MATH 210, MANIFOLDS III, Spring 2006

Homework Assignment IV: Transversality and degree, due Wednesday 05/17-2006

1. Let M be a submanifold of $\mathbb{R}^n$. Prove that almost all affine subspaces in $\mathbb{R}^n$ parallel to a given one intersect M transversally.

2*. Let M be a submanifold of $\mathbb{R}^n$. Prove that almost all linear subspaces in $\mathbb{R}^n$ of a given dimension k intersect M transversally, provided that $0 \notin M$ or $k + \dim M \geq n$.

3. Let M and N be closed, oriented manifolds and let $F\colon M \times [0,1] \to N$ be a smooth map. Set $f_0 = F|_{M\times\{0\}}$ and $f_1 = F|_{M\times\{1\}}$. Assume that y is a regular value of all three maps f_0, f_1 and F. Prove that $\deg(f_0, y) = \deg(f_1, y)$.

Remark. This assertion is a part of the proof of homotopy invariance of degree. In class we proved this for $\deg_2$. When doing this problem you need to account for orientations and signs.

4. The next two questions show how the notions of degree and winding number are used or can be used in elementary complex analysis. Prove the following:

(a) Rouché's theorem: Let f and g be holomorphic functions on a neighborhood of the closed unit disc D^2. Assume that $|f| > |g|$ and $f \neq 0$ on ∂D^2. Then the number of zeros of f in D^2 is equal to the number of zeros of $f + g$ in D^2.

(b) The argument principle: Let f be a function meromorphic on a neighborhood of the closed unit disc D^2, with no zeros or poles on ∂D^2. Then

$$W_0(f|_{\partial D^2}) = \#(\text{zeros of } f \text{ in } D^2) - \#(\text{poles of } f \text{ in } D^2).$$

Hints. Let $f\colon D^2 \to \mathbb{C}$ be a smooth map and let $w \notin f(\partial D^2)$ be a regular value of f. Then $W_w(f|_{\partial D^2}) = \deg(f|_{D^2}, w)$. Furthermore, Let U be an open subset of $\mathbb{C}$ and let $z \in U$ be a regular point of a holomorphic function $f\colon U \to \mathbb{C}$. Then, in the notation of the lectures, $\sigma_z(f) = 1$ (not -1!), where $\mathbb{C}$ is equipped with its standard orientation. In other words, a holomorphic map is necessary orientation preserving at its regular points. Combining these two observations, we see, for instance, that $W_0(f|_{\partial D^2}) = \#(\text{zeros of } f \text{ in } D^2)$ in (a), provided that 0 is a regular value.

5. Let M^m and N^n be disjoint, closed, oriented submanifolds of $\mathbb{R}^{k+1}$ such that $m + n = k$. (For instance, M and N are two disjoint circles in $\mathbb{R}^3$.) The linking number $Lk(M, N)$ is the degree of the map $F\colon M \times N \to S^k$ defined by $F(x, y) = (x - y)/\|x - y\|$.

(a) Prove that $Lk(M, N) = (-1)^{(n+1)(m+1)} Lk(N, M)$.

(b) Assume that M is the boundary of a compact, oriented submanifold X contained in $\mathbb{R}^{k+1} \setminus N$ or that $\dim M > 0$ and M is contractible to a point in $\mathbb{R}^{k+1} \setminus N$. Prove that then $Lk(M, N) = 0$.

(c) Find the linking number in $S^3 \setminus \text{point} = \mathbb{R}^3$ of two distinct fibers of the Hopf fibration.