Lagrangian Floer Homology

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Motivation

Motivation: Floer's proof of an Arnold-type Conjecture:

Theorem (Floer)

Let (M, ω) be a closed symplectic manifold, L a compact Lagrangian submanifold of M, and ψ a Hamiltonian diffeomorphism of (M, ω). Assume that the symplectic area of any topological disc in M with boundary in L vanishes. Assume moreover that $\psi(L)$ and L intersect transversely. Then the number of intersection points of L and $\psi(L)$ satisfies the lower bound

$$\#(\psi(L) \cap L) \geq \sum_i \dim H^i(L; \mathbb{Z}/2).$$

Floer's Approach

Idea: Given a tranversely intersecting pair of Lagrangian submanifolds L_0, L_1, construct a chain complex $CF(L_0, L_1)$ freely generated by the intersection points $\in L_0 \cap L_1$, satisfying:

- $\partial^2 = 0$, so that the (Lagrangian) Floer (co)homology

$$HF(L_0, L_1) := \ker \partial / \operatorname{im} \partial$$

  is well-defined
- if L_1 and L_1' are Hamiltonian isotopic, then

$$HF(L_0, L_1) \simeq HF(L_0, L_1')$$

- if L_1 is Hamiltonian isotopic to L_0, then (with suitable coefficients)

$$HF(L_0, L_1) \simeq H^*(L_0)$$

Smooth Trajectories Between Two Lagrangians

Let L_0, L_1 be compact Lagrangian submanifolds of symplectic manifold (M, ω) that intersect transversely (and hence at finitely many points).

- The space of smooth trajectories from L_0 to L_1, endowed with the C^∞-topology:

$$\mathcal{P}(L_0, L_1) := \{\gamma : [0,1] \to M \mid \gamma \text{ smooth}, \gamma(0) \in L_0, \gamma(1) \in L_1\}$$

- constant paths $\in \mathcal{P}(L_0, L_1) \longleftrightarrow$ intersection points $\in L_0 \cap L_1$
- For a fixed $\hat{\gamma} \in \mathcal{P}(L_0, L_1)$, let $\mathcal{P}(\hat{\gamma})$ denote the connected component of $\mathcal{P}(L_0, L_1)$ containing $\hat{\gamma}$.

Action Functional

- The universal cover of $\mathcal{P}(\hat{\gamma})$ is

  $$\widetilde{\mathcal{P}(\hat{\gamma})} = \{(\gamma, [\Gamma]) : \gamma \in \mathcal{P}(\hat{\gamma}), \Gamma : \text{smooth homotopy from } \hat{\gamma} \text{ to } \gamma\}$$

  (i.e. $\Gamma : [0,1] \times [0,1] \to M$ smooth with $\Gamma(s, \cdot) \in \mathcal{P}(\hat{\gamma})$ for all $s \in [0,1]$, $\Gamma(0, \cdot) = \hat{\gamma}$, and $\Gamma(1, \cdot) = \gamma$.)

- The action functional $\mathcal{A}$ is defined on $\widetilde{\mathcal{P}(\hat{\gamma})}$ by

$$\begin{aligned} \mathcal{A}(\gamma, [\Gamma]) &:= -\int_\Gamma \omega \\ &= -\int_{[0,1]\times[0,1]} \Gamma^*\omega \\ &= -\int_0^1 \int_0^1 \omega\left(\frac{\partial \Gamma}{\partial s}, \frac{\partial \Gamma}{\partial t}\right) \mathrm{d}s \wedge \mathrm{d}t \end{aligned}$$

Action Functional

WARNING: Actually, I lied: $\mathcal{A}$ is not well-defined!

We impose the following assumption (which will also avoid other complications later):

> Assume that for each $i = 0, 1$,
>
> $$[\omega] \cdot \pi_2(M, L_i) = 0,$$
>
> i.e. any topological disk in M with boundary in L_i has vanishing symplectic area.

Note that, in particular, this implies that M is symplectically aspherical, i.e.

$$[\omega] \cdot \pi_2(M) = 0.$$

Action Functional

To see that $\mathcal{A}$ is well-defined: Let $\gamma \in \widetilde{\mathcal{P}(\hat{\gamma})}$ and Γ, Γ' be homotopic smooth homotopies from $\hat{\gamma}$ to γ.

- We have the cylinder $\overline{\Gamma}\#\Gamma' : S^1 \times [0,1] \to M$, where $\overline{\Gamma}\#\Gamma'(s,i)$ is a loop in L_i, for $i = 0, 1$.
- Γ, Γ' are homotopic $\Rightarrow \Gamma(s,i), \Gamma'(s,i)$ are homotopic in $L_i \Rightarrow$ there exist topological disks whose boundaries are $\overline{\Gamma}\#\Gamma'(s,i) \subset L_i$
- L_i Lagrangian $\Rightarrow$ these disks have symplectic area 0
- symplectic asphericity $\Rightarrow$ symplectic area of the cylinder is also 0:

$$0 = \int_{\overline{\Gamma}\#\Gamma'} \omega = \int_{\Gamma} \omega - \int_{\Gamma'} \omega$$

Riemannian Metric

To do Morse theory, we need a gradient flow, and hence we need a Riemannian metric on $\widetilde{\mathcal{P}(\hat{\gamma})}$.

Recall: Let $J \in \mathrm{End}(TM)$ be an almost complex structure on (M, ω).

- J is ω-compatible if
$$\omega(v, Jv) > 0, \quad \forall v \in T_xM, \forall x \in M,$$
and
$$\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot).$$
- If J is ω-compatible, then it induces a Riemannian metric on M:
$$g_J(\cdot, \cdot) := \omega(\cdot, J\cdot)$$
- The space $\mathcal{J}(M, \omega)$ of ω-compatible almost complex structures on M is non-empty and contractible.

Riemannian Metric

Now, let $(\gamma, [\Gamma]) \in \widetilde{\mathcal{P}(\hat{\gamma})}$.

- The tangent space to $\widetilde{\mathcal{P}(\hat{\gamma})}$ at $(\gamma, [\Gamma])$ is the space of vector fields ξ along γ with initial and final vectors in the tangent spaces to L_0 and L_1, respectively, i.e.

$$T_{(\gamma,[\Gamma])}\widetilde{\mathcal{P}(\hat{\gamma})} = \{\xi \in \Gamma(\gamma^*(TM)) : \xi(t) \in T_{\gamma(t)M}, \xi(i) \in T_{\gamma(i)}L_i\}$$

- Choose a smooth family $J = \{J_t\}_{0\leq t\leq 1} \subset \mathcal{J}(M,\omega)$, which gives rise to a smooth family of Riemannian metrics $\{g_t\}_{0\leq t\leq 1}$.
- We define a Riemannian metric on $\widetilde{\mathcal{P}(\hat{\gamma})}$ as follows: If $\xi, \eta \in T_{(\gamma,[\Gamma])}\widetilde{\mathcal{P}(\hat{\gamma})}$, put

$$\langle \xi, \eta \rangle := \int_0^1 g_t(\xi(t), \eta(t))\mathrm{d}t.$$

Gradient

We are now able to make sense of the gradient flow. Let $(\gamma, [\Gamma]) \in \widetilde{\mathcal{P}(\hat{\gamma})}$ and $\xi \in T_{(\gamma,[\Gamma])}\widetilde{\mathcal{P}(\hat{\gamma})}$. Let $\{\gamma_\eta(t)\}_{1-\epsilon_0 \leq \eta \leq 1+\epsilon_0}$ be a smooth homotopy such that $\gamma_1 = \gamma$ and

$$\frac{\partial}{\partial \eta}\bigg|_{\eta=1} \gamma_\eta(t) = \xi(t),$$

and let $\Gamma : [0, 1+\epsilon_0] \times [0,1] \to M$ be a smooth homotopy for which

$$\begin{aligned} \Gamma(0,t) &= \hat{\gamma}(t) \\ \Gamma(1,t) &= \gamma(t) \\ \Gamma(\eta,t) &= \gamma_\eta(t), \quad \forall \eta \in [1-\epsilon_0, 1+\epsilon_0]. \end{aligned}$$

For each η, we have the homotopy $\Gamma_\eta : [0,1] \times [0,1] \to M$ from $\hat{\gamma}$ to γ_η defined by

$$\Gamma_\eta(s,t) = \Gamma(\eta s, t)$$

Gradient

$$
\begin{aligned}
\mathrm{d}\mathcal{A}_{(\gamma,[\Gamma])}(\xi) &= \frac{\partial}{\partial\eta}\bigg|_{\eta=1}\mathcal{A}(\gamma_\eta,[\Gamma_\eta]) \\
&= -\frac{\partial}{\partial\eta}\bigg|_{\eta=1}\int_{\Gamma_\eta}\omega \\
&= -\frac{\partial}{\partial\eta}\bigg|_{\eta=1}\int_0^1\int_0^1\omega\left(\frac{\partial\Gamma_\eta}{\partial s},\frac{\partial\Gamma_\eta}{\partial t}\right)\mathrm{d}s\wedge\mathrm{d}t \\
&= -\frac{\partial}{\partial\eta}\bigg|_{\eta=1}\int_0^1\int_0^\eta\omega\left(\frac{\partial\Gamma}{\partial\tilde{s}},\frac{\partial\Gamma}{\partial t}\right)\mathrm{d}\tilde{s}\wedge\mathrm{d}t \\
&= -\int_0^1\omega\left(\xi(t),\frac{\partial\gamma}{\partial t}\right)\mathrm{d}t \\
&= \int_0^1\omega\left(\frac{\partial\gamma}{\partial t},\xi(t)\right)\mathrm{d}t
\end{aligned}
$$

Gradient

With the family of ω-compatible almost complex structures J_t, we then have

$$\begin{aligned} \mathrm{d}\mathcal{A}_{(\gamma,[\Gamma])}(\xi) &= \int_0^1 \omega\left(J_t\frac{\partial\gamma}{\partial t}, J_t\xi(t)\right)\mathrm{d}t \\ &= \int_0^1 g_t\left(J_t\frac{\partial\gamma}{\partial t}, \xi(t)\right)\mathrm{d}t \\ &= \left\langle J_t\frac{\partial\gamma}{\partial t}, \xi\right\rangle \end{aligned}$$

That is, with respect to the Riemannian metric we defined,

$$\operatorname{grad}\mathcal{A}(\gamma,[\Gamma]) = J_t\frac{\partial\gamma}{\partial t}$$

Critical Points

$$\operatorname{grad} \mathcal{A}(\gamma, [\Gamma]) = J_t \frac{\partial \gamma}{\partial t}$$

- Since J_t is an automorphism of TM for each t, $\operatorname{grad} \mathcal{A}(\gamma, [\Gamma]) = 0$ if and only if $\dfrac{\partial \gamma}{\partial t} \equiv 0$, i.e. $\gamma \in \mathcal{P}(L_0, L_1)$ is a constant trajectory.

That is:

The critical points of $\mathcal{A}$ are the intersection points $\in L_0 \cap L_1$.

Gradient Flowlines

$$\operatorname{grad} \mathcal{A}(\gamma, [\Gamma]) = J_t \frac{\partial \gamma}{\partial t}$$

- If $p, q \in L_0 \cap L_1$, a flowline of $-\operatorname{grad} \mathcal{A}$ from p to q is a smooth function $\tilde{u} : \mathbb{R} \to \widetilde{\mathcal{P}(\hat{\gamma})}$ satisfying

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}s} = -\operatorname{grad} \mathcal{A}, \quad \lim_{s \to -\infty} \tilde{u}(s) = p, \quad \text{and} \quad \lim_{s \to +\infty} \tilde{u}(s) = q.$$

Equivalently, we have a strip $u : \mathbb{R} \times [0, 1] \to M$ satisfying:

$$\begin{aligned} \frac{\partial u}{\partial s} + J_t \frac{\partial u}{\partial t} &= 0, \\ \lim_{s \to -\infty} u(s, t) &= p, \\ \lim_{s \to +\infty} u(s, t) &= q. \end{aligned}$$

J-Holomorphic Strips

$$\begin{aligned} \frac{\partial u}{\partial s} + J_t \frac{\partial u}{\partial t} &= 0, \\ \lim_{s \to -\infty} u(s,t) &= p, \\ \lim_{s \to +\infty} u(s,t) &= q. \end{aligned}$$

- Note that the first equation above is the Cauchy-Riemann equation $\overline{\partial}_J(u) = 0$ for u, with respect to the family $J = \{J_t\}$ of almost complex structures.
- A map u satisfying $\overline{\partial}_J(u) = 0$ is called a J-holomorphic strip in (M, ω). That is, a connecting gradient flowline is a J-holomorphic strip.
- Riemann Mapping Theorem $\Rightarrow \mathbb{R} \times [0,1]$ is biholomorphic to $D^2 - \{\pm 1\}$, and hence a J-holomorphic strip can also be regarded as a closed Whitney disk with two points removed (corresponding to the two critical points p, q)

Finite Energy

- If M is not compact, we need to impose an extra condition: The J-holomorphic strips $u : \mathbb{R} \times [0,1] \to M$ must have finite energy, i.e.

$$E(u) := \int_u \omega = \int_0^1 \int_0^1 \left| \frac{\partial u}{\partial s} \right|^2 \mathrm{d}s\mathrm{d}t < \infty.$$

- If M is compact, the limiting conditions

$$\lim_{s \to -\infty} u(s,t) = p, \quad \lim_{s \to +\infty} u(s,t) = q$$

turn out to be equivalent to the finite energy condition.

Moduli Space of J-Holomorphic Strips

- $\widehat{\mathcal{M}}(p,q;J)$ denotes the space of all smooth J-holomorphic strips $u : \mathbb{R} \times [0,1] \to M$ for which $\lim_{s\to-\infty} u(s,t) = p$, $\lim_{s\to+\infty} u(s,t) = q$, $u(s,\cdot) \in \mathcal{P}(L_0, L_1)$, and $E(u) < \infty$.
- $\mathcal{M}(p,q;J)$ denotes the quotient of $\widehat{\mathcal{M}}(p,q;J)$ by the action of $\mathbb{R}$ by reparametrization, i.e. $a \in \mathbb{R}$ acts by

$$u \mapsto u_a(s,t) := u(s-a,t)$$

- If $\beta \in \pi_2(M, L_0 \cup L_1)$, $\widehat{\mathcal{M}}(p,q;\beta,J)$ denotes the space of all smooth J-holomorphic strips $u \in \widehat{\mathcal{M}}(p,q;J)$ with $[u] = \beta$.
- $\mathcal{M}(p,q;\beta,J)$ denotes the quotient of $\widehat{\mathcal{M}}(p,q;\beta,J)$ by the usual action of $\mathbb{R}$ by reparametrization.

Maslov Index: One Lagrangian

To address the dimension of the moduli space of holomorphic strips, we look at the *Maslov index*. For the moment, consider a single Lagrangian submanifold L of M.

- If $u : (D^2, \partial D^2) \to (M^{2n}, L)$ is smooth, there is a trivial symplectic fibration $u^*(TM) \to D^2$ (since D^2 is contractible).
- Restricting to $\partial D^2 = S^1$, we have a loop of Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$.
- That is, u induces a loop in the Lagrangian Grassmannian of $\mathbb{R}^{2n}$, i.e. a map
$$u_L : S^1 \to \Lambda(\mathbb{R}^{2n})$$
- The Maslov index of u is the winding number
$$\mu_L(u) := (u_L)_*(1) \in \pi_1(\Lambda(\mathbb{R}^{2n})) \simeq \mathbb{Z}$$
- The Maslov index only depends on the homotopy class, i.e. we have a well-defined homomorphism $\mu_L : \pi_2(M, L) \to \mathbb{Z}$.

Maslov Index: Two Lagrangians

Now, back to the two Lagrangian submanifolds L_0 and L_1 of M:

- If $u : \mathbb{R} \times [0,1] \to M$ is a smooth J-holomorphic strip in $\widehat{\mathcal{M}}(p, q; J)$, then $u^*(TM)$ is again a trivial symplectic fibration since $\mathbb{R} \times [0,1]$ is contractible.
- Restricting to both L_0 and L_1, we have two paths $\ell_0 := (u|_{\mathbb{R}\times 0})^*(TL_0)$ and $\ell_1 := (u|_{\mathbb{R}\times 1})^*(TL_1)$ of Lagrangian subspaces, one connecting T_pL_0 to T_qL_0 and the other connecting T_pL_1 to T_qL_1.
- In general, for two paths $\ell_0, \ell_1 : [0,1] \to \Lambda(\mathbb{R}^{2n})$ of Lagrangian subspaces such that $\ell_0(i)$ and $\ell_1(i)$ are transverse ($i = 0, 1$), the Maslov index of ℓ_1 relative to ℓ_0 is the (signed) count of instances t at which $\ell_0(t)$ and $\ell_1(t)$ are NOT tranverse to each other.
- The index of the holomorphic strip u is the above Maslov index.

Maslov Index: Two Lagrangians

Example: In $\mathbb{C}^n$, consider the paths of Lagrangian subspaces

$$\begin{aligned} \ell_0(t) &= \mathbb{R}^n, \\ \ell_1(t) &= (e^{i\theta_1(t)\mathbb{R}}) \times \cdots \times (e^{i\theta_n(t)\mathbb{R}}) \end{aligned}$$

where each $\theta_i(t)$ sweeps through 0 and is within π.

The Maslov index of ℓ_1 relative to ℓ_0 is n.

Maslov Index: Two Lagrangians

Example: In $\mathbb{R}^2$, consider the strip $u : \mathbb{R} \times [0,1] \to \mathbb{R}^2$ enclosed by

$$\begin{aligned} L_0 &= x\text{-axis}, \\ L_1 &= \{(x, x^2 - x) : x \in \mathbb{R}\}. \end{aligned}$$

The index of the strip u is $\operatorname{ind}([u]) = 1$.

The Moduli Space as a Manifold

Theorem

Let L_0, L_1 be compact Lagrangian submanifolds of a closed symplectic manifold (M, ω). If L_0 and L_1 intersect transversely, then there exists a dense subset $\mathcal{J}_{\text{reg}}(L_0, L_1) \subset C^\infty([0,1], \mathcal{J}(M, \omega))$ such that for $J = \{J_t\}_{0 \leq t \leq 1} \in \mathcal{J}_{\text{reg}}(L_0, L_1)$, p and q in $L_0 \cap L_1$, and $\beta \in \pi_2(M, L_0 \cup L_1)$, the moduli space $\widehat{\mathcal{M}}(p, q; \beta, J)$ is a smooth manifold. Moreover, $\widehat{\mathcal{M}}(p, q; \beta, J)$ has dimension $\operatorname{ind}(\beta)$.

In particular, if $\operatorname{ind}(\beta) = 1$, then $\widehat{\mathcal{M}}(p, q; \beta, J)$ has dimension 1, and hence $\mathcal{M}(p, q; \beta, J)$ has dimension 0.

The Floer Complex

- The Floer complex $CF(L_0, L_1)$ is the free Λ-module generated by the intersection points $\in L_0 \cap L_1$:

$$CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \Lambda \cdot p.$$

- compact and tranverse Lagrangians $L_0, L_1 \Rightarrow$ finitely many generators

Transversality

What if L_0 and L_1 are not transverse to each other? In particular, what if $L_0 = L_1$?

- Introduce an inhomogeneous Hamiltonian perturbation to the Cauchy-Riemann equation:

  $$\frac{\partial u}{\partial s} + J_t(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right) = 0,$$

  where X_t is the Hamiltonian vector field associated to a time-dependent Hamiltonian $H_t : M \to \mathbb{R}$.
- Strips now converge (as $s \to \pm\infty$) to trajectories of the flow of X_t starting on L_0 and ending on L_1, and hence the generators of $CF(L_0, L_1)$ are now flowlines $\gamma : [0,1] \to M$, $\dot{\gamma}(t) = X_t(\gamma(t))$ such that $\gamma(i) \in L_i$.

Novikov Coefficients

To define the Floer differential, we would like to work with *Novikov coefficients*:

- The Novikov ring over a field $\mathbb{K}$ is

$$\Lambda_0 = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} : a_i \in \mathbb{K}, \lambda_i \in \mathbb{R}_{\geq 0}, \lim_{i \to \infty} \lambda_i = +\infty \right\}$$

- The Novikov field Λ is the field of fractions of Λ_0, i.e.

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} : a_i \in \mathbb{K}, \lambda_i \in \mathbb{R}, \lim_{i \to \infty} \lambda_i = +\infty \right\}$$

- The point: There can be only at most finitely many i for which $\lambda_i < 0$ and $a_i \neq 0$.

The Floer Differential

- The Floer differential $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$ is the Λ-linear map defined by

$$\partial(p) = \sum_{q \in L_0 \cap L_1, \text{ind}([u])=1} (\#\mathcal{M}(p,q;[u],J)) T^{\omega([u])} q,$$

  where $\#\mathcal{M}(p,q;[u],J) \in \mathbb{Z}$ (or $\mathbb{Z}/2$) is the signed (or unsigned) count of points in the moduli space of J-holomorphic strips from p to q in the class $[u]$, and $\omega([u]) = \int u^*\omega$ is the symplectic area of u.
- In general, this sum can be infinite. But Gromov's Compactness Theorem $\Rightarrow$ given any energy bound E_0, there are only finitely many $[u] \in \pi_2(M, L_0 \cup L_1)$ with $\omega([u]) \leq E_0$ for which $\mathcal{M}(p,q;[u],J) \neq \emptyset$.
- Thus, with Novikov coefficients, the sum is well-defined.

The Floer Differential

In general, we don't always have $\partial^2 = 0$.

- Floer proved that $\partial^2 = 0$ for $\mathbb{K} = \mathbb{Z}/2$, and under our assumption $\pi_2(M, L_i) = 0$ for $i = 0, 1$.
- One can do away with Novikov coefficients in certain cases, such as when one has *exact* Lagrangian submanifolds in an *exact* symplectic manifold.

Compactness

Another issue in showing $\partial^2 = 0$: compactness of the moduli space

- Gromov's Compactness Theorem $\Rightarrow$ any sequence of J-holomorphic curves with uniformly bounded energy admits a subsequence which uniformly converges, up to reparametrization, to a nodal *tree* of J-holomorphic curves.
- unbounded derivatives, thus leading to energy blowing up in the limit, result in bubbling phenomena

Compactness

Three possible limiting behaviors:

- strip breaking: when energy concentrates at either end $s \to \pm\infty$, i.e. p, q
- disk bubbling: when energy concentrates at a point on the boundary of the strip
- sphere bubbling: when energy concentrates at an interior point of the strip

Compactness

- Strip breaking corresponds to broken trajectories in ordinary Morse theory.
- As in ordinary Morse theory, if the only limiting behavior is strip breaking, one can show that $\partial^2 = 0$.
- The assumption $[\omega] \cdot \pi_2(M, L_i) = 0$ ensures that disk and sphere bubbles never occur.
- Another way to avoid disk and sphere bubbles: If one can guarantee that such bubbles have index > 2
- Still another way: If ambient and Lagrangian submanifolds are exact, energy is constant on a fixed homotopy class $[u]$, and so no disk nor sphere bubbling ocurs.

$\partial^2 = 0$

Sketch of proof that $\partial^2 = 0$, assuming we exclude disk and sphere bubbling: Fix Lagrangians L_0, L_1, a generic almost complex structure J, and a Hamiltonian perturbation H (to ensure transversality).

- Given $p, q \in L_0 \cap L_1$ and $[u] \in \pi_2(M, L_0 \cup L_1)$ with $\mathrm{ind}([u]) = 2$, we have $\dim \mathcal{M}(p, q; [u], J) = 1$
- Gromov compacness, no disk nor sphere bubbles $\Rightarrow$ the moduli space can be compactified to the space $\overline{\mathcal{M}}(p, q; [u], J)$ of broken strips
- $\overline{\mathcal{M}}(p, q; [u], J) \leftrightarrow \mathcal{M}(p, r; [u'], J) \times \mathcal{M}(r, q; [u''], J)$, where r is any generator of the Floer complex and $[u] = [u'] + [u'']$
- index additivity $\Rightarrow \mathrm{ind}[u'] + \mathrm{ind}[u''] = 2$
- tranversality $\Rightarrow$ non-constant strips must have $\mathrm{ind} \geq 1 \Rightarrow$ strips can only break into two components

$\partial^2 = 0$

- Gluing Theorem $\Rightarrow$ every broken strip is locally the limit of a unique family of index 2 strips, and $\overline{\mathcal{M}}(p, q; [u], J)$ is a 1-manifold with boundary

$$\partial\overline{\mathcal{M}}(p, q; [u], J) = \coprod \left(\mathcal{M}(p, r; [u'], J) \times \mathcal{M}(r, q; [u''], J)\right)$$

  taken over all $r \in L_0 \cap L_1$ and $[u] = [u'] + [u'']$ with $\mathrm{ind}([u']) = \mathrm{ind}([u'']) = 1$
- Choice of orientations and spin structures on L_0 and L_1 equips moduli spaces with natural orientations

$\partial^2 = 0$

- Oriented number of boundary points of the compact 1-manifold $\overline{\mathcal{M}}(p, q; [u], J)$ is $0 \Rightarrow$

$$\sum (\#\mathcal{M}(p, r; [u'], J))\,(\#\mathcal{M}(r, q; [u''], J))\,T^{\omega([u])} = 0$$

  where the sum is taken over all $r \in L_0 \cap L_1$ and $[u] = [u'] + [u'']$ with $\operatorname{ind}([u']) = \operatorname{ind}([u'']) = 1$
- Noting that $\omega([u]) = \omega([u']) + \omega([u''])$, summing over all possible $[u]$ gives us that the coefficient of q in $\partial^2(p)$ is 0.

Problem with Disk or Sphere Bubbling

Why we should worry about disk bubbling:

Consider $M = T^*(S^1) \cong S^1 \times \mathbb{R}$, L_0 a simple closed curve going around M once, and L_1 a homotopically trivial loop intersecting L_0 transversely at two points p, q

- $CF(L_0, L_1) = \Lambda p \oplus \Lambda q$
- If u is the upper strip and v is the lower strip, then

$$\partial(p) = \pm T^{\omega([u])} \quad \text{and} \quad \partial(q) = \pm T^{\omega([v])} p.$$

- $\Rightarrow \partial^2(p) \neq 0$

Lagrangian Floer Homology

- The Lagrangian Floer homology of (L_0, L_1) is the homology of the Floer complex:

$$HF(L_0, L_1) := \frac{\ker \partial}{\operatorname{im} \partial}$$

Theorem

Assume that $[\omega] \cdot \pi_2(M, L_i) = 0$ for $i = 0, 1$. Moreover, when $\operatorname{char} \mathbb{K} \neq 2$, assume that L_0, L_1 are oriented and equipped with spin structures. Then, the Floer differential ∂ is well-defined, satisfies $\partial^2 = 0$, and the Floer cohomology

$$HF(L_0, L_1) := H^*(CF(L_0, L_1), \partial)$$

is, up to isomorphism, independent of the chosen almost complex structure J and invariant under Hamiltonian isotopies of L_0 or L_1.

Lagrangian Floer Homology

- Having shown $\partial^2 = 0$, one still has to show that $HF(L_0, L_1)$ is independent of J and is invariant under Hamiltonian isotopies of L_0 and L_1.
- Once the invariance has been established, we can now define, for a single Lagrangian submanifold

$$HF(L, L) := HF(L, \psi(L)),$$

where ψ is any Hamiltonian diffeomorphism such that L and $\psi(L)$ intersect transversely.

Lagrangian Floer Homology

Proposition

Let L be a compact Lagrangian submanifold of (M, ω). If $[\omega] \cdot \pi_2(M, L) = 0$, then

$$HF^*(L, L) \simeq H^*(L; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} \Lambda.$$

Thus, if $\psi \in \mathrm{Ham}(M, \omega)$,

$$\begin{aligned} \#(L \cap \psi(L)) &\geq \dim_\Lambda HF(L, \psi(L)) \\ &= \dim_\Lambda HF(L, L) \\ &= \dim_\Lambda \Lambda \otimes_{\mathbb{Z}/2} H^*(L; \mathbb{Z}/2) \\ &= \sum_j \mathrm{rank}\, H^j(L; \mathbb{Z}/2) \end{aligned}$$

Arnold's Conjecture

Finally, we have Floer's result, a version of the Arnold conjecture, proved using Lagrangian Floer homology:

Theorem

Let L be a compact Lagrangian submanifold of a closed symplectic manifold (M, ω) and $\psi \in \mathrm{Ham}(M, \omega)$. Assume that $[\omega] \cdot \pi_2(M, L) = 0$. Then

$$\#(L \cap \psi(L)) \geq \sum_j \mathrm{rank}\, H^j(L; \mathbb{Z}/2)$$

Arnold's Conjecture

Now let $\psi \in \mathrm{Ham}(M, \omega)$ with non-degenerate fixed points.

- $\mathrm{Graph}(\psi)$ is a Lagrangian submanifold of the symplectic manifold $(M \times M, \pi_1^*(\omega) - \pi_2^*(\omega))$
- nondegeneracy $\Rightarrow$ $\mathrm{Graph}(\psi)$ is transverse to the diagonal $\Delta = \{(x, x) : x \in M\}$
- assuming $\mathrm{Graph}(\psi), \Delta$ satisfy the hypotheses for L_0, L_1,

$$\begin{aligned}
\#(\Delta \cap \mathrm{Graph}(\psi)) &= \#(\Delta \cap (1 \times \psi)(\Delta)) \\
&\geq \sum_{j=0}^{2n} \mathrm{rank}\, H^j(\Delta; \mathbb{Z}/2) \\
&= \sum_{j=0}^{2n} \mathrm{rank}\, H^j(M; \mathbb{Z}/2)
\end{aligned}$$

Arnold's Conjecture

Corollary

Let (M, ω) be a closed symplectic manifold for which $\pi_2(M) = 0$. If $\psi \in \mathrm{Ham}(M, \omega)$ has non-degenerate fixed points, then

$$\#(\mathrm{Fix}(\psi)) \geq \sum_{j=0}^{2n} \mathrm{rank}\, H^j(M; \mathbb{Z}/2).$$

(Not Exactly) An Example

Example:
Consider $M = S^2$ and L an equator in M.

- This DOES NOT satisfy $[\omega] \cdot \pi_1(M, L) = 0$. However, it is a monotone Lagrangian, and Oh showed that Lagrangian Floer homology can still be defined in this case.
- Then, for any Hamiltonian diffeomorphism ψ of M,

$$\#(\psi(L) \cap L) \geq \sum_i \operatorname{rank} H^i(S^1) = 2.$$

- In particular, this shows that L is a non-displaceable Lagrangian in M.

References

- D. Auroux, *A Beginner's Introduction to Fukaya Categories*
- C. Gerig, *Lagrangian Intersection Floer Homology*
- A. Pedroza, *A Quick View of Lagrangian Floer Homology*

END

Thank you for listening!