2) Instead of looking for φ_t look for a generating time-dependent v.f.
v_t : $\qquad t \in [0,1]$

$$\frac{d}{dt}\varphi^t(x) = v_t(\varphi^t(x))$$

Differentiating $(*)$ in t

$$(*) \underset{\text{FTC}}{\Leftarrow} \frac{d}{dt} f_t \circ \varphi_t = 0$$

$$\Leftrightarrow \frac{df_t}{dt}(\varphi_t(x)) + L_{v_t} f_t(\varphi_t(x)) = 0$$

$$\Leftrightarrow \boxed{\frac{df_t}{dt} + L_{v_t} f_t = 0} \quad (**)$$

Apply φ_t^{-1} or just $y = \varphi^t(x)$ $\qquad$ ptwise

Want: solve (**) for $v_t = (v_1, \dots, v_n) = \vec{v}_t$

suppress t

Notation $f(x) = Q(x) + R(x)$

$$f(x) = \sum x_i x_j \underbrace{g_{ij}}_{a_{ij} + R_{ij}(x)}(x)$$

Hessian

$R_{ij}(x)$: $= 0$ at 0

$$= \underbrace{\sum x_i x_j a_{ij}}_{Q(x)} + \underbrace{\sum x_i x_j R_{ij}(x)}_{R(x)}$$

$$\boxed{\begin{aligned} f_t(x) &= Q(x) + t R(x) \\ &= \sum x_i x_j a_{ij} + t \sum x_i x_j R_{ij} \end{aligned}}$$

one eq

under-determined

(**) $\Leftrightarrow$

probably equal

$$-\sum_{i,j} x_i x_j R_{ij}(x) = \sum_{i,j} x_i v_j \left(2a_{ij} + t(R_{ij} + R_{ji}) + \dots\right)$$

some more junk

$$-\sum_{i,j} x_i x_j R_{ij}(x) = \sum_{i,j} x_i v_j \left(2a_{ij} + 2t\hat{R}_{ij}\right)$$

vanishing at 0

$$\Leftarrow \left\{ \underbrace{-\sum_j x_j R_{ij}(x)}_{r_i(x)} = \sum 2\left(a_{ij} + t\hat{R}_{ij}(x)\right) v_j \right.$$

system of linear equations at (x, t)

$i = 1, \dots, n$

$$2(a_{ij} + t\underbrace{\hat{R}_{ij}(x)}_{=0 \text{ at } 0}) \leftarrow \text{matrix } A_t(x)$$

$$\vec{r} = A_t \vec{v}_t$$

$$\underbrace{A_t(0) = 2\,\text{Hessian}}_{\text{non-deg}}$$

$\Longleftarrow$ non-deg

non-deg for $t \in [0,1]$
and all x near 0

set $\vec{v}_t(x) = A_t^{-1}(x)\,\vec{r}(x)$

Nuance (important): need to make sure

- φ^t is defined for $t \in [0,1]$
- $\varphi^t(0) = 0$

(Existence & uniqueness is local in t)

$$v_t(0) = 0 \Leftarrow \vec{r}(0) = 0$$

$$r_i(x) = \underbrace{\sum x_j R_{ij}(x)}_{=0 \text{ at } 0}$$

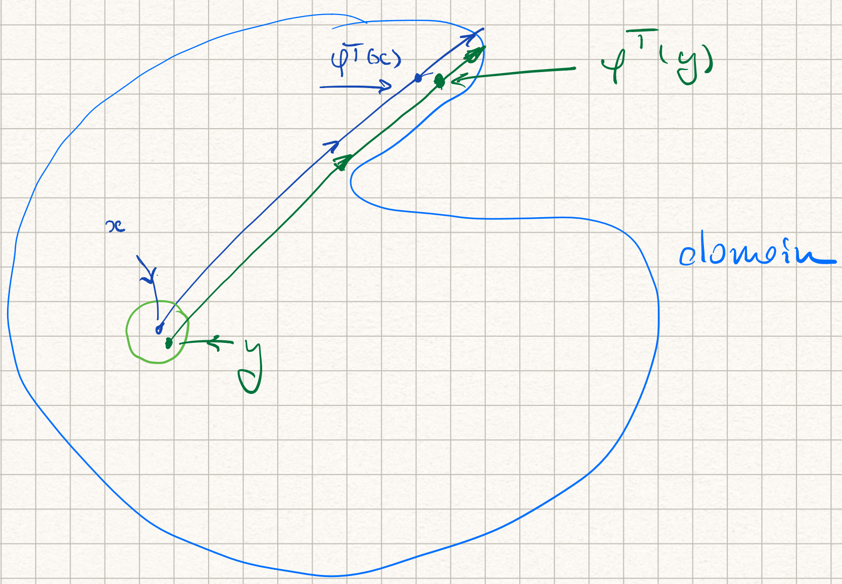
Reason: • the existence time is lower semicontinuous

• ∞ at $x = 0$

$\triangleleft$

25a

Pictorial Pf of Lower Semi-continuity



Main pt: $\varphi^t(y)$ exists and is close to $\varphi^t(x)$ as long as $t \in [0, T]$ if y is close to x. $\triangleleft$

Digression - Broader Context - Lecture 3
Normal Forms & Singularities of Functions

01/12

⇒ Consider a class of objects:

(a) • matrices (= linear transf)

(b) • symmetric matr (= quadr forms

(c) • functions near a pt $p = 0 \in \mathbb{R}^n$

⇒ With an equiv relation coming from a gp action = "coord changes"

(a) • $A \rightsquigarrow BAB^{-1}$

(b) • $A \rightsquigarrow BAB^{T}$

(c) • $f \rightsquigarrow f \circ \varphi$ $\quad \varphi$ = diffeo near p, $\varphi(p) = p$

Normal form = a simple form all (or some) objects can be brought to

roughly speaking

E.g (a) • Jordan normal form

(b) • diagonal matrix with ± 1 or 0 on diagonal

<u>What about (c)?</u>

Is there a normal form for functions near p?

Not really, but YES under additional conditions

• <u>Fact</u>: $df_p \neq 0$ $\quad p = 0$

$\Leftrightarrow \exists\ (x_1, \dots, x_n)$ s.t. $f(x) = x_1 + c$ near c

This is Ex on p 9

In other words: all functions with $df_p \neq 0$ are equivalent to each other (up to a const)

• <u>Next step</u>: Focus on functions with

$$df_p = 0$$

<u>Morse Lemma</u> = Normal form, provided that $d^2_p f$ is non-deg: the only invariant is Morse index

• One can go further and study what happens when $d^2_p f$ degenerates

<u>E.g.</u> For hol functions of one-variable
Ex the normal form is $z^k + c$

For smooth functions the situation quickly becomes complicated
Subject: "Singularities of smooth functions"

Ex. $f: \mathbb{R} \xrightarrow{C^\infty} \mathbb{R}$ near 0

$$f(0) = \dots = f^{(k-1)}(0) = 0$$
$$f^{(k)}(0) \neq 0$$

$\Rightarrow \exists$ a change of variable $\varphi : (\mathbb{R}, 0) \circlearrowleft$ near 0 s.t.

$$(f \circ \varphi)(y) = \pm y^k$$

Hint: Use Hadamard's Lemma

The situation becomes much more involved when $n > 1$

See, e.g., Arnold – Varchenko – Gusein-Zade

Pf.

Hadamard $\Rightarrow f(x) = \pm x^k \cdot g(x)$, $g(0) > 0$

$$= \pm \Big[\underbrace{x\, g(x)^{1/k}}_{y = \varphi(x)} \Big]^k$$

$\leftarrow$ diffeo $\triangleleft$

$$x \longmapsto x g(x) \qquad g(0) \neq 0$$
$$\overset{\varphi}{\longmapsto}$$