Finding Simple geodesics using Morse Bott and Equivariant Morse theory

In this talk we want to explore the question of how many simple, that is to say uniterated, geodesics a metric g has on a manifold M.

In attempting to "count" these geodesics we will introduce the basics of Morse Bott & Equivariant Morse theory.

First let us recall the Morse inequality:

Morse Inequality

$$M_t(f) - P_t(M) = (1+t)\,Q_t(f)$$

where M is a manifold, f is a Morse function $Q_t(f)$ is a polynomial with nonnegative coef.

and we recall the Morse Polynomial

$$\mathcal{M}_t(f) = \sum_{p \in Crit(f)} t^{\lambda(p)}$$

$\lambda(p)$ is the Morse index of p.

as well as the Poincare series

$$P_t(M) = \sum t^k \left| H_k(W, F) \right|$$

Obiviously from this we have that

$$\mathcal{M}_t(f) \geq P_t(M)$$

term by term.

Now Naively we might like to assert that for a manifold M that

$$\underline{\mathcal{M}_t(E)} \geq \underline{P_t(\Lambda M)}.$$

where E is the energy functional and ΛM is the Loop space on M.

There are two main issues with this approach.

1) The energy functional E is not Morse. If γ is a critical point of E corresponding to a nontrivial closed geodesic, then every isometry of S^1 is going to produce a constant speed reparameterization of the geodesic. The isometry group of S^1 is $\underline{O(2)}$. Thus the set of critical points of E has a natural $O(2)$ action in the following sense $\varphi \in O(2)$ acts on $\gamma \in crit(E)$ by

$$\underline{\varphi * \gamma(t) = \gamma(\varphi(t))}.$$

Thus for any critical value of E you have two circles worth of critical points in ΛM. This implies E is not Morse, even if we restrict our attention to P_k. ~~the~~

2) Even if E were Morse every manifold that has one closed geodesic automatically has infinitely many closed geodesics. If γ is a closed geodesic on M then γ^k (the k-th iteration of γ) also is a closed geodesic. If we wish to show that (M, g) has multiple closed simple geodesics we will have to account for the contribution γ^k is making to homology if we want to apply a Morse inequality.

We will introduce Morse Bott theory and Equivariant Morse theory to try to address some of these issues.

Morse - Bott theory

We start by broadening our notion of a function being nondegenerate. Let $P \subseteq \mathrm{Crit}(f)$ be a submanifold of M. We say P is <u>Morse-Bott nondegenerate</u> if $d^2f|_{NP}$ is non singular at every point in P. The Hessian of f is well defined on P because $P \subseteq \mathrm{crit}(f)$. Furthermore The normal bundle of P decomposes as

$$NP = N^+P \oplus N^-P.$$

Where on N^+P and N^-P respectively follow the positive and negative Eigen directions of the Hssian of f. Let $\underline{\lambda^P}$ denote the fiber dimension of N^-P. Finally Let $\underline{\theta^-}$ denote the orientation bundle of N^-P.

Then the assertion is that the correct way to "count" P in the Morse theory sense is

$$\underline{t^{\lambda_P} P_t(P, \theta)}.$$

We say a function f is <u>Morse-Bott nondegenerate</u> if all its critical manifolds are MB-nondeg. If f is MB-nondeg. then we have the following extension of Morse inequality.

$$\underline{\mathcal{M}_t^B(f)} \geq P_t(M; K)$$

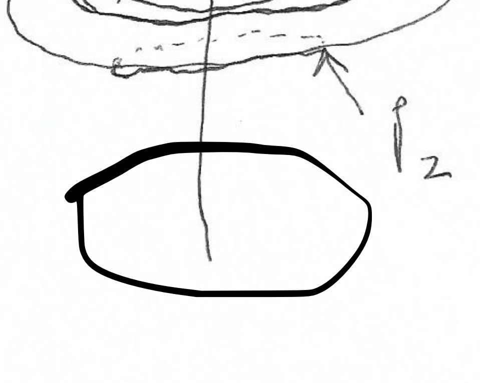
where M is compact, K is the coefficient field and the <u>MB series</u>

$$\underline{\mathcal{M}_t^B(f)} = \sum_{N \subset \mathrm{Crit}(f)} t^{\lambda_P} P_t(\underline{N}; \theta \otimes K).$$

The reason for this inequality should become clear when we discuss how to define a homology theory.

<u>Example</u>

Consider the standard embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$ given by rotating a circle around the Z axis. Then the Z coordinate is a height function on $\mathbb{T}^2$ with two critical circles on the top and bottom of $\mathbb{T}^2$



for P_1 $\text{Dim}(N^- P_1) = 1$

for P_2 $\text{Dim}(N^- P_2) = 0$

$$\mathcal{M}^{3}_t(Z) = (1+t) + t(1+t)$$
$$= 1 + 2t + t^2$$

Ex 2

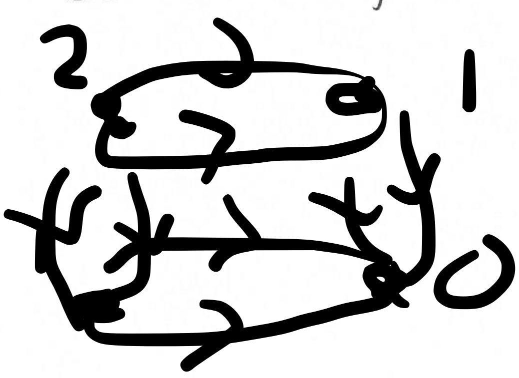
An isolated geodesic in M corresponds to a MB- nondeg. ~~fix.~~ point set in ΛM.

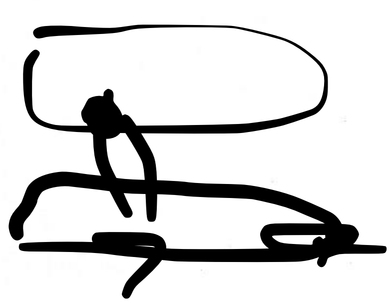
In this perspective positive and negative normal bundles take the place of stable & unstable manifolds, and cells are replaced with ceell bundles.

Another aproach to MB (homology)

One way we could prove the above Morse ineq. is to give an associated Homology complex. There are several ways to do this Here is one method. First each MB function f has finitely my critical maifolds $P^1, \ldots, P^n$ by MB nodegeneracy.

for each P^i fix a morse function $\underline{h}^i$ on P^i. Then the generators of the complex are $\bigcup_{i\in 1}^{n} \mathrm{crit}(h^i)$. The index of a critical point is the sum of index wrt h^i and $\mathrm{Dim}(N^- P^i)$. and the differential is the direct sum of the morse differential on P^i for h^i plus counting the trajectoris from $c \in (\mathrm{crit}\ h^i)$ to $c' \in \mathrm{crit}(h^j)$ where index (c) - index $(c') = 1$. One has to be a bit careful about how one "count" trajectoris betwen critical manifolds





Proposition

for a generic metric g on a manifold M. The ~~E~~energy functional E is MB-nondeg on the peicewise geodesic space $P_n \subseteq \Lambda M$.

Theorem

Let S^n, $n \geq 2$, be the n-sphere then the ~~simple~~ closed geodesics are represented by MB-nondeg. critical manifolds in any P_n in which they occure.

Further more these manifolds are of two types;

$N_0 = S^n$ corresponding to point orbits

$N_K = T_1 S^n$ the unit tangent bundle of S^n corresponding to a great circle starting at a unit vector iterated K-times.

Finally there MB index is given by

$$\text{index } N_k = (2k-1)(n-1) = \lambda_{N_k}$$

From this we can deduce the Morse (mod 2) polynomial of E on P_m

$$M_t(E : P_m) = \overbrace{(1+t^n)}^{N_0} + \sum_{k=1}^{p} \overbrace{(1+t^n)(1+t^{n-1})}^{N_k}\, t^{\lambda_{N_k}}$$

where $p \to \infty$ as $m \to \infty$

the Poincare Series is

$$P_t(\Lambda S^n) = (1+t^n) + \frac{t^{n-1}}{1-t^{2(n-1)}}(1+t^n)(1+t^{n-1}).$$

this shows that we have ∞-many closed geodesics but they may not be simple.

<u>Theorem</u>; Let M be a compact simply connected manifold such that $b_i(\Lambda M, K) \to \infty$ as $i \to \infty$ the ∞ many prime geodesics.

This ~~Th~~ follows from tw facts first
firs a geodesic γ together with
its iterats (γ^k) give the followy
contribution to the morse polynomial

$$2(1+t)\sum_{n=1}^{\infty} t^{\cancel{\lambda(\gamma^m)}} \lambda(\gamma^m)$$

along with the fact that

$$na-b \leq \lambda(\gamma^n) \leq na+b$$

for $a, b \in \mathbb{Z}$.

Thus any simple geodesic contributes
at most a finite amount to the coefficients
~~of~~ of the Morse polynomial and
as such there must be infinitely many.

Equivariant Theory

One failure of the theory to this point is that it fails to explain the following

Example consider the following embedding of the n-sphere into $\mathbb{R}^{n+1}$

$$S^n = \left(\sum_{i=1}^{n+1} (a_i x_i)^2 = 1 \right) \quad (1)$$

where $a_1 < a_2 < \dots < a_{n+1}$ the first critical manifold $T_1 S^n$, decomposes into $\frac{n(n+1)}{2} = \binom{n}{2}$ simple geodesics given by the intersection of the coordinate planes with (1). On the other hand $T_1 S^n$ contributes

$$t^{n-1} P_t(T_1 S^n) = t^{n-1}(1+t^{n-1})(1+t^n)$$

to the Morse Series of ΛS^n

Thus under small perturbations we would expect $T_1 S^n$ to contribute no more than 4 critical points.

The reason for this discrepency is due to the fact that the energy function has a built in symmetry

This symmetry is due to the fact that the energy functional

$$E = \int_0^1 |\dot{r}|^2 \, dt$$

is invariant under isometries of S^1 as mentioned before.

This is of course relative to the domain to the domain of the energy functional. If the domain is H^1 maps from S^1 to M then E is apriori invariant under the $O(2)$ action.

On the other hand if the domain is P_n then E is apriori invariant under the D_{2n} action of cyclicly permuting and reflecting the verticies (marked points)

So our goal is to extend Morse theory in a way that accounts for the action of a compact Lie group G on a manifold M.

First lets look at the case where G acts freely on M. In this case a G-Equivariant function f decends to a smooth function

$f/G : M/G \to \mathbb{R}$.

in this situation we can simply do usual Morse theory of f/G.

This motivates what we do in the general case which is that the action of G is <u>not free</u>. In this case M/G is not a manifold and we cant do Morse theory without extending it to non manifolds. The strategy is to extend ~~case~~ the free case from a topological perspective rather than an analytical one.

First note that if I cross M with a smooth manifold U s.t. G acts freely on U then G acts freely on $M \times U$.

The goal then is to find a U such that

1) G acts freely on U

2) U is contractible

We then define $M_G = M \times U / G$ and the G-equivariant morse ~~theory~~ homology is defined as follow

$$H_*^G(M;k) := H_*(M_G;k)$$

A few notes: 1) if G acts freely on M then the projection

$$M_G \to M/G \quad \text{is a homotopy equivalance}$$

2) The projection

$$M_G \to U/G \quad \text{is a fibration with fiber } M.$$

3) $H_*^G(\text{Point}) \cong H_*(BG)$ here $BG = U/G$

which genrally means it doesn't compute singular homology.

and one sees that BG play an essential role in any equivariant Morse inequality namely if N is a critical manifold such that $N/G = \{x\}$

then $P_t^G(N/G) = P_t(BG)$ by 3).

A final question remains which is do such U exist. The answer is that they do and are essentially unique.

Examples

G	U	$U/G = BG$	$P_t(BG)$
$\mathbb{Z}$	$\mathbb{R}$	S^1	$(1+t)$
$\mathbb{Z}^n$	$\mathbb{R}^n$	T^n	$(1+t)^n$
$\mathbb{Z}_2$	S^∞	$\mathbb{R}P^\infty$	$\frac{1}{1-t}$
S^1	S^∞	$\mathbb{C}P^\infty$	$\frac{1}{1-t^2}$

Example of S^1 action

Let $S^2 = \{x^2+y^2+z^2 = 1\}$ S^1 action is rotation around z axis. S^1 does not act freely on $z = \pm 1$. The function f is the height function. Here $S^2/S^1 = I$ and $P_t(I) = 1$ but there are two critical points. OTOH

$$P^G(S^2) = \frac{1+t^2}{1-t^2}$$ seen via fibration.