MATH 210, MANIFOLDS III, Spring 2006

Homework Assignment V: Euler characteristic
due Wednesday 05/24-2006

1. Let M be a smooth manifold. Furthermore, let us identify the diagonal Δ in $M \times M$ with M by using, say, the projection to the first factor. Show that the normal bundle to Δ in $M \times M$ is isomorphic to TM.

2. Prove that $\chi(M) = 0$, when M is an odd-dimensional, closed manifold.

3. Let v be a vector field on $\mathbb{R}^n$ with a non-degenerate zero at the origin. Thus, $v(x) = Ax + \ldots$, where A is an invertible matrix and the dots denote higher order terms. Show that the origin is an isolated zero of v. Furthermore, recall that the index $\sigma(v)$ of v at the origin is, by definition, $sign(\det A) = (-1)^\nu$, where ν stands for the number of real, negative, eigenvalues of A. Consider the map $f = v/\|v\|$ from a small sphere S^{n-1}_ϵ centered at the origin to S^{n-1}. Prove that $\deg f = \sigma(v)$.

4*. Let M^n be an orientable, closed manifold. Prove that $\chi(M) = 0$ if and only if there exists a non-vanishing vector field v on M.

Remark. Of course, the non-trivial statement is that v exists whenever $\chi(M) = 0$. One may approach this as follows. First show that, under no assumptions on $\chi(M)$, there exists a vector field w such that all zeros of w are contained in a small ball B in a coordinate neighborhood. Now consider the map $f = w/\|w\|$ on ∂B. Prove that the degree of $f\colon \partial B \to S^{n-1}$ is equal to $\chi(M)$. In particular, $\deg f = 0$ when $\chi(M) = 0$. Due to the homotopy classification of maps $S^{n-1} \to S^{n-1}$ by degree, f extends to a map $F\colon B \to S^{n-1}$. Use this map F to modify w inside B to obtain a non-vanishing vector field.

5. Show that $\chi(\mathbb{C}P^n) = n + 1$ by constructing a vector field v on $\mathbb{C}P^n$ with exactly $n + 1$ zeros each of which is non-degenerate and has index one.

Remark. Here is a good candidate for v. Consider $F(z_0, \ldots, z_n) = \sum \lambda_k |z_k|^2$ on $\mathbb{C}^{n+1}$, where all coefficients λ_k are distinct. This function is invariant with respect to the diagonal action of S^1. Thus the restriction of F to the unit sphere descends to a smooth function f on $\mathbb{C}P^n$. Then v is the gradient of f with respect to a suitable (in fact, any) metric.