§ 29 Application I: Bott's Periodicity (for $U(n)$)

Lecture 17

03/02

Top Digression

Facts

$\boxed{\begin{array}{l} U(n) \\ = \{A \mid A^* = A^{-1}\} \\ = \text{unitary transf} \\ \quad \text{of } \mathbb{C}^n \end{array}}$

1) $U(n) \hookrightarrow U(n+1)$

$\rightsquigarrow$ $\pi_k(U(n)) \longrightarrow \pi_k(U(n+1))$

$\cong$ $k \leq 2n-1$

2)
$$\begin{tikzcd} SU(n) \arrow[r, hook] & U(n) \arrow[d, "\det"] \\ & S^1 \subset \mathbb{C} \end{tikzcd}$$

Long exact seq

$\Downarrow$

$\Rightarrow$ $\pi_k(SU(n)) \cong \pi_k(U(n))$ $k \geq 2$

Pf of 1) : $U(n+1)$ acts on $S^{2n+1} \subset \mathbb{C}^{n+1}$

and $U(n) = \text{Stab}(\underbrace{\text{North Pole}}_{p_+})$

$\Rightarrow$
$$\begin{tikzcd} U(n) \arrow[r, hook] & U(n+1) \arrow[d] & \ni A \arrow[d, mapsto] \\ & S^{2n+1} & Ap_+ \end{tikzcd}$$

$$\cancelto{0}{\pi_{k+1} S^{2n+1}} \to \pi_k U(n) \xrightarrow[\cong]{} \pi_k U(n+1) \to \cancelto{0}{\pi_k S^{2n+1}} \to \pi_{k-1} U(n)$$

$k+1 < 2n+1$

$\triangleleft$

Set $$U = \bigcup U(n)$$
$$= \varinjlim U(n)$$
$$= \left\{ \begin{pmatrix} \boxed{}^{\,U(n)} & & 0 \\ & 1 & \\ 0 & & \ddots\, 1 \end{pmatrix} \right\} \text{ for some } n$$

$$\underbrace{\pi_k U = \pi_k U(n)}_{\text{stable homotopy gps}}; \quad 2n > k$$

<u>Thm (Bott's periodicity for U)</u>

$$\pi_1 U = \mathbb{Z},\ \pi_2 U = 0,\ \pi_3 U = \mathbb{Z},\ \pi_4 U = 0,\ \dots$$

<u>Rmk</u> ∃ a similar periodicity for $SO(n)$ but it's 8-periodic, & more involved

<u>Pf.</u>
- Following Milnor
- skipping one diff. geometry step

Note: can replace $U(n)$ by $SU(n)$, $k>1$
$$\pi_k U = \pi_k SU$$

<u>A bit more topology:</u>

$$Gr_m(2m) = \{ m\text{-dim } \mathbb{C}\text{-subspaces in } \mathbb{C}^{2m} \}$$

<u>Fact:</u> $$\pi_i Gr_m(2m) \cong \pi_{i-1} U(m)$$
$$i \leq 2m$$

↑

Pf || Two fibrations:

a)

$$\begin{tikzcd} U(m) \arrow[r, hook] & U(2m) \arrow[d] \\ & U(2m)/U(m) \end{tikzcd}$$

induces $\cong$ in π_i $i \leq \ldots$

complex Stiefel manifold: subspaces with frames

$\Rightarrow$ $\pi_i(U(2m)/U(m)) = 0$
$i \leq 2m$

b)

$$\begin{tikzcd} U(m) \arrow[r, hook] & U(2m)/U(m) \arrow[d] \\ & U(2m)/U(m)\times U(m) = Gr_m(2m) \end{tikzcd}$$

$\Rightarrow$ $\pi_i\, Gr_m(2m) \cong \pi_{i-1} U(m)$
$i \leq 2m$

$\lhd$

Lower-dim calculations

- $\pi_1 U(\underset{\substack{\vee\\ 1}}{n}) \cong \pi_1(\underset{\substack{\shortparallel \\ U(1)}}{S^1}) = \mathbb{Z}$

$\underbrace{k \leq 2n-1}_{\text{stable range}}$

- $\pi_2 U(\underset{\substack{\vee\\ 2}}{n}) \cong \pi_2 \underbrace{SU(2)}_{\substack{\shortparallel \\ S^3}} = 0$

Rmk. $\pi_2(\text{any Lie gp}) = 0$

Now the key part: Morse Theory

Metric on SU(n) or U(n)

Equip SU(n) with the Killing metric

- biinvariant
- On $T_I SU(n) = su(n)$ (or $T_I U$):

$$= \{ A \mid A^* = -A, \operatorname{tr} A = 0 \}$$

$$\langle A, B \rangle = \operatorname{tr} AB^*$$

Conj. invariant ⇒ right (or left) translation is bi-inv

- $\exp(A) = I + A + \frac{1}{2} A^2 + \frac{1}{3!} A^3 + \dots$

$$\Omega(I, -I, SU(2m)) \sim \Omega_I(SU(2m))$$

Lemma 1: The space of minimizing geodesics from I to $-I$ is homeo (diffeo) to $Gr_m(2m)$

Lemma 2: Every non-minizing geodesic from I to $-I$ has Morse index $\geq 2m+2$

Rmk $-I$ is conj to I

index $:= \max_{V} \dim V$

$$d^2_\gamma E|_V < 0$$ ⟵ accounting for deg

- We'll prove Lemma 1.
- Lemma 2 ⟶ Milnor (need more diff. geom..)

  Sounds reasonable: think of $SU(2m)$ as something like the sphere: longer geodesics get a lot of conj. pts.

  (See the pf
- Lemmas 1 & 2 ⇒ Bott's periodicity

$$\pi_k\, \Omega(I,-I,SU(2m)) = \pi_k\, \Omega_I(SU(2m))$$

Lemmas ⟶ $\cong$ $k \le 2m$ $\qquad\qquad\qquad \cong$

$$\pi_k\, Gr_m(2m) \qquad\qquad \pi_{k+1}\, SU(2m)$$

$\cong$ $k \le 2m$ $\cancel{\sim}$

$$\pi_{k-1}\, SU(2m)$$

Using the calculation of $\pi_k U$ for $k=1$ & 2 we learn what these groups are.

Pf of Lemma 1

$n = 2m$

$\gamma_A(t) = \exp(tA)$

$A \in T_I SU(n)$: $A^* = -A$, $\operatorname{tr} A = 0$

$\Rightarrow$ A is diagonalizable by a unitary transf

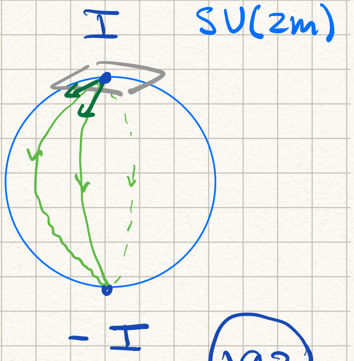
$$BAB^{-1} = \begin{pmatrix} i\pi a_1 & & 0 \\ & \ddots & \\ 0 & & i\pi a_n \end{pmatrix}, \quad B \in U(n)$$

Let's say A itself has this form

$$\gamma_A(1) = \exp(A) = \begin{pmatrix} e^{i\pi a_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\pi a_n} \end{pmatrix}$$

- $\exp(A) = -I \iff$ all a_j = odd integers
- $\ell(\gamma_A) = \int_0^1 \|A\| dt = \|A\|$

$= \pi\sqrt{a_1^2 + \dots + a_n^2}$



$\Rightarrow$ γ_A is a minimizing geodesic
$\Leftrightarrow$ all $a_j = \pm 1$

- But $trA = \Sigma a_j = 0$, $n = 2m$
  $\Rightarrow$ m of $a_j = -1$ & m are $+1$

Rmk This is the reason to work with $SU(2m)$ but not $U(n)$

$A \rightsquigarrow L \subset \mathbb{C}^{2m}$, $L \in Gr_m(2m)$

$\uparrow$ span of eigenvectors with eigenvalue -1

$$\begin{tikzcd}
\left\{\begin{matrix}\text{minimizing}\\ \text{geodesics}\end{matrix}\right\} \arrow[d, leftrightarrow] & \\
\{A\} \arrow[r, leftrightarrow] & Gr_m(2m)
\end{tikzcd}$$

$\triangleleft$

Rmk This also suggests why Lemma 2 is true: then $|a_j| \geq 3$ for at least one a_j. This turns out to imply $\exists$ mult conj pts along such a geodesic. (Non-obvious)

§ 30 Application II: Lefschetz Hyperplane Section Thm

Also in Milnor but here we do a slightly different pf

should really be in Part I

Recall from complex analysis:

- $F : \underset{\mathbb{C}^m}{\overset{\cap}{U}} \longrightarrow \underset{\mathbb{C}^n}{\overset{\cap}{V}}$ is holomorphic if

$$DF \circ J = J \circ DF \quad @ \text{ every pt} \qquad \text{" } J = i \text{ "}$$

$$\Leftrightarrow \quad DF \in M_{m \times n}(\mathbb{C})$$

$\Leftrightarrow$ all components of F satisfy CR with respect to every variable (and F is C^1 ?)

- complex manifolds are just like smooth manifolds but smooth maps, charts, etc are replaced by hol maps

Rmk Complex manifold never have boundary: either open or closed.

E.g. 1) The system of equations

$$\begin{cases} f_1 = 0 \\ \vdots \\ f_k = 0 \end{cases}$$

$f_j : \mathbb{C}^n \longrightarrow \mathbb{C}$ hol

gives a complex submanifold, provided that df_j are lin. ind at every pt (over $\mathbb{C}$)

2) Same true in $\mathbb{C}P^n$ when f_j a hom. polynomials

$f(\lambda z) = \lambda^d f(z)$

We'll also need

Fact

$M \subset \mathbb{R}^n$

The function $f(x) = \|x - p\|^2$ is Morse on M for allmost all $p \in \mathbb{R}^n$

In particular this is true for complex submanifolds of $\mathbb{C}^n$.

In what follows we'll always assume that $p = 0$ and $f(x) = \|x\|^2$ is Morse.

In fact we have proved this
See next page

$$\begin{tikzcd} M \subset \mathbb{R}^n \arrow[d, "f"] \\ \mathbb{R} \end{tikzcd}$$

$h_v(x) = \langle x, \underset{\mathbb{R}^n}{\underset{\cap}{v}} \rangle$: proj to $v\mathbb{R}$

What we have shown is that:

For almost all $v \in \mathbb{R}^n$

$f + h_v$ is Morse on M

But $f(x) = \| x \|^2$

$$f(x) + h_v(x) = \langle x, x \rangle + \langle x, v \rangle$$
$$= \langle x + \tfrac{1}{2} v, x + \tfrac{1}{2} v \rangle - \tfrac{1}{4} \| v \|^2$$

$\Rightarrow$ $\langle x + \frac{1}{2} v, x + \frac{1}{2} v \rangle$ is Morse

Can replace 0 by any pt.

Thm $M^m \subset \mathbb{C}^n$ complex submanifold

$\Rightarrow$ every critical pt of $f(x) = \|x\|^2$ on M has index $\leq m$

Cor Assume that M is proper
($M \cap$ any compact is compact)
$\Rightarrow$ M has homotopy type of m-dim CW complex (perhaps infinite)

Morse theory

Such complex submanifolds of $\mathbb{C}^n$ are called **Stein** manifolds.

Disussion

- Here $m = \dim_{\mathbb{C}} M$, $2m = \dim_{\mathbb{R}} M$
- $\mathbb{C}^n$ has no closed complex submanifolds (if it did, Thm would be wrong)

Max principle: proj of M to any coord is a hol function

M closed $\Rightarrow$ $f = \text{const}$ by max principle

- The assertion is essentially local. Only need to have a nbd of a critical pt.

Pf I

Calculation

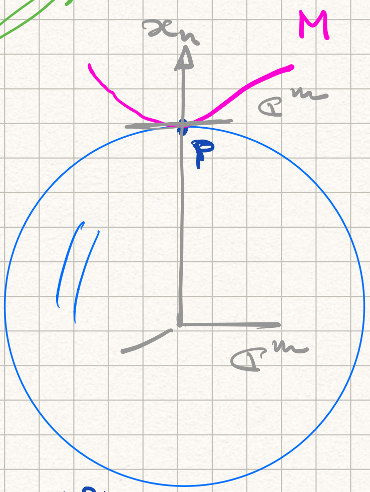
- $f: \underset{\mathbb{C}^n}{\overset{\cap}{M^m}} \longrightarrow \mathbb{R}$ $\qquad f(x) = \|x\|^2$, Morse

- $p \in \mathrm{Crit}(f)$

WLOG:

$p = (0, \dots, 1)$ NP

$\|p\| = 1$



$T_pM \subset T_pS^{2n-1}$

complex $\uparrow$ $\qquad$ $\uparrow$ real hyperplane

$\Downarrow$

$S^{2n-1} = \{\|x\| = 1\}$

$z_k = x_k + i y_k$

$\underbrace{T_pM}_{\mathbb{C}^m} \subset \mathbb{C}^{n-1} \subset T_pS^{2n-1}$

$\{(\underbrace{z_1, \dots, z_{n-1}}, y_n)\}$

$\mathbb{C}^m \uparrow (z_1, \dots, z_m)$

- Near p: M is a graph of a map $\mathbb{C}^m \xrightarrow[\text{hol}]{} \mathbb{C}^{n-m}$

$\xi = (z_1, \dots, z_m) \longmapsto (\overset{z_{m+1}}{g_1}, \dots, \overset{z_n}{g_{n-m}})$ $g_{n-m} \longrightarrow$ to z_n

- $g_k(0) = 0$, $\quad k < n-m$ $\qquad dg_k(0) = 0$
- $g_{n-m}(0) = 1$

$$f(\zeta) = \underbrace{\sum_{j=1}^{m} |z_j|^2}_{Q_0} + \sum |g_k|^2 = \sum_{j=1}^{n} |z_j|^2$$

$k < n-m \Rightarrow g_k(\zeta) = O(|\zeta|^2)$

$\Rightarrow |g_k|^2 = O(|\zeta|^4)$

$\Rightarrow$ does not contribute to $d^2 f$

$k = n-m$, $g_{n-m} = 1 + O(|\zeta|^2)$

$$g = g_{n-m} = 1 + \underbrace{\sum c_{eq} z_e z_q} + \dots$$

$H : \mathbb{C}^m \longrightarrow \mathbb{C}$ a complex quadratic form

$$|g|^2 = 1 + \underbrace{H + \overline{H}}_{Q_1} + \dots$$

$Q_1 = H + \overline{H} : \mathbb{R}^{2m} \longrightarrow \mathbb{R}$

$Q_1(z) = H(z) + \overline{H(z)}$

$= 2 \operatorname{Re} H(z)$

$$d_p^2 f = Q_0 + Q_1$$

Claim: index $Q_1 \leq m$

To be more precise $\mathbb{R}^{2m} = \underbrace{V_0 \oplus V_+ \oplus V_-}_{\text{ortogonal for } Q_1}$

and $Q_1|V_0 = 0$

$Q_1|V_\pm \gtrless 0$

$\dim V_+ = \dim V_-$

Claim $\Rightarrow$ Thm

$$\underset{\underset{0}{\vee}}{Q_0} + \underset{\underset{0}{\vee\!/}}{Q_1} \Big|_{V_0 \oplus V_+} > 0$$

and $\quad \dim (V_0 + V_+) \geqslant m$

$\Rightarrow \quad \operatorname{ind}(Q_0 + Q_1) \leqslant m$

Pf of the Claim

Diagonalize H:

$$BHB^T = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{pmatrix}, \qquad B \in GL(n, \mathbb{C})$$

$L =$ eigenspace

- eigenvalue 0: $\quad Q_1|_L = 0$

  L goes into V_0

- eigenvalue 1: $\quad L = \mathbb{C} = \mathbb{R}^2, \quad z = x + iy$

$$Q_1(z) = z^2 + \bar{z}^2 = 2\operatorname{Re} z^2$$
$$= x^2 - y^2$$

x-axis goes into V_+

y-axis goes into V_-

$\lhd$

$\lhd$

Pf II - Symplectic geometrical

- Real symplectic form on $\mathbb{R}^{2n} = \mathbb{C}^n$

$$\omega = \Sigma\, dx_k \wedge dy_k = \frac{i}{2}\,\Sigma\, dz_k \wedge d\bar{z}_k$$

$$= d\lambda \quad , \lambda = \frac{1}{2}\,\Sigma\, x_k dy_k - y_k dx_k$$

- $f = \frac{1}{4}\Sigma(x_k^2 + y_k^2) = \frac{1}{4}\Sigma\, |z_k|^2$

$$df = \frac{1}{2}\,\Sigma\left(x_k dx_k + y_k dx_k\right)$$

- J acts on $T(\mathbb{R}^{2n} = \mathbb{C}^n)$ as multiplication by i

$$J = \begin{pmatrix} \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & & 0 \\ & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & \\ 0 & & \ddots \end{pmatrix}$$

$\Rightarrow J$ acts on $T^*(\mathbb{R}^{2n} = \mathbb{C}^n)$ as $(J\alpha)(v) = -\alpha(Jv)$

$$J\,df = \frac{1}{2}\,\Sigma\left(x_k dy_k - y_k dx_k\right) = \lambda$$

Checking: $(J\,df)(\partial_{x_k}) = -df(J\partial_{x_k}) = -df(\partial_{y_k})$

$(J\,df)(\partial_{y_k}) = -df(J\partial_{y_k}) = -df(-\partial x_k)$

$$\Rightarrow \quad \omega = d(J\,df) \quad \text{on} \quad \mathbb{C}^n = \mathbb{R}^{2n}$$

- $M \subset \mathbb{C}^n$ complex submanifold

$$J : TM \circlearrowleft$$

⇒ $\boxed{\omega|_M = d(Jdf|_M)}$ $\qquad \lambda|_M = Jdf|_M$

- Liouville v.f.

X on M for $\lambda|_M$

$$i_X \omega = \lambda$$

⇒ $L_X \omega = \omega$: $L_X\omega = di_X\omega + \cancelto{0}{i_X d\omega}$

$$= d\lambda = \omega$$

The flow φ_t of X stretches ω:

$$\varphi_t^* \omega = e^t \omega$$

- $\lambda = Jdf$: $i_X\omega = Jdf$

$\omega(X, v) = -df(\underbrace{Jv}_{w})$ $\qquad \langle\, ,\, \rangle = \omega(\cdot, J\cdot)$

$\underbrace{\omega(X, +Jw)} = +df(w)$

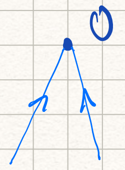
$\langle X, w \rangle = \overset{\shortparallel}{df(w)}$: $\langle X, \cdot \rangle = df$

⇒ $\boxed{X = \nabla f}$

Punchline

$p \in \mathrm{Crit}(f \text{ on } M)$, ω_p

$D\varphi_t^* \omega_p = e^t \omega_p$ on $T_pM = \mathbb{R}^{2m}$



$v, w \in T_pM$

$$\omega(D\varphi_t v, D\varphi_t w) = e^t \omega(v, w) \quad (*)$$

$V \subset T_pM$ be such that $d_p^2 f|_V \leq 0$ $\Big\}$ "stable manifold"

$v, w \in V \Rightarrow D\varphi_t v, D\varphi_t w \longrightarrow 0$ if < 0

or at most grow polynomially as $t \to \infty$

$\Rightarrow \omega(D\varphi_t v, D\varphi_t w)$ cannot grow exp in $(*)$

$\Rightarrow \omega(v, w) = 0 \quad \forall v, w \in V$

$\Rightarrow V$ is isotropic

$\Rightarrow \dim V \leq m$

$\lhd$

Rmk Both pfs don't need non-deg.

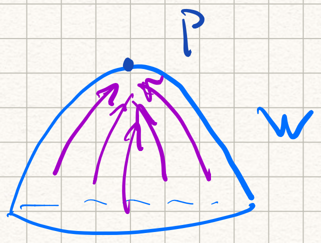
In both cases $\mathrm{ind} := \max_{d^2 f|_E > 0} \dim V \geq m$

Geometrically, assuming non-degeneracy

- $W^s(p) =: W$ = stable manifold of p for $X = \nabla f$
- φ_t = grad flow = flow of X
- $\varphi_t^* \omega|_W = e^t \omega|_W$
- But φ^t contracts W to a pt

$\hookrightarrow$ $\omega|_W = 0$ i.e. W is isotropic

$\Rightarrow$ $\dim W \leq n$



Back to the Lefschetz hyperplace section thm

Setting

$M^n \subset \mathbb{C}P^n$ complex submanifold (closed)
(by def. algebraic)

$\cup$

$H = \mathbb{C}P^{n-1} \leftarrow$ hyperplane

$$M \pitchfork H \Rightarrow \underbrace{M \cap H}_{\text{complex submanifolds in } H=\mathbb{C}P^{n-1}} \subset H$$

The Lefschetz hyperplane section thm:

Thm M is obtained from $M \cap H$ by attaching cells of $\dim \geq m$

$\Downarrow$

Thm

- $H_k(M \cap H) \xrightarrow{\simeq} H_k(M)$ if $k < m-1$
- $H_{m-1}(M \cap H) \underset{\text{onto}}{\twoheadrightarrow} H_{m-1}(M)$ (both over $\mathbb{Z}$)
- $\pi_r(M, M \cap H) = 0$ $r < m$

long exact sequence (+ Lefschetz duality for $k=m-1$?)

Pf. • Recall

$\mathbb{CP}^n \setminus (H = \mathbb{CP}^{n-1}) = \mathbb{C}^n$ holomorphically

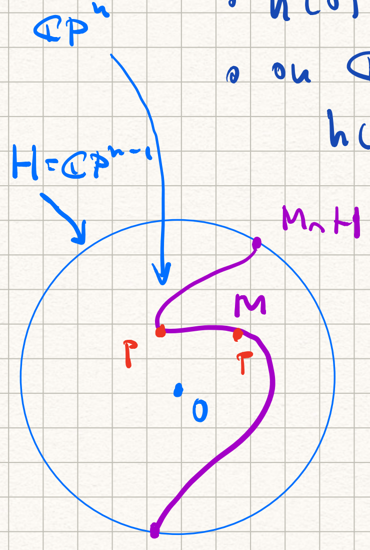
$\sim B^n$ "metrically" (not literally)

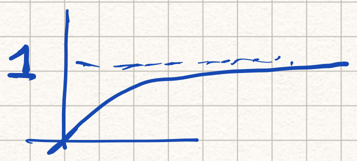
Pick $0 \in \mathbb{CP}^n \setminus H$ (generic)

- $\exists\ h : \mathbb{CP}^n \longrightarrow \mathbb{R}$ such that
  - $h|_H = \min h = 0$
  - $h(0) = \max h = 1$
  - on $\mathbb{C}^n = \mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$

    $h(z) = - g \circ f(z)$, $f(z) = \|z\|^2$

E.g. $(\text{dist to } H)^2$ would do.





- $h|_M$
  - attains min on $M \cap H$ $= 0$
  - Morse outside $M \cap H$ (for a generic chose of 0)

    with only finitely many crit pts.

<u>Rmk</u> <u>Can make sure $h|_M$ is Morse Bott</u>

- M is obtained from

  $\{h \le \varepsilon\}$ = small tub. nbd of $M \cap H$

  $\sim M \cap H$

  by attaching a cell of

  dim = index (h at p)

  for every $p \in \underbrace{\operatorname{Crit}(h \text{ on } M \setminus (M \cap H))}$

  $= \operatorname{Crit}(f \text{ on } M \setminus (M \cap H))$

  and $\operatorname{ind} h = 2m - \underbrace{\operatorname{ind} f}_{\le m} \ge m$

  ◁