MATH 210, MANIFOLDS III, Spring 2006

Homework Assignment III: Fundamental group, due Wednesday 05/03-2006

Throughout this assignment, X and Y stand for sufficiently nice (metrizable, locally path connected) *path connected* topological spaces.

1. Let $[S^1, X]$ be the set of homotopy classes of continuous maps $S^1 \to X$. Prove that $[S^1, X]$ is canonically in one-to-one correspondence with conjugacy classes of elements of $\pi_1(X, x_0)$ (for any x_0).

2. Prove that $\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$. As a consequence, $\pi_1(T^n) = \mathbb{Z}^n$.

3. Prove that $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$.

4. Let Σ_g be the sphere with g handles. Prove that $\pi_1(\Sigma_g)$ is the group with $2g$ generators, say $a_1, b_1, \ldots, a_g, b_g$, and the relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

Hint: use the Seifert – van Kampen theorem.

5. Let $\pi\colon Y \to X$ be a covering map (with X and Y path connected), $x_0 \in X$ and $y_0 \in \pi^{-1}(x_0)$. Let $\gamma\colon [0,1] \to X$ be a loop with $x_0 = \gamma(0) = \gamma(1)$. Consider the unique lift $\tilde{\gamma}\colon [0,1] \to Y$ such that $\tilde{\gamma}(0) = y_0$ and set $y_1 = \tilde{\gamma}(0)$. Prove that $y_1 = y_0$ iff the class $[\gamma] \in \pi_1(X, x_0)$ is in the subgroup $\pi_*(\pi_1(Y, y_0))$.

Remark. This fact is central to many results concerning covering maps. Hence, proving it you should rely only on definitions and the existence and uniqueness of the lift $\tilde{\gamma}$.

6. Describe (e.g., sketch) the universal covering of the figure eight space.

7. Let $X = S^1$ and $k \geq 2$. Prove that every map $f\colon S^k \to X$ is homotopic to a constant map, i.e., a map sending S^k to one point. Prove the same result when X is the figure eight space. (Hint: lift f to the universal covering $\tilde{X}$ of X and use the fact that $\tilde{X}$ is contractible. Try to prove a similar result for a sphere with $g \geq 1$ handles.)