Morse-Novikov homology

M closed manifold

α closed 1-form on M

Goal: Define "Morse homology" of α.

Plan:

- critical points of α
- "negative gradient" flow
- transversality

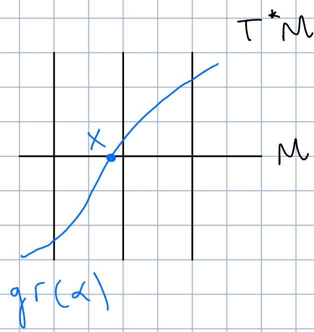
} same as in the exact case $\alpha = df$.

- compactness of moduli spaces } not the same.

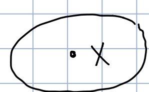
Ref: Hutchings: Lectures notes on Morse homology. (Chap 7)

Critical points

1. $\mathrm{Crit}(\alpha) := \{ x \in M \mid \alpha_x \equiv 0 \}$



(i) $x \in \mathrm{Crit}(\alpha)$ is called non-deg if $gr(\alpha) \pitchfork M$ at x.

(ii) 

$U \subset M$

$\alpha = df \Rightarrow x \in \mathrm{Crit}(f)$

$x \in \mathrm{Crit}(\alpha)$ is non-deg $\Longleftrightarrow$ the Hessian $d_x^2 f$ is non-deg.

2. α is called Morse if all $\mathrm{Crit}(\alpha)$ are non-deg.

in that case $|\mathrm{Crit}(\alpha)| < \infty$.

3. $x \in \mathrm{Crit}(\alpha)$

$\mathrm{ind}(x) :=$ # of negative eigenvalues of $d_x^2 f$.

Gradient flow

choose a metric g on M

$$g(V, \cdot) := -\alpha$$

"negative gradient"

φ^t is the flow of V

Morse

Thm: For a generic g, $\forall\, x, y \in \mathrm{Crit}(\alpha)$

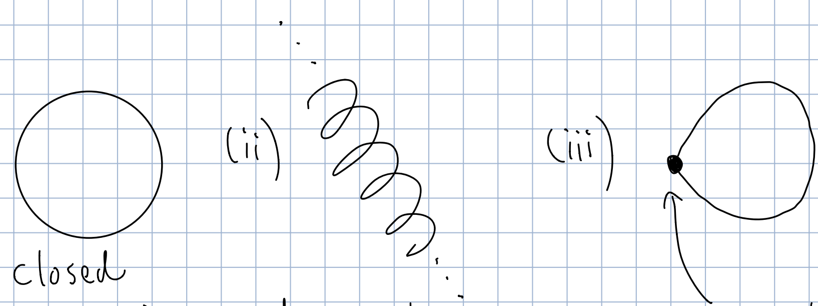
$$W_x^u \pitchfork W_y^s .$$

$$W_x^u = \{p \in M \mid \lim_{t\to-\infty} \varphi^t(p) = x\} \qquad W_y^s = \{p \in M \mid \lim_{t\to\infty} \varphi^t(p) = y\}$$

Not allowed

Ex:

(i) (ii) (iii)



closed (no crit pt)

does not converge to a crit pt

crit pt

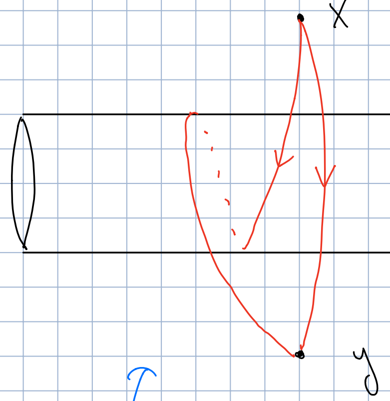
these are not covered by stable/unstable mfds.

Cor: $M(x,y) := W_x^u \cap W_y^s / \mathbb{R}$ is a mfd of dimension $\mathrm{ind}(x) - \mathrm{ind}(y) - 1$.

flow lines connecting x to y.

Remark: $\mathcal{M}(x,y)$ might be zero dim but infinite.

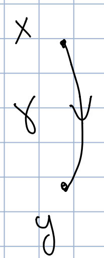
$\operatorname{ind}(x) - \operatorname{ind}(y) = 1$



← hole in M

doesn't happen when $\alpha = df$.

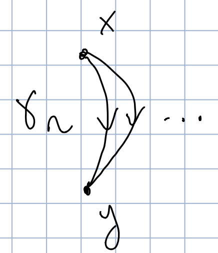
Compactness



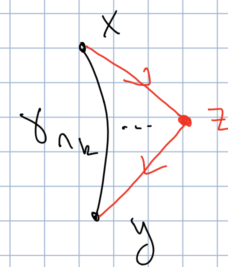
flow line of V

$$E(\gamma) := \int_\gamma -\alpha$$

energy of γ.

Thm: 

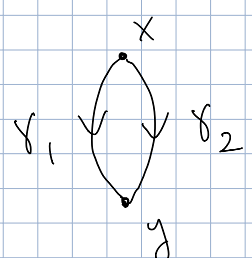
with $E(\gamma_n) < C$

$\Rightarrow \exists$ a subseq. γ_{n_k} that converges. (possibly to a broken trajectory) 

Observations:

① Exact case $\alpha = df$:

$$E(\gamma_n) = \int_{\gamma_n} -df = -(f(y) - f(x)) < C.$$

② 

$$E(\gamma_1) - E(\gamma_2) = \int_{\gamma_1 \# \gamma_2^-} -\alpha = 0$$

$\Leftrightarrow \gamma_1 \# \gamma_2^-$ is in the kernel of

$$P_\alpha : \pi_1(M) \to \mathbb{R}$$
$$\gamma \mapsto \int_\gamma -\alpha .$$

period map

Passing to a covering

$$\pi : \tilde{M} \longrightarrow M$$

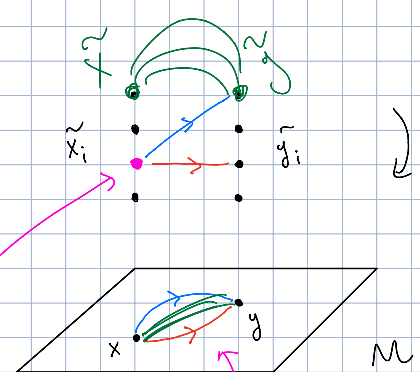
minimal abelian covering of M

s.t. $\pi^* \alpha = d\tilde{f}$

$$\tilde{f} : \tilde{M} \to \mathbb{R}$$

one can also take a larger covering (but minimal is better)

Try: Do Morse homology of $(\tilde{f}, \pi^* g)$



H abelian

(minimal $\Rightarrow$ free action)

Choose

the loop is not in $\ker P_\alpha$

$$H = H_1(M) / H_1(\tilde{M})$$

$$P_\alpha : \pi_1(M) \to \mathbb{R}$$

$$H = \pi_1(M) / \ker P_\alpha$$

Observations:

① $\forall\, \tilde{x}, \tilde{y} \in \mathrm{Crit}(\tilde{f})$

compactness thm.

$$\mathrm{ind}(\tilde{x}) - \mathrm{ind}(\tilde{y}) = 1 \Rightarrow |\mathcal{M}(\tilde{x}, \tilde{y})| < \infty$$

(2) $\forall\, h \in H$, $\forall\, \tilde{x}, \tilde{y} \in \mathrm{Crit}(\tilde{f})$

$$|\mathcal{M}(\tilde{x}, \tilde{y})| = |\mathcal{M}(h\tilde{x}, h\tilde{y})|$$

(3) $$CM_k(\tilde{f}) = \bigoplus_{\substack{\tilde{x} \in \mathrm{Crit}(\tilde{f}) \\ \mathrm{ind}(\tilde{x}) = k}} \mathbb{Z}_2 \langle \tilde{x} \rangle$$

$$\partial_k(\tilde{x}) = \sum_{\mathrm{ind}(\tilde{y}) = k-1} |\mathcal{M}(\tilde{x}, \tilde{y})|\ \tilde{y}$$

sum might be infinite
(e.g. $|\mathcal{M}(x,y)| = \infty$)

Novikov ring

G abelian group

$N : G \to \mathbb{R}$ group homo.

$$Nov(G;N) := \left\{ \underbrace{\sum_{\substack{g \in G \\ a_g \in \mathbb{Z}_2}} a_g \cdot g}_{\text{formal sum}} \;\middle|\; \forall R \in \mathbb{R},\ \exists \text{ finitely many } a_g \neq 0 \text{ with } N(g) < R \right\}$$

Ex:

1) $i : \mathbb{Z} \to \mathbb{R}$

$\mathbb{Z} = \{ \dots, t^{-2}, t^{-1}, 1, t, t^2, \dots \}$

$i(t^{-2}) = -2$

$$Nov(\mathbb{Z}; i) = \left\{ \sum_{i \in \mathbb{Z}} a_i t^i \;\middle|\; \forall R \in \mathbb{R}\ \exists \text{ finitely many } a_i \neq 0 \text{ with } i < R \right\}$$

0 OR 1

$$\simeq \mathbb{Z}_2((t)).$$

take $R = 1$

there has to be finitely many terms

2) $0 : \mathbb{Z} \to \mathbb{R}$

$$Nov(\mathbb{Z}; 0) = \mathbb{Z}_2[t, t^{-1}]$$

Morse - Novikov complex

$$P_\alpha : H \to \mathbb{R} \; ; \quad \Lambda_\alpha := \mathrm{Nov}(H; P_\alpha)$$

$$CN_k(\alpha, g) := CM_k(\tilde{f}) \otimes_{\underbrace{\mathbb{Z}_2[H]}_{\text{group ring}}} \Lambda_\alpha$$

OR equivalently, $\quad \pi : \tilde{M} \to M$

$\forall x \in Crit(\alpha)$, choose a rep $\tilde{x} \in \pi^{-1}(x)$

$$CN_k(\alpha, g) := \bigoplus_{ind(\tilde{x}) = k} \Lambda_\alpha \langle \tilde{x} \rangle$$

rep

$$\partial_k(\tilde{x}) := \sum_{\substack{ind(\tilde{y}) = k-1 \\ h \in H}} \underbrace{|\mathcal{M}(\tilde{x}, h\tilde{y})|}_{\text{mod } 2} \, h\tilde{y}$$

Check: $\displaystyle\sum_{h \in H} |\mathcal{M}(\tilde{x}, h\tilde{y})| \cdot h \in \Lambda_\alpha$

Note that $|\mathcal{M}(\tilde{x}, h\tilde{y})| \neq 0$ $\quad h'(hy)$

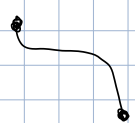
$\Rightarrow \tilde{f}(\tilde{x}) > \tilde{f}(h\tilde{y})$. $\quad P_\alpha(h') \leq 0$

$$\underline{\text{Need}}: \quad \nexists \quad \underbrace{P_\alpha(h_i)}_{\displaystyle = \int_{h_i} -\alpha} \longrightarrow -\infty$$

Pα is injective

such that $|\mathcal{M}(\tilde{x}, h_i\tilde{y})| \neq \emptyset$.

$\tilde{y}$



$\tilde{\eta}_i \subset \tilde{M}$

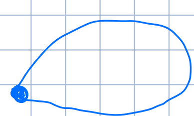
$h_i\tilde{y}$

$d\tilde{f} = \pi^*\alpha$

$$\tilde{f}(h_i\tilde{y}) - \tilde{f}(\tilde{y}) = \int_{\tilde{\eta}_i} d\tilde{f}$$

$\pi(\tilde{\eta}_i) \subset M$

$$= \int_{\underbrace{\pi(\tilde{\eta}_i)}_{h_i}} \alpha = -P_\alpha(h_i)$$



y

$\underline{\text{Thm}}$:

1. $\partial^2 = 0$.
2. $HN_*(\alpha)$ does not depend on $\alpha \in [\alpha]$ but depends on H.
3. $HN_*(\alpha) \cong H_*\left(C_*^{cell}(\tilde{M}) \otimes_{\mathbb{Z}_2[H]} \Lambda_\alpha\right)$

Remark:

① X symplectic vector field on (M, ω)

$$\Rightarrow \quad \omega(X, \cdot) =: \alpha \quad \text{closed}$$

$HN_*(\alpha)$ gives info about fixed points of the flow of X.

② The construction is very similar to Floer homology when

$$\omega\big|_{\pi_2} \neq 0 \quad \text{and} \quad c_1\big|_{\pi_2} = 0.$$

(as doing $HN_*(\alpha)$ on M without introducing $\pi : \tilde{M} \to M$.)

Examples:

① $d\theta \in H^1(S^1; \mathbb{R})$

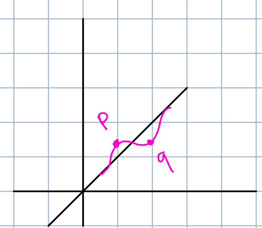
(i) $\pi : \mathbb{R} \to S^1$ covering

$P_{d\theta} : \overset{=\mathbb{Z}}{\pi_1(S^1)} \to \mathbb{R}$ identity

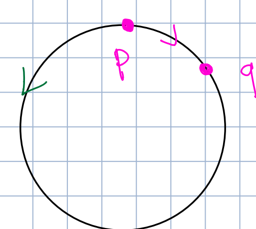
$\Lambda = \mathbb{Z}_2((t))$

(ii) has no crit points $\Rightarrow HN_*(d\theta) = 0.$

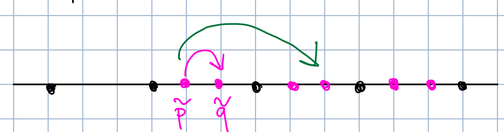
②



perturb $d\theta \longrightarrow \alpha \in H^1(S^1; \mathbb{R})$



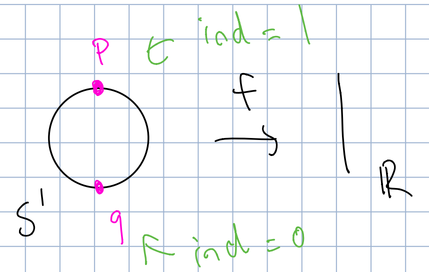
$\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$



$CN_*(\alpha) = \overset{\mathbb{Z}_2((t))}{\Lambda}\langle \tilde{p} \rangle \oplus \Lambda \langle \tilde{q} \rangle \Rightarrow HN_*(\alpha) = 0.$

$\partial \tilde{p} = \underbrace{(1+t)}_{\text{invertible in } \mathbb{Z}_2((t))} \tilde{q}$

(3)



, $df \in H^1(S^1; \mathbb{R})$

(i) $H = 0 \Rightarrow HN_*(df) = HM_*(f)$

(ii) $H = \mathbb{Z}$, $P_{df} : H \longrightarrow \mathbb{R}$ is zero.

$$\Lambda = \mathrm{Nov}(\mathbb{Z}; P_{df}) = \mathbb{Z}_2[t, t^{-1}].$$

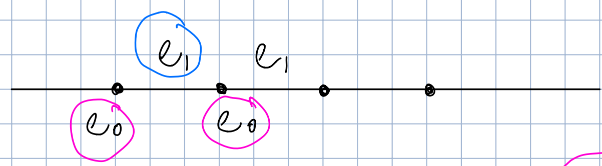
$$CN_*(df) = \Lambda\langle \tilde{p} \rangle \oplus \Lambda \langle \tilde{q} \rangle$$

$$\partial \tilde{p} = (1+t)\,\tilde{q}$$

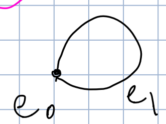
$$HN_1(\alpha) = 0, \quad HN_0(\alpha) = \mathbb{Z}_2[t, t^{-1}] \big/ \langle (1+t) \rangle$$

④ $d\theta \in H^1(S^1; \mathbb{R})$, $\pi : \mathbb{R} \to S^1$

$$C_*^{cell}(\mathbb{R}) \otimes_{\mathbb{Z}_2[\mathbb{Z}]} \mathbb{Z}_2((t))$$



different lifts



$$\partial e_1 = (1+t)\, e_0$$

homology is zero.

Circled valued Morse Theory

Morse - Novikov	Circle valued
$f^* d\theta \in H^1(S^1; \mathbb{R})$	① $f : M \to S^1$
$\mathrm{Crit}(f^* d\theta)$ $\longleftrightarrow$	$\mathrm{Crit}(f)$
② $\alpha \in H^1(M; \mathbb{R})$ integral	$f : M \to S^1$
	 $f(x) = e^{2\pi i \int_\gamma \alpha}$
$\mathrm{Crit}(\alpha)$ $\longleftrightarrow$	$\mathrm{Crit}(f)$