MATH 210, MANIFOLDS III, Spring 2006

Homework Assignment II: Transversality, due Wednesday 4/26-2006

1. Prove that transversality is a C^1-open condition. More specifically, assume that M is a compact manifold and $\partial N = \emptyset$, and let $L \subset N$ be a submanifold that is closed as a subset of N. Let $f\colon M \to N$ be transverse to L. Then every smooth map $g\colon M \to N$ which is C^1-close to f is also transverse to L.

2. Assume that M is compact, N is a manifold without boundary and $L \subset N$ is a submanifold that is closed as a subset of N. Let $f\colon M \to N$ be a map transverse to L. Prove that

(a) $f^{-1}(L)$ is a closed submanifold of M of dimension $\dim f^{-1}(L) = \dim M + \dim L - \dim N$, provided that M is closed;

(b) more generally, $f^{-1}(L)$ is a submanifold of M such that $\partial f^{-1}(L) \subset \partial M$ and again $\dim f^{-1}(L) = \dim M + \dim L - \dim N$, provided that $f|_{\partial M}$ is transverse to L.

Remarks. The assertion of the problem implies that a transverse intersection of smooth submanifolds is again a smooth submanifold. Part (b) is one of a few possible extensions of (a) to the case where the manifolds have boundary. The dimension formula for $f^{-1}(L)$ takes a much simpler form when interpreted in terms of codimension: $\operatorname{codim} f^{-1}(L) = \operatorname{codim} L$.

3. The goal of this problem is deriving the transversality theorem for sections of a fiber bundle from that for maps. Assume for the sake of simplicity that M is a closed manifold.

(a) Let $\varphi_0\colon M \to M$ be a diffeomorphism and let $\varphi\colon M \to M$ be a smooth map which is C^1-close to φ. Prove that φ is a diffeomorphism.

(b) Let $\pi\colon E \to M$ be a fiber bundle and let $L \subset E$ be a smooth closed submanifold. Prove the transversality theorem for sections of E: arbitrarily close (in the C^∞ sense) to any section f of E there exists a section g which is transverse to L.

Remarks and hints. The assertion that requires some attention in part (a) is that φ is one-to-one and onto. (Note also that (a) fails when φ is only required to be C^0-close to φ_0. Why?) To derive (b) from the transversality theorem for maps, consider a map $\tilde{g}\colon M \to E$ transverse to L and C^∞-close to f, whose existence is guaranteed by the transversality theorem for maps. The map $\tilde{g}$ need not be a section. (Why?) Note that $\varphi = \pi \circ \tilde{g}\colon M \to M$ is C^∞-close to id, and hence, by (a), is a diffeomorphism. Then $g = \tilde{g} \circ \varphi^{-1}$ is a section of E that is C^∞-close to f and transverse to L.

4. Let l_0 and l_1 be two disjoint embedded circles in $\mathbb{R}^4$. Prove that l_0 and l_1 are not linked, i.e., there exists a map $f\colon D^2 \to \mathbb{R}^4$ whose boundary is l_0 (i.e., $f|_{S^1}$ is an embedding with image l_0) and such that $f(D^2) \cap l_1 = \emptyset$.